Two models of Lévy random fields and their multifractal analysis

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Outline of the talk

Multifractal properties of Lévy processes

Lévy fields in the sense of Mori, and their multifractal properties

Lévy fields in the sense of Adler et al., and their multifractal properties
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Lévy processes and Lévy-Itô decomposition

**Definition**

A **Lévy process** is a random process:

- with stationary and independent increments,
- with càdlàg sample paths,
- which vanishes at zero.

**Theorem (Lévy-Itô decomposition)**

Let $Y = \{Y(t), t \geq 0\}$ be a real-valued Lévy process. Then, $Y$ may be decomposed in the form

$$Y(t) = \underbrace{at}_{\text{drift term}} + \underbrace{\sigma B(t)}_{\text{Brownian component}} + \underbrace{L_\nu(t)}_{\text{jump component}},$$

where $\nu$ is the **Lévy measure** of $Y$, i.e. a nonnegative Borel measure $\nu$ on $\mathbb{R}^*$ such that

$$\int_{x \in \mathbb{R}^*} (1 \wedge x^2) \nu(dx) < \infty.$$
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\int_{x \in \mathbb{R}^*} (1 \wedge x^2) \nu(dx) < \infty.
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Hölder exponent of the Brownian component

Definition

If $f : \mathbb{R}^d \to \mathbb{R}$ is locally bounded, the **Hölder exponent** $\alpha_f(t_0)$ is the supremum of all $\alpha \geq 0$ such that

$$\exists P_{t_0} \text{ polynomial} \quad \exists C \quad \forall t \text{ near } t_0 \quad |f(t) - P_{t_0}(t)| \leq C\|t - t_0\|^\alpha.$$

Theorem

$$Y(t) = \underbrace{at}_{\text{drift term}} + \underbrace{\sigma B(t)}_{\text{Brownian component}} + \underbrace{L_\nu(t)}_{\text{jump component}}.$$ 

The Brownian component is **mono-Hölder**:

$$a.s. \quad \forall t \geq 0 \quad \alpha_B(t) = 1/2.$$
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Multifractal analysis of the jump component

**Definition**

If $f : \mathbb{R}^d \to \mathbb{R}$ is locally bounded, the **iso-Hölder sets** of $f$ are

$$E_f(h) = \{ t \in \mathbb{R}^d | \alpha_f(t) = h \}.$$

and its **local spectrum of singularities** is

$$d_f(h, W) = \dim_H(E_f(h) \cap W).$$

**Theorem (S. Jaffard, 1999)**

- The jump component $L_\nu$ is a **homogeneous multifractal process**: its spectrum is deterministic and independent on $W$.
- Its spectrum is **linear**, and governed by the Lévy measure $\nu$ only through its Blumenthal-Getoor index

$$\beta_\nu = \inf \left\{ \gamma > 0 \left| \int_{|x| \leq 1} |x|^{\gamma} \nu(dx) < \infty \right. \right\}.$$
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**Theorem (S. Jaffard, 1999)**

Assume that $0 < \beta_\nu < 2$ (generic case). With probability one, for any nonempty open set $W \subseteq (0, \infty)$,

$$d_{L_\nu}(h, W) = \begin{cases} 
\beta_\nu h & \text{if } h \in [0, 1/\beta_\nu] \\
-\infty & \text{if } h > 1/\beta_\nu.
\end{cases}$$
Singularity sets of Lévy processes

Definition

If $f : \mathbb{R}^d \to \mathbb{R}$ is locally bounded, the **singularity sets** of $f$ are

$$E'_f(h) = \{ t \in \mathbb{R}^d \mid f \text{ is continuous at } t \text{ and } \alpha_f(t) \leq h \}.$$ 

The singularity sets of the jump component belong to some of the classes $\mathcal{G}^s$ of sets with large intersection introduced by K. Falconer.

Theorem (K. Falconer, 1994)

For any real $s \in (0, d]$, the class $\mathcal{G}^s(\mathbb{R}^d)$ is the maximal class of subsets of $\mathbb{R}^d$ that is:

- composed of $G_\delta$-sets with Hausdorff dimension at least $s$;
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If $f : \mathbb{R}^d \to \mathbb{R}$ is locally bounded, the **singularity sets** of $f$ are

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**Theorem (D., 2009)**

Assume that $0 < \beta_\nu < 2$ (generic case). With probability one, for all $h \in [0, 1/\beta_\nu]$, 

$$E'_L(h) \in G^{\beta_\nu h}((0, \infty))$$

and for any nonempty open $W \subseteq (0, \infty)$, 

$$\dim_H(E'_L(h) \cap W) = \beta_\nu h.$$
Connection with a random covering problem

Let \((S_n, X_n), n \geq 1\), denote the atoms of a Poisson random measure on \((0, \infty) \times \mathbb{R}^*\) with intensity \(\text{Leb} \otimes \nu\). We have

\[
L_\nu(t) = \sum_{|X_n| > 1} X_n 1\{|S_n \leq t\} + \lim_{\varepsilon \to 0} \left( \sum_{\varepsilon < |X_n| \leq 1} X_n 1\{|S_n \leq t\} - t \int_{\varepsilon < |x| \leq 1} x \nu(dx) \right).
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The sets \(E_{L_\nu}(h)\) and \(E'_{L_\nu}(h)\) may be expressed in terms of the sets \(J_\nu = \{S_n, n \geq 1\}\) and \(K_\nu(\alpha) = \left\{ t \in [0, \infty) \mid |t - S_n| < |X_n|^{1/\alpha} \text{ for infinitely many } n \geq 1 \right\}\).

Proposition

Almost surely, for all \(h \in [0, 1/\beta_\nu]\),

\[
E'_{L_\nu}(h) = (\mathbb{R} \setminus J_\nu) \cap \bigcap_{\alpha > h} K_\nu(\alpha) \quad \text{and} \quad E_{L_\nu}(h) \setminus J_\nu = E'_{L_\nu}(h) \cup \bigcup_{\alpha < h} K_\nu(\alpha).
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**Lévy random fields in the sense of Mori**

There are several ways of extending the notion of Lévy process to the multivariate setting, i.e. that of a real valued random field $Y = \{ Y(t), t \in \mathbb{R}^d \}$.

**Definition (T. Mori, 1992)**

A random field $Y = \{ Y(t), t \in \mathbb{R}^d \}$ is a Lévy field in the sense of Mori if:

- it is stochastically continuous and $Y(0) = 0$ a.s.;
- it has stationary increments: $Y(a + \cdot) - Y(a) \overset{d}{=} Y$ for any $a \in \mathbb{R}^d$;
- its finite-dimensional marginals are infinitely divisible;
- for any $a, b \in \mathbb{R}^d$, the increments of $\{ Y(a + \lambda b), \lambda \in \mathbb{R} \}$ are independent.

- **Stability under trace**: The restriction of a $d$-dimensional Lévy field to a $d'$-dimensional linear subspace is a $d'$-dimensional Lévy field. In particular, the restriction of $Y$ to any half-line is a Lévy process.

- **Stability under linear transforms of coordinates**: If $Y(t)$ is a Lévy field and $M$ is an invertible deterministic linear mapping, then $Y(Mt)$ is a Lévy field.
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**Stability under linear transforms of coordinates:** If \( Y(t) \) is a Lévy field and \( M \) is an invertible deterministic linear mapping, then \( Y(Mt) \) is a Lévy field.
Theorem (T. Mori, 1992)

Every Lévy field \( Y = \{ Y(t), t \in \mathbb{R}^d \} \) may be represented in the following manner:

\[
Y \overset{d}{=} \langle a, \cdot \rangle + B_\mu + L_\nu
\]

\( \text{linear drift} \quad \text{Gaussian field} \quad \text{jump field} \)

where:

- the drift is given by a vector \( a \in \mathbb{R}^d \);
- \( \mu \) is a finite nonnegative symmetric (i.e. invariant under \( s \mapsto -s \)) Borel measure defined on the unit sphere \( S^{d-1} \) of \( \mathbb{R}^d \);
- \( \nu \) is a nonnegative Borel measure on \( S^{d-1} \times \mathbb{R}^* \) which is symmetric (i.e. invariant under \( (s, x) \mapsto (-s, -x) \)) and such that

\[
\int_{s \in S^{d-1}} (1 \wedge x^2) \nu(ds, dx) < \infty ;
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\( \nu \) is the analog of the Lévy measure: \( \nu(ds, dx) \) describes the amount of hyperplanes orthogonal to \( s \) where a jump of size \( x \) occurs.
### Lévy-Itô decomposition of Mori’s Lévy fields

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The Gaussian component $B_\mu$

The increments $B_\mu(t) - B_\mu(t')$ are normally distributed with mean zero and variance

$$\varrho_\mu(t, t')^2 = \frac{1}{2} \int_{s \in S^{d-1}} |\langle s, t' - t \rangle| \mu(ds).$$

In the **isotropic case** (when $\mu$ is rotation invariant), the field $B_\mu$ is the **Lévy Brownian motion** and the above representation is essentially due to Chentsov (1957). In that situation,

$$\varrho_\mu(t, t')$$

is proportional to $\|t' - t\|^{1/2}$.

Moreover, $B_\mu$ is 1/2-sssis:

- $B_\mu$ is selfsimilar with index $1/2$, i.e.

  $$\forall a > 0 \quad \{B_\mu(at), \ t \in \mathbb{R}^d\} \overset{d}{=} \{a^{1/2}B_\mu(t), \ t \in \mathbb{R}^d\};$$

- $B_\mu$ has stationary increments in the strong sense, i.e. for any direct isometry $g$,

  $$\{B_\mu(g(t)) - B_\mu(g(0)), \ t \in \mathbb{R}^d\} \overset{d}{=} \{B_\mu(t) - B_\mu(0), \ t \in \mathbb{R}^d\}.$$
Regularity of the Gaussian component $B_\mu$

**Theorem**

\textit{In the general case (not necessarily isotropic), the field $B_\mu$ is mono-Hölder: with probability one,}

$$\forall t \in \mathbb{R}^d \quad \alpha_{B_\mu}(t) = 1/2.$$ 

- The lower bound relies on Dudley's entropy bound for the pseudometric $\varrho_\mu$. This actually yields a uniform modulus of continuity for $B_\mu$: almost surely, for all $A > 0$,

$$\limsup_{\delta \to 0} \frac{1}{(\delta \log(1/\delta))^{1/2}} \sup_{t, t' \in [-A, A]^d, \|t' - t\| \leq \delta} |B_\mu(t') - B_\mu(t)| < \infty.$$ 

- The upper bound relies on the following fact, which may be proven by adapting an idea of Dvoretzky: there exists $\kappa_{d,\mu} > 0$ such that almost surely,

$$\forall t \in \mathbb{R}^d \quad \forall \delta > 0 \quad \exists t' \in \overline{B}(t, \delta) \quad |B_\mu(t') - B_\mu(t)| > \kappa_{d,\mu} \|t' - t\|^{1/2}.$$
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$$\forall t \in \mathbb{R}^d \quad \alpha_{B_\mu}(t) = 1/2.$$  

*The lower bound relies on Dudley's entropy bound for the pseudometric $d_\mu$. This actually yields a uniform modulus of continuity for $B_\mu$: almost surely, for all $A > 0$,*

$$\limsup_{\delta \to 0} \frac{1}{(\delta \log(1/\delta))^{1/2}} \sup_{t, t' \in [-A, A]^d, \|t' - t\| \leq \delta} |B_\mu(t') - B_\mu(t)| < \infty.$$  

*The upper bound relies on the following fact, which may be proven by adapting an idea of Dvoretzky: there exists $\kappa_{d, \mu} > 0$ such that almost surely,*

$$\forall t \in \mathbb{R}^d \quad \forall \delta > 0 \quad \exists t' \in \overline{B}(t, \delta) \quad |B_\mu(t') - B_\mu(t)| > \kappa_{d, \mu} \|t' - t\|^{1/2}.$$
The jump component $L_\nu$

We consider a symmetric nonnegative Borel measure $\nu$ on $S^{d-1} \times \mathbb{R}^*$ such that

$$\int_{s \in S^{d-1}} \int_{x \in \mathbb{R}^*} (1 \wedge x^2) \nu(ds, dx) < \infty.$$ 

We consider a Poisson random measure on $(0, \infty) \times S^{d-1} \times \mathbb{R}^*$ with intensity $\text{Leb} \otimes \nu$; its atoms are denoted by

$$(P_n, S_n, X_n), \quad n \geq 1.$$ 

**Jumps of magnitude $> 1$: multivariate compound Poisson processes.**

- For all $t \in \mathbb{R}^d$, let $L_{\nu,1}(t) = \sum_{|X_n| > 1} X_n \mathbbm{1}_{\{P_n < \langle S_n, t \rangle\}}$.

- $L_{\nu,1}$ is piecewise constant with jumps of magnitude $|X_n| > 1$ located on the hyperplanes

$$H_n = \{t \in \mathbb{R}^d \mid P_n = \langle S_n, t \rangle\}.$$
The jump component $L_\nu$

We consider a symmetric nonnegative Borel measure $\nu$ on $\mathbb{S}^{d-1} \times \mathbb{R}^*$ such that

$$\int_{s \in \mathbb{S}^{d-1}, x \in \mathbb{R}^*} (1 \wedge x^2) \nu(ds, dx) < \infty.$$ 

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$$H_n = \{t \in \mathbb{R}^d \mid P_n = \langle S_n, t \rangle\}.$$
The jump component \( L_\nu \)

We consider a symmetric nonnegative Borel measure \( \nu \) on \( \mathbb{S}^{d-1} \times \mathbb{R}^* \) such that

\[
\int_{s \in \mathbb{S}^{d-1}} \int_{x \in \mathbb{R}^*} (1 \wedge x^2) \nu(ds, dx) < \infty.
\]

We consider a Poisson random measure on \((0, \infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^*\) with intensity \( \text{Leb} \otimes \nu \); its atoms are denoted by

\[ (P_n, S_n, X_n), \quad n \geq 1. \]

**Jumps of magnitude \( > 1 \):** multivariate compound Poisson processes.

- For all \( t \in \mathbb{R}^d \), let \( L_{\nu, 1}(t) = \sum_{|X_n| > 1} X_n 1_{\{|P_n < \langle S_n, t \rangle\}}. \)

- \( L_{\nu, 1} \) is piecewise constant with jumps of magnitude \( |X_n| > 1 \) located on the hyperplanes

\[ H_n = \{ t \in \mathbb{R}^d \mid P_n = \langle S_n, t \rangle \}. \]
The jump component $L_\nu$

$$L_{\nu,1}(t) = \sum_{|X_n|>1} X_n \mathbb{1}_{\{P_n < \langle S_n, t \rangle\}}$$
The jump component $L_{\nu}$

**Jumps of magnitude $\leq 1$:** multivariate compensated sums of jumps. For $\varepsilon > 0$ and $t \in \mathbb{R}^d$, let

$$L_{\nu,\varepsilon}(t) = \sum_{\varepsilon < |X_n| \leq 1} X_n 1\{P_n < \langle S_n, t \rangle\} - \int_{s \in \mathbb{S}^{d-1}, \varepsilon < x \varepsilon \leq 1} x\langle s, t \rangle \nu(ds, dx)$$

piecewise constant with jumps on hyperplanes

linear compensator

**Considering all the jumps:** the whole field $L_{\nu}$

$$L_{\nu}(t) = L_{\nu,1}(t) + \lim_{\varepsilon \to 0} L_{\nu,\varepsilon}(t)$$

- Because of the integrability condition on $\nu$, convergence holds in $L^2$, and also a.s., for any fixed $t$.
- Under a further admissibility condition on $\nu$, convergence holds a.s. for all $t$ simultaneously.
The jump component $L_\nu$

**Jumps of magnitude $\leq 1$:** multivariate compensated sums of jumps. For $\varepsilon > 0$ and $t \in \mathbb{R}^d$, let

$$L_{\nu,\varepsilon}(t) = \sum_{\varepsilon < |X_n| \leq 1} X_n \mathbb{1}_{\{P_n < \langle S_n, t \rangle\}} - \int_{s \in \mathbb{S}^{d-1}} \int_{\varepsilon < x \leq 1} x \langle s, t \rangle \nu(ds, dx)$$

piecewise constant with jumps on hyperplanes

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Jumps of magnitude $\leq 1$: multivariate compensated sums of jumps. For $\varepsilon > 0$ and $t \in \mathbb{R}^d$, let

$$L_{\nu, \varepsilon}(t) = \sum_{\varepsilon < |X_n| \leq 1} X_n \mathbb{1}_{\{P_n < \langle S_n, t \rangle \}} - \int_{s \in \mathbb{S}^{d-1}} x \langle s, t \rangle \nu(ds, dx)$$

piecewise constant with jumps on hyperplanes

linear compensator

Considering all the jumps: the whole field $L_\nu$

$$L_\nu(t) = L_{\nu, 1}(t) + \lim_{\varepsilon \to 0} L_{\nu, \varepsilon}(t)$$

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- Under a further admissibility condition on $\nu$, convergence holds a.s. for all $t$ simultaneously.
The jump component $L_\nu$

**Jumps of magnitude $\leq 1$:** multivariate compensated sums of jumps. For $\varepsilon > 0$ and $t \in \mathbb{R}^d$, let

$$L_{\nu,\varepsilon}(t) = \sum_{\varepsilon < |X_n| \leq 1} X_n 1\{P_n < \langle S_n, t \rangle\} - \int_{s \in \mathbb{S}_{d-1} \atop \varepsilon < x \leq 1} x\langle s, t \rangle \nu(ds, dx)$$

piecewise constant with jumps on hyperplanes

**linear compensator**

**Considering all the jumps:** the whole field $L_\nu$

$$L_{\nu}(t) = L_{\nu,1}(t) + \lim_{\varepsilon \to 0} L_{\nu,\varepsilon}(t)$$

- Because of the integrability condition on $\nu$, convergence holds in $L^2$, and also a.s., for any fixed $t$.
- Under a further admissibility condition on $\nu$, convergence holds a.s. for all $t$ simultaneously.
We define the index of $\nu$ by

$$\beta_\nu = \inf \left\{ \gamma > 0 \left| \int_{S^{d-1}} \int_{x \in (0,1]} x^\gamma \nu(ds, dx) < \infty \right. \right\}.$$ 

(Analog of the Blumenthal-Getoor index in the multivariate case)

From now on, we restrict our attention to the generic case where $0 < \beta_\nu < 2$. 
Index of a Lévy field

We define the **index** of \( \nu \) by

\[
\beta_\nu = \inf \left\{ \gamma > 0 \left| \int_{s \in S^{d-1}} \int_{x \in (0,1]} x^\gamma \nu(ds, dx) < \infty \right. \right\}.
\]

(Analog of the Blumenthal-Getoor index in the multivariate case)

From now on, we restrict our attention to the generic case where \( 0 < \beta_\nu < 2 \).
A sample path of the jump component; jumps of magnitude $> 2^{-8}$
Isotropic and stable case

Suppose that

\[ \nu(ds, dx) = \sigma(ds) \otimes \frac{dx}{|x|^{\alpha+1}}, \quad 0 < \alpha < 2. \]

uniform on \( S^{d-1} \)

- The jump component \( L_\nu \) is usually called a Lévy-Chentsov field (as e.g. in G. Samorodnitsky and M. Taqqu’s book).
- It is selfsimilar with index \( 1/\alpha \) and has stationary increments in the strong sense (1/\( \alpha \)-sssis).
- Its local boundedness and the existence of local times were studied by N.-R. Shieh (1995).
- In that situation, \( \beta_\nu = \alpha. \)
Isotropic and $\alpha$-stable case; jumps of magnitude $> 2^{-8}$

$\alpha = 0.5$  
$\alpha = 1$  
$\alpha = 1.5$  
$\alpha = 1.7$  
$\alpha = 1.9$  
$\alpha = 1.99$
An anisotropic example

\[ \nu(ds, dx) = f(s) \sigma(ds) \otimes \frac{dx}{|x|^\alpha+1} \]

\[ \alpha = 0.5 \quad \alpha = 1 \quad \alpha = 1.5 \]
Another anisotropic example

\[ \nu(ds, dx) = \frac{dx}{|x|^{\alpha(s)+1}} \sigma(ds) \]
Multifractal spectrum of the jump component $L_{\nu}$

**Theorem (D., Jaffard, 2012)**

*The jump component is a homogeneous multifractal process:* with probability one, for any nonempty open $W \subseteq \mathbb{R}^d$,

$$d_{L_{\nu}}(h, W) = \begin{cases} 
  d - 1 + \beta_{\nu} h & \text{if } h \in [0, 1/\beta_{\nu}] \\
  -\infty & \text{if } h > 1/\beta_{\nu}.
\end{cases}$$
Multifractal spectrum of the jump component $L_{\nu}$

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The jump component is a **homogeneous multifractal process**: with probability one, for any nonempty open $W \subseteq \mathbb{R}^d$,

$$d_{L_{\nu}}(h, W) = \begin{cases} d - 1 + \beta_{\nu} h & \text{if } h \in [0, 1/\beta_{\nu}] \\ -\infty & \text{if } h > 1/\beta_{\nu}. \end{cases}$$
Behavior of traces

Let \( e = (e_1, \ldots, e_{d'}) \) denote an arbitrary orthonormal system of \( \mathbb{R}^d \) with \( 1 \leq d' \leq d \), and let \( Y \) denote a Lévy field defined on \( \mathbb{R}^d \). Then,

\[
Y^e(t_1, \ldots, t_{d'}) = Y(t_1 e_1 + \ldots + t_{d'} e_{d'})
\]

is a Lévy field on \( \mathbb{R}^{d'} \). In the isotropic case, i.e. when

\[
\mu(ds) = \sigma(ds) \quad \text{and} \quad \nu(ds, dx) = \sigma(ds) \otimes \pi(dx)
\]

uniform on \( S^{d-1} \)

measure on \( \mathbb{R}^* \)

the index \( \beta_\nu \) depends on \( \pi \) only, so that the previous results lead to

\[
a.s. \quad \forall h \quad \dim E_{Y^e}(h) = \dim E_Y(h) - (d - d').
\]

Hence, the spectrum is lowered by the codimension, as generally expected: J.-M. Aubry, D. Maman and S. Seuret showed that this holds in a prevalent manner in a Besov space (2010).
Behavior of traces

Let $e = (e_1, \ldots, e_{d'})$ denote an arbitrary orthonormal system of $\mathbb{R}^d$ with $1 \leq d' \leq d$, and let $Y$ denote a Lévy field defined on $\mathbb{R}^d$. Then,

$$Y^e(t_1, \ldots, t_{d'}) = Y(t_1 e_1 + \ldots + t_{d'} e_{d'})$$

is a Lévy field on $\mathbb{R}^{d'}$. In the isotropic case, i.e. when

$$\mu(ds) = \sigma(ds)$$

uniform on $S^{d-1}$

and

$$\nu(ds, dx) = \sigma(ds) \otimes \pi(dx)$$

measure on $\mathbb{R}^*$

the index $\beta_\nu$ depends on $\pi$ only, so that the previous results lead to

$$\text{a.s. } \forall h \quad \dim E_{Y^e}(h) = \dim E_Y(h) - (d - d').$$

Hence, the spectrum is lowered by the codimension, as generally expected: J.-M. Aubry, D. Maman and S. Seuret showed that this holds in a prevalent manner in a Besov space (2010).
Size and large intersection properties of the singularity sets of \( L_\nu \)

Recall that the singularity sets are

\[ E'_{L_\nu}(h) = \{ t_0 \in \mathbb{R}^d \mid L_\nu \text{ is continuous at } t_0 \text{ and } \alpha_{L_\nu}(t_0) \leq h \}. \]

**Theorem (D., Jaffard, 2012)**

With probability one, for all \( h \in [0, 1/\beta_\nu] \),

\[ E'_{L_\nu}(h) \in G^{d-1+\beta_\nu h}(\mathbb{R}^d) \]

and for any nonempty open \( W \subseteq \mathbb{R}^d \),

\[ \dim_H(E'_{L_\nu}(h) \cap W) = d - 1 + \beta_\nu h. \]
Size and large intersection properties of the singularity sets of $L_\nu$

Recall that the singularity sets are

$$E'_{L_\nu}(h) = \{ t_0 \in \mathbb{R}^d \mid L_\nu \text{ is continuous at } t_0 \text{ and } \alpha_{L_\nu}(t_0) \leq h \}.$$ 

**Theorem (D., Jaffard, 2012)**

*With probability one, for all $h \in [0, 1/\beta_\nu]$,*

$$E'_{L_\nu}(h) \subseteq \mathcal{G}^{d-1+\beta_\nu h}(\mathbb{R}^d)$$

*and for any nonempty open $W \subseteq \mathbb{R}^d$,*

$$\dim_H(E'_{L_\nu}(h) \cap W) = d - 1 + \beta_\nu h.$$
Recall that there is a jump of size $X_n$ on the hyperplane $H_n$, and that $H_n$ and $X_n$ are distributed according to a Poisson random measure controlled by $\nu$.

**Proposition**

With probability one, for all $h \in [0, 1/\beta_\nu)$,

$$E_{L_\nu}(h) = (\mathbb{R}^d \setminus J_\nu) \cap \bigcap_{\alpha > h} K_\nu(\alpha),$$

with

$$J_\nu = \bigcup_{n \geq 1} H_n,$$

the set of jump locations of $L_\nu$.

and

$$K_\nu(\alpha) = \left\{ t \in \mathbb{R}^d \mid \text{dist}(t, H_n) < |X_n|^{1/\alpha} \text{ for infinitely many } n \geq 1 \right\}.$$
Connection with a covering problem

Recall that there is a jump of size $X_n$ on the hyperplane $H_n$, and that $H_n$ and $X_n$ are distributed according to a Poisson random measure controlled by $\nu$.

**Proposition**

With probability one, for all $h \in [0, 1/\beta]\nu$,

$$E_{L,\nu}^I(h) = (\mathbb{R}^d \setminus J_{\nu}) \cap \bigcap_{\alpha > h} K_{\nu}(\alpha),$$

with

$$J_{\nu} = \bigcup_{n \geq 1} H_n, \quad \text{the set of jump locations of } L_{\nu}.$$

and

$$K_{\nu}(\alpha) = \left\{ t \in \mathbb{R}^d \mid \text{dist}(t, H_n) < |X_n|^{1/\alpha} \text{ for infinitely many } n \geq 1 \right\}.$$
Connection with a covering problem


As noted in Chapter 8, it is regrettable that the Cantor dust should be so hard to illustrate directly. However, it can be visualized indirectly as the intersection of the triadic Koch curve with its base. And in the same way the Lévy dust can be imaged indirectly. On this plate, the black street-like stripes are placed at random, and in particular their directions are isotropic. Their widths follow a hyperbolic distribution and rapidly become so thin that they cannot be drawn. Asymptotically, the white remainder set (the “blocks of houses”) is of zero area and of dimension $D$ less than 2.

As long as the remaining blocks of houses have a dimension $D>1$, their intersection by an arbitrary line is a Lévy dust of dimension $D-1$. On the other hand, if $D<1$, the intersection is almost certainly empty. This result is, however, not very apparent here because the construction could not be carried far enough.

Chapter 33 provides a better illustration. When the tremas subtracted from the plane are random discs as exemplified by Plates 306 to 309, the trem fractals’ intersections with straight lines are Lévy dusts.
Multifractal properties of Lévy processes

Lévy fields in the sense of Mori, and their multifractal properties

Lévy fields in the sense of Adler et al., and their multifractal properties
Lévy random fields in the sense of Adler et al.

**Definition**

An independently scattered homogeneous Lévy random measure on $\mathbb{R}^d$, or **Lévy noise**, with underlying Lévy process $\xi$ is a stochastic process $Y = \{Y(B), \ B \in \mathcal{B}(\mathbb{R}^d)\}$ such that:

- for any Borel subset $B$ of $\mathbb{R}^d$, we have $Y(B) \overset{d}{=} \xi(\text{Leb}(B));$
- for any disjoint Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}^d$, the variables $Y(B_1), \ldots, Y(B_n)$ are independent and $Y\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} Y(B_i)$.

**Definition**

A Lévy field in the sense of R. Adler, D. Monrad, R. Scissors and R. Wilson, or **Lévy sheet**, is a stochastic process $Y = \{Y(t), \ t \in [0, \infty)^d\}$ of the form

$$Y(t) = Y([0, t_1] \times \ldots \times [0, t_d]), \quad t = (t_1, \ldots, t_d),$$

where $Y$ is a Lévy noise on $\mathbb{R}^d$. 

Arnaud Durand

Lévy random fields

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Lévy random fields in the sense of Adler et al.

Definition

An independently scattered homogeneous Lévy random measure on $\mathbb{R}^d$, or Lévy noise, with underlying Lévy process $\xi$ is a stochastic process $\mathcal{Y} = \{ \mathcal{Y}(B), \ B \in \mathcal{B}(\mathbb{R}^d) \}$ such that:

- for any Borel subset $B$ of $\mathbb{R}^d$, we have $\mathcal{Y}(B) \overset{d}{=} \xi(\text{Leb}(B))$;
- for any disjoint Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}^d$, the variables $\mathcal{Y}(B_1), \ldots, \mathcal{Y}(B_n)$ are independent and $\mathcal{Y} \left( \bigcup_{i=1}^{n} B_i \right) = \sum_{i=1}^{n} \mathcal{Y}(B_i)$.

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$$\mathcal{Y}(t) = \mathcal{Y}([0, t_1] \times \ldots \times [0, t_d]), \quad t = (t_1, \ldots, t_d),$$

where $\mathcal{Y}$ is a Lévy noise on $\mathbb{R}^d$. 
Lévy random fields in the sense of Adler et al.

**Definition**

An independently scattered homogeneous Lévy random measure on $\mathbb{R}^d$, or Lévy noise, with underlying Lévy process $\xi$ is a stochastic process $\mathcal{Y} = \{\mathcal{Y}(B), \ B \in \mathcal{B}(\mathbb{R}^d)\}$ such that:

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- for any disjoint Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}^d$, the variables $\mathcal{Y}(B_1), \ldots, \mathcal{Y}(B_n)$ are independent and $\mathcal{Y}\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mathcal{Y}(B_i)$.

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A Lévy field in the sense of R. Adler, D. Monrad, R. Scissors and R. Wilson, or Lévy sheet, is a stochastic process $\mathcal{Y} = \{\mathcal{Y}(t), \ t \in [0, \infty)^d\}$ of the form

$$\mathcal{Y}(t) = \mathcal{Y}([0, t_1] \times \ldots \times [0, t_d]), \quad t = (t_1, \ldots, t_d),$$

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Lévy random fields in the sense of Adler et al.

**Definition**

An independently scattered homogeneous Lévy random measure on $\mathbb{R}^d$, or Lévy noise, with underlying Lévy process $\xi$ is a stochastic process $\mathcal{Y} = \{\mathcal{Y}(B), \ B \in \mathcal{B}(\mathbb{R}^d)\}$ such that:

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- for any disjoint Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}^d$, the variables $\mathcal{Y}(B_1), \ldots, \mathcal{Y}(B_n)$ are independent and $\mathcal{Y} \left( \bigcup_{i=1}^{n} B_i \right) = \sum_{i=1}^{n} \mathcal{Y}(B_i)$.

**Definition**

A Lévy field in the sense of R. Adler, D. Monrad, R. Scissors and R. Wilson, or Lévy sheet, is a stochastic process $\mathcal{Y} = \{\mathcal{Y}(t), \ t \in [0, \infty)^d\}$ of the form

$$\mathcal{Y}(t) = \mathcal{Y}([0, t_1] \times \ldots \times [0, t_d]), \quad t = (t_1, \ldots, t_d),$$

where $\mathcal{Y}$ is a Lévy noise on $\mathbb{R}^d$. 
Lévy-Itô decomposition of a Lévy noise

**Theorem**

*Every Lévy noise* $\mathcal{Y} = \{\mathcal{Y}(B), B \in \mathcal{B}(\mathbb{R}^d)\}$ *may be represented in the following manner:*

$$
\mathcal{Y} \overset{d}{=} \underbrace{a \text{Leb}}_{\text{drift}} + \underbrace{\sigma \mathcal{W}}_{\text{Gaussian component}} + \underbrace{\mathcal{L}_\nu}_{\text{jump component}}
$$

*where:*

- the drift is given by a real number $a$ times the Lebesgue measure;
- $\sigma$ is a nonnegative real and $\mathcal{W}$ is the Gaussian noise based on the Lebesgue measure: $W(B) \sim \mathcal{N}(0, \text{Leb}(B))$;
- $\nu$ is a the Lévy measure of the underlying Lévy process $\xi$ and

$$
\mathcal{L}_\nu(B) = \int_{s \in B, |x| > 1} x \, N(ds, dx) + \int_{s \in B, 0 < |x| \leq 1} x \, N^*(ds, dx).
$$

*where* $\mathcal{N}$ *is a Poisson random measure with intensity* $\text{Leb} \otimes \nu$, *and* $\mathcal{N}^* = \mathcal{N} - \text{Leb} \otimes \nu$ *is the associated compensated measure.*
Lévy-Itô decomposition of a Lévy noise

**Theorem**

Every Lévy noise \( Y = \{Y(B), B \in B(\mathbb{R}^d)\} \) may be represented in the following manner:

\[
Y \overset{d}{=} \underbrace{a \text{Leb}}_{\text{drift}} + \underbrace{\sigma Y}_{\text{Gaussian component}} + \underbrace{L'_{\nu}}_{\text{jump component}}
\]

where:

- **the drift is given by a real number** \( a \) **times the Lebesgue measure**;
- \( \sigma \) **is a nonnegative real and** \( W \) **is the Gaussian noise based on the Lebesgue measure**: \( W(B) \sim \mathcal{N}(0, \text{Leb}(B)) \);
- \( \nu \) **is the Lévy measure of the underlying Lévy process** \( \xi \) and

\[
L'_{\nu}(B) = \int_{s \in B} x \ N(ds, dx) + \int_{s \in B} x \ N^*(ds, dx).
\]

where \( N \) **is a Poisson random measure with intensity** \( \text{Leb} \otimes \nu \), and \( N^* = N - \text{Leb} \otimes \nu \) **is the associated compensated measure**.
Lévy-Itô decomposition of a Lévy noise

**Theorem**

Every Lévy noise $\mathcal{Y} = \{\mathcal{Y}(B), B \in \mathcal{B}(\mathbb{R}^d)\}$ may be represented in the following manner:

$$\mathcal{Y} \overset{d}{=} \underbrace{a \text{Leb}}_{\text{drift}} + \underbrace{\sigma \mathcal{W}}_{\text{Gaussian component}} + \underbrace{\mathcal{L}_\nu}_{\text{jump component}}$$

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$$\mathcal{L}_\nu(B) = \int_{s \in B} \chi_{|x| > 1} x \, N(ds, dx) + \int_{s \in B} \chi_{0 < |x| \leq 1} x \, N^*(ds, dx).$$

where $N$ is a Poisson random measure with intensity $\text{Leb} \otimes \nu$, and $N^* = N - \text{Leb} \otimes \nu$ is the associated compensated measure.
**Theorem**

Every Lévy noise $\mathcal{Y} = \{\mathcal{Y}(B), B \in \mathcal{B}(\mathbb{R}^d)\}$ may be represented in the following manner:

$$\mathcal{Y} \overset{d}{=} a \text{Leb} + \sigma \mathcal{W} + \mathcal{L}_\nu$$

where:

- the drift is given by a real number $a$ times the Lebesgue measure;
- $\sigma$ is a nonnegative real and $\mathcal{W}$ is the Gaussian noise based on the Lebesgue measure: $W(B) \sim \mathcal{N}(0, \text{Leb}(B))$;
- $\nu$ is the Lévy measure of the underlying Lévy process $\xi$ and

$$\mathcal{L}_\nu(B) = \int_{|x| > 1} x \, N(ds, dx) + \int_{0 < |x| \leq 1} x \, N^*(ds, dx).$$

where $N$ is a Poisson random measure with intensity $\text{Leb} \otimes \nu$, and $N^* = N - \text{Leb} \otimes \nu$ is the associated compensated measure.
Lévy-Itô decomposition of a Lévy sheet

**Theorem (R. Adler et al., 1983)**

*Every Lévy sheet* $Y = \{Y(t), t \in [0, \infty)^d\}$ *admits a modification with sample paths in a multivariate Skorokhod space which is given by:*

$$Y(t) = a t_1 \cdots t_d + \sigma W(t) + L_\nu(t)$$

*where:*

- $W$ is the standard Brownian sheet, whose pointwise regularity properties are well-known (see Orey-Pruitt, 1973);
- letting $(S_n, X_n)$, $n \geq 1$, denote the atoms of a Poisson random measure on $(0, \infty)^d \times \mathbb{R}^*$ with intensity $\text{Leb} \otimes \nu$, we have

$$L_\nu(t) = \sum_{|X_n| > 1} X_n 1_{\{S_n \leq t\}} + \lim_{\varepsilon \to 0} \left( \sum_{\varepsilon < |X_n| \leq 1} X_n 1_{\{S_n \leq t\}} - t_1 \cdots t_d \int_{\varepsilon < |x| \leq 1} x \nu(dx) \right),$$

where convergence holds a.s. on every compact, and $S_n \leq t$ means $S_n \in [0, t_1] \times \cdots \times [0, t_d]$. 

Theorem (R. Adler et al., 1983)

Every Lévy sheet \( Y = \{ Y(t), t \in [0, \infty)^d \} \) admits a modification with sample paths in a multivariate Skorokhod space which is given by:

\[
Y(t) = a t_1 \cdots t_d + \sigma W(t) + L_\nu(t)
\]

where:

- \( W \) is the standard Brownian sheet, whose pointwise regularity properties are well-known (see Orey-Pruitt, 1973);
- letting \( (S_n, X_n), n \geq 1 \), denote the atoms of a Poisson random measure on \((0, \infty)^d \times \mathbb{R}^* \) with intensity \( \text{Leb} \otimes \nu \), we have

\[
L_\nu(t) = \sum_{|X_n| > 1} X_n \mathbb{1}_{\{S_n \leq t\}} + \lim_{\epsilon \to 0} \left( \sum_{\epsilon < |X_n| \leq 1} X_n \mathbb{1}_{\{S_n \leq t\}} - t_1 \cdots t_d \int_{\epsilon < |x| \leq 1} x \nu(dx) \right),
\]

where convergence holds a.s. on every compact, and \( S_n \leq t \) means \( S_n \in [0, t_1] \times \cdots \times [0, t_d] \).
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where convergence holds a.s. on every compact, and \( S_n \leq t \) means \( S_n \in [0, t_1] \times \cdots \times [0, t_d] \).
The jump component $L_\nu$

$$\sum_{|X_n| > 1} X_n 1_{\{s_n \leq t\}}$$
Sample paths when $\nu(dx) = dx/|x|^{\alpha+1}$; jumps of magnitude $> 2^{-8}$
Multifractal spectrum of the jump component $L_\nu$

**Theorem (D., 2012)**

The jump component is a **homogeneous multifractal process** in $(0, \infty)^d$: with probability one, for any nonempty open $W \subseteq (0, \infty)^d$,

$$d_{L_\nu}(h, W) = \begin{cases} 
    d - 1 + \beta_\nu h & \text{if } h \in [0, 1/\beta_\nu] \\
    -\infty & \text{if } h > 1/\beta_\nu.
\end{cases}$$
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**Problem**

What happens on the boundary of $[0, \infty)^d$, i.e. when at least one of the coordinates vanishes?
Multifractal spectrum of the jump component $L_\nu$; case where $d = 2$

**Theorem (D., 2012)**

*The jump component is a homogeneous multifractal process in $(0, \infty)^2$: with probability one, for any nonempty open $W \subseteq (0, \infty)^2$,

$$d_{L_\nu}(h, W) = \begin{cases} 1 + \beta_\nu h & \text{if } h \in [0, 1/\beta_\nu] \\ -\infty & \text{if } h > 1/\beta_\nu. \end{cases}$$

*With probability one, $\alpha_{L_\nu}(0) = 2/\beta_\nu$.\*
Multifractal spectrum of the jump component $L_\nu$; case where $d = 2$

**Theorem (D., 2012)**

- *The jump component is a homogeneous multifractal process in $(0, \infty)^2$: with probability one, for any nonempty open $W \subseteq (0, \infty)^2$,*

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d_{L_\nu}(h, W) = \begin{cases} 
1 + \beta_\nu h & \text{if } h \in [0, 1/\beta_\nu] \\
-\infty & \text{if } h > 1/\beta_\nu.
\end{cases}
\]

- *With probability one, $\alpha_{L_\nu}(0) = 2/\beta_\nu$. 

![Diagram](image.png)
Multifractal spectrum of the jump component \( L_\nu \); case where \( d = 2 \)

**Theorem (D., 2012)**

- The jump component is a **homogeneous multifractal process** in \((0, \infty)^2\):
  
  With probability one, for any nonempty open \( W \subseteq (0, \infty)^2 \),
  
  \[
  d_{L_\nu}(h, W) = \begin{cases} 
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- With probability one, \( \alpha_{L_\nu}(0) = 2/\beta_\nu \).