Sets avoiding distance 1 in $\mathbb{R}^n$

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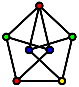
- The Moser graph has chromatic number 4:



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- Asymptotically:

$$(1.2)^n \lesssim \chi(\mathbb{R}^n) \lesssim 3^n$$

  The lower bound: Frankl Wilson 1981 a sequence of graphs with vertices in $\{0,1\}^n$ and their intersection theorem.
  The upper bound: Larman Rogers 1972 construction using Voronoi cells of lattices.

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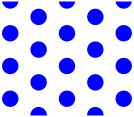

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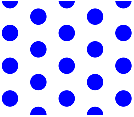
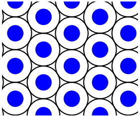


Open disks of diameter 1, centers at distance at least 2 apart.

- If A is measurable, we can measure its size by its (upper) density $\delta(A)$.



$$\delta = \frac{\Delta_2}{4} = \frac{\pi}{8\sqrt{3}} \approx 0.226$$

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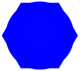

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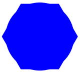

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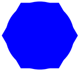

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$$\chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)}.$$

- Obviously $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$. We know $\chi_m(\mathbb{R}^2) \geq 5$ (Falconer 1981).

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$$m_1(\mathbb{R}^2) < 0.258$$

- More generally Moser, Larman and Rogers conjectured that

$$m_1(\mathbb{R}^n) < 1/2^n.$$

For *sets with block structure*, this is proved by Keleti, Matolcsi, Oliveira, Ruzsa 2015.

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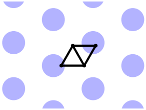
- We have:

$$m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|}$$

Proof: let A be a subset of $\mathbb{R}^n$ avoiding 1. A translated copy of G has on average $|V|\delta(A)$ vertices in A, but also at most $\alpha(G)$ vertices in A.



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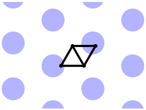

- Larman Rogers 1972: good graphs for small dimensions.
  Improved by Szekely Wormald 1989.
  Frankl Wilson 1981, Raigorodskii 2000:

$$m_1(\mathbb{R}^n) \lesssim (1.239)^{-n}$$

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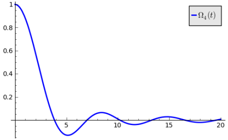
- It is well known that

$$\widehat{\omega}(u) = \Omega_n(\|u\|) = \Gamma(n/2)(2/\|u\|)^{n/2-1} J_{n/2-1}(\|u\|)$$



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- Let A be 1-avoiding and L-periodic. The auto-correlation function associated to A:

$$f_A(x) = \frac{1}{\mathrm{vol}(L)} \int_{\mathbb{R}^n/L} \mathbf{1}_A(x+y)\,\mathbf{1}_A(y)\,dy.$$

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$\mu := \omega - m\delta_{0^n}$. We have $\widehat{\mu} = \widehat{\omega} - m \geq 0$.

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- We compute in two different ways

$$\int f_A(x)d\mu(x) = -mf_A(0^n) = -m\delta(A)$$
$$= \sum_{u \in L^\#} \widehat{f_A}(u)\widehat{\mu}(u) \geq \widehat{f_A}(0^n)\widehat{\mu}(0^n) = \delta(A)^2(1-m).$$

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- If G is a finite graph embedded in $\mathbb{R}^n$, with vertices $V = \{v_1, \ldots, v_M\}$,

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- If $\|v_i\| = r$ and if z_0, z_1, z_2 satisfy:

$$\begin{cases} z_2 \geq 0 \\ z_0 + z_1\Omega_n(t) + z_2\Omega_n(rt) \geq 0 \\ z_0 + z_1 + z_2 = 1 \end{cases}$$

  then

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- Proof: consider the measure

$$\mu = z_0\delta_{0^n} + z_1\omega + z_2\frac{1}{M}\sum_{i=1}^{M}\delta_{v_i}$$

Results

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- Improvements for $2 \leq n \leq 24$

$n = 2$	$m_1(\mathbb{R}^2) \leq 0.258$		[KOMR 2015]
$n = 3$	$m_1(\mathbb{R}^3) \leq 0.165$	$\chi_m(\mathbb{R}^3) \geq 7$	[OV 2010]
$n = 4$	$m_1(\mathbb{R}^4) \leq 0.100$	$\chi_m(\mathbb{R}^4) \geq 10$	[BPT 2013]
$\vdots$			$\vdots$
$n = 24$	$m_1(\mathbb{R}^{24}) \leq 1.8e-04$	$\chi_m(\mathbb{R}^{24}) \geq 5424$	[BPT 2013]

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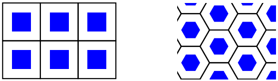
$$A = \cup_{x \in L}(x + P/2)$$



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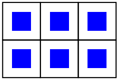

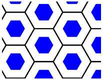

Conjecture

(B., Sinai Robins) If P is a polytope that tiles $\mathbb{R}^n$ by translations then

$$m_P(\mathbb{R}^n) = 1/2^n$$

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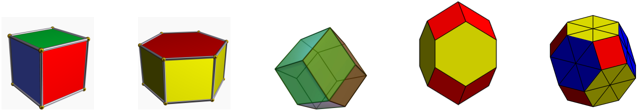

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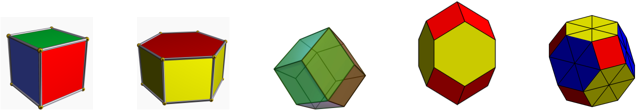

- (Minkowski 1897, Venkov 1954, McMullen 1980): A polytope *P* tiles space by translations iff it is centrally symmetric, its facets are centrally symmetric, and its belts of codimension 2 have 4 or 6 elements.

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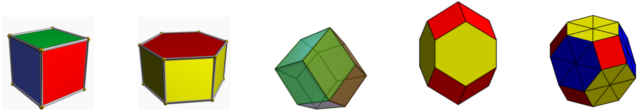

- (Minkowski 1897, Venkov 1954, McMullen 1980): A polytope P tiles space by translations iff it is centrally symmetric, its facets are centrally symmetric, and its belts of codimension 2 have 4 or 6 elements.
- Dirichlet-Voronoï cells of lattices tile space by translation.
(Voronoï conjecture 1908 : a translative polytope is the affine image of the D-V cell of a lattice).

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- The eigenvalue-Fourier bound: Let μ be a probability measure supported on δP,

$$m_P(\mathbb{R}^n) \leq \frac{-\min \widehat{\mu}(u)}{1 - \min \widehat{\mu}(u)}.$$

So far, the combinatorial bound gave better results.

Partial results

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- The root lattices $A_n := \mathbb{Z}^{n+1} \cap \{\sum_{i=1}^{n} x_i = 0\}$:

Theorem (B, Shiryaev 2015)

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- The root lattices $D_n := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \ : \ \sum_{i=1}^n x_i = 0 \bmod 2\}$

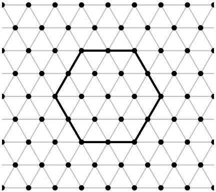
Theorem (B, Bellitto, Pêcher 2015)

If P is the Dirichlet-Voronoï cell of the root lattice D_n, $n \geq 4$, then

$$m_P(\mathbb{R}^n) \leq 1/((3/4)2^n + n - 1)$$

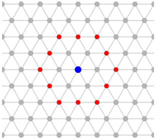
For D_4, the upper bound is $1/15$.
The upper bound is $O(1/2^n)$.

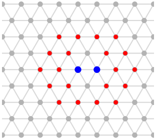
The hexagon

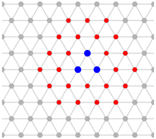


$$\|x - y\|_P = 1 \iff d_G(x, y) = 2$$

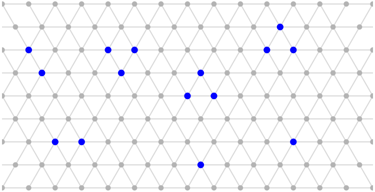
The cliques of G:



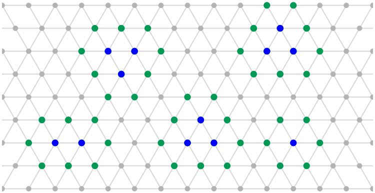




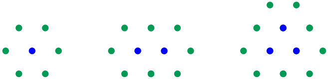
The sets avoiding the hexagon-distance 1 are disjoint unions of cliques of G:



The cliques augmented by their vertex boundaries must be disjoint



so the density of a set avoiding hexagon-distance 1 cannot be better than:



$$\frac{1}{7} \qquad \frac{2}{10} = \frac{1}{5} \qquad \frac{3}{12} = \frac{1}{4}$$

The Dirichlet-Voronoï cells of the lattices A_n and D_n

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$$\{(0,\dots,0),\ (1/2,\dots,1/2),\ (-1/2,1/2,\dots,1/2),\ (0,1,0,\dots,0)\}$$