The Colourful Tverberg Theorem with Equal Coefficients

Tverberg

Any $(k-1)(d+1)+1$ points in $\mathbf{R}^d$ can be partitioned into k classes whose convex hulls intersect.

Once the partition is chosen the intersection is (generically) a single point.

If $d=2$ and $k=3$ we have 7 points: two triangles and a point or two lines and a triangle.

Bárány and Larman conjectured a colourful version:

Any $d+1$ colour classes of k points can be partitioned into k rainbow simplices that intersect.

In this case the intersection is not a single point.

(Conjecture) Any $d+1$ colour classes of k points can be partitioned into k rainbow simplices that intersect.

Vrećica and Zivaljević proved that if each colour class contains $2k-1$

points and k is prime then we can find k disjoint rainbow sets that intersect.

The restriction on k comes from the use of topological methods.

Blagojević, Matschke and Ziegler made a breakthrough on the optimal case: in which each colour class has k points.

Their proof is topological and requires that $k+1$ be prime.

The key first step is to introduce an additional point, in a colour class of its own, so that we now have the Tverberg number $k(d+1)+1$ and find a family of $k+1$ intersecting rainbow sets.

This makes the intersection a single point.

The problem in all these cases is to show that a certain linear programme is feasible. The difficulty lies in finding a partition which creates a feasible linear programme.

The original proof of Tverberg's Theorem was complicated.

The standard proof today depends upon the Colourful Caratheodory Theorem of Bárány and an idea of Sarkaria.

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Proof Assume there is no such simplex. Choose the rainbow simplex to which the origin is closest.

Let x be the closest point: there is a colour not used to make the facet to which x belongs. Choose a point of that colour whose inner product with x is negative.

Swap.

Proof of Tverberg We are given $(k-1)(d+1)+1$ points in $\mathbf{R}^d$. Call them (z_j). For each one form a new point w_j in $\mathbf{R}^{d+1}$ as $(z_j, 1)$.

Let $u_1, u_2, \ldots, u_k$ be the vertices of a regular simplex in $\mathbf{R}^{k-1}$. Observe that

$$\sum \lambda_i u_i = 0$$

if and only if the λ_i are equal.

Now for each j form a colour class of k points in $\mathbf{R}^{(k-1)(d+1)}$ of the form $u_i \otimes w_j$.

We have $n = (k-1)(d+1)+1$ colour classes in $\mathbf{R}^{(k-1)(d+1)}$ each of which is $u_1 \otimes w, \ldots, u_k \otimes w$ so its convex hull contains 0.

So there is a colourful set of $(k-1)(d+1)+1$ points with

$$\sum_{j=1}^{n} \lambda_j u_* \otimes w_j = 0.$$

Put w_j into A_i if $* = i$.

We have positive combinations

$$\sum_{w_j \in A_i} \lambda_j w_j$$

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The total weight of λ is the same for each i because of the 1 that we appended.

So, after scaling, we have convex combinations of the w_j in each class that are equal.

Lovász Any $d+1$ pairs of points can be partitioned into two intersecting rainbow simplices.

Proof Let the pairs be (x_i, y_i) for $1 \leq i \leq d+1$. The $d+1$ vectors $x_i - y_i$ are linearly dependent.

So $\sum \alpha_i(x_i - y_i) = 0$ and by switching the pairs as necessary we can assume that all the α_i are positive and their sum is 1.

Then

$$\sum \alpha_i x_i = \sum \alpha_i y_i$$

is a solution.

The theorem of Lovász solves the original coloured Tverberg for $k = 2$ in a strong way: the convex combinations that we find have equal coefficients in the sense that for each colour class, its weight is the same in both combinations.

The proof is similar to the Sarkaria trick but you don't see it because the u_i in 1 dimension are just the numbers 1 and -1.

Based on this, Pablo Soberón asked whether there might be an equal coefficient version of Colourful Tverberg for k-partitions.

Soberón Given $n = (k-1)d + 1$ colour classes of k points, there is partition into k rainbow sets whose convex hulls intersect with equal coefficients.

The number of colour classes is optimal for the equal coefficient problem.

The proof uses the Sarkaria trick but in a completely different way. We apply Colourful Caratheodory to colour classes of sums.

We have $n = (k-1)d + 1$ colour classes of k points.

For each j between 1 and n we form a new class of points in $\mathbf{R}^{(k-1)d}$

$$\sum_1^k u_i \otimes z_{j\sigma(i)}$$

where $z_{j1}, z_{j2}, \ldots, z_{jk}$ are the points in the j^{th} original class and σ is a permutation.

So we look at all matrices that we can build by assigning one of the $z_{j.}$ to each u_i and adding up.

The new class contains the origin in its convex hull because the sum of its points is zero.

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where $z_{j1}, z_{j2}, \ldots, z_{jk}$ are points in the j^{th} original class and σ is a permutation.

Now we apply Colourful Caratheodory to find a rainbow set whose convex hull contains the origin.

So we have convex weights α_j with

$$\sum_{j=1}^n \alpha_j \sum_i u_i \otimes z_{j\sigma_j(i)} = 0.$$

Then the vectors

$$\sum_{j=1}^{n} \alpha_j z_{j\sigma_j(i)}$$

are the same point, w say, for all i.

The i^{th} set in the partition is the rainbow set $z_{1\sigma_1(i)}, \ldots, z_{n\sigma_n(i)}$.

Their convex hulls each contain the point w (and with equal coefficients).

The general case of the original problem (without the restriction on primality) is still open.

(Conjecture) Any $d+1$ colour classes of k points can be partitioned into k rainbow simplices that intersect.

There is something wrong with our inability to prove this using an analytic/geometric argument. Topological methods are powerful: but the primality can't possibly be essential, can it?

In some ways this conjecture is the most natural problem of its type.

On the other hand perhaps we do need topology. Look at the case $d = 1$.

We have k red points on the line and k blue ones. We want to marry them so that each couple straddles a fixed point p.

Choose p so that there are k points on either side. Then the number of red points on the left is equal to the number of blue points on the right.

This is a topological argument.

Or is it?

Choose p to minimise

$$\sum |x_i - p|.$$