Busemann's Theorem for Entropy?

Keith Ball, Piotr Nayar and Tomasz Tkocz

Busemann's Theorem states that if K is a symmetric convex body in $\mathbf{R}^n$ then the function

$$\theta \mapsto \frac{1}{\text{vol}\left(K \bigcap \langle\theta\rangle^\perp\right)}$$

defined for unit vectors θ, can be extended by homogeneity to a norm on $\mathbf{R}^n$.

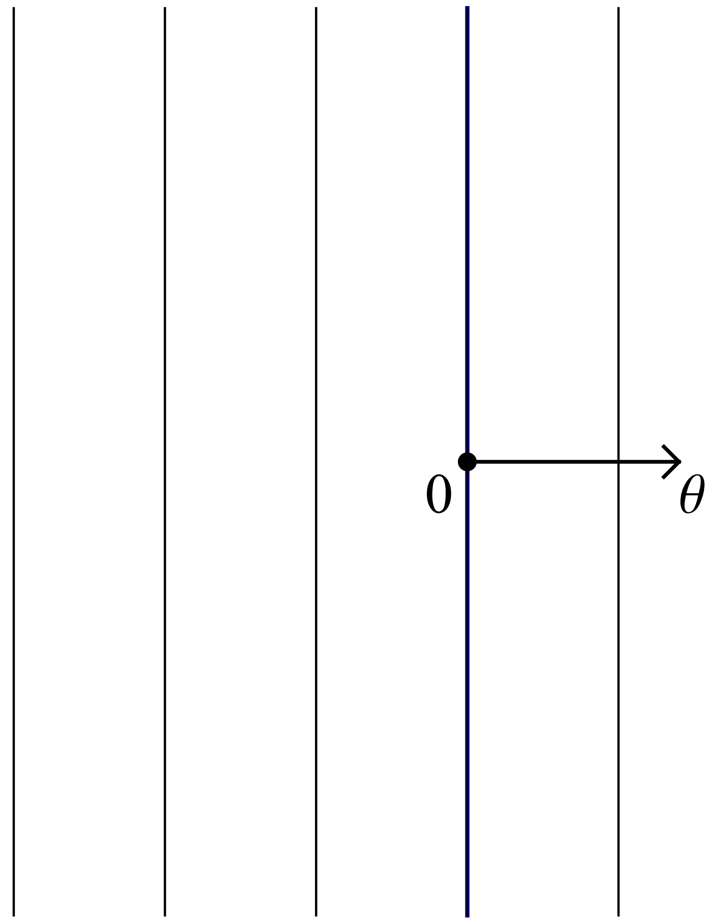
The point is to prove a triangle inequality relating the volumes of 3 slices of K.

The three directions lie in a plane so we can replace K by a function on this plane: the logarithmically concave function whose value at x is the $(n-2)$-dimensional volume of the fibre of K above x.

Let's assume that the volume of K is 1. Then the fibre function is a density on the plane: the density of a random variable X (say).

For each θ let f_θ be the density of the marginal $\langle X, \theta \rangle$.

Then the volume of the slice $K \bigcap \langle\theta\rangle^{\perp}$ is just $f_\theta(0)$.



Busemann's Theorem therefore says that if X is a symmetric log-concave random vector in the plane and for each unit vector θ we let f_θ be the density of $\langle X, \theta \rangle$ then the function

$$\theta \mapsto \frac{1}{f_\theta(0)}$$

extends by homogeneity to a norm on $\mathbf{R}^2$.

It is easy to check that this is the same as saying that if f_x is the density of $\langle X, x \rangle$ for $x \in \mathbf{R}^2$ then

$$x \mapsto \frac{1}{f_x(0)}$$

is a norm.

Some years ago I observed that if f is an even, log-concave density on the line then

$$-\log f(0) \leq -\int f \log f \leq -\log f(0) + C.$$

The best constant here (in each dimension) was found by Fradelizi.

Thus the function

$$x \mapsto e^{\mathrm{Ent}\langle X,x\rangle}$$

looks like the norm

$$x \mapsto \frac{1}{f_x(0)}.$$

Question Suppose $f : \mathbf{R}^n \to [0, \infty)$ is an even log-concave density. Is it true that the function

$$x \mapsto e^{\mathsf{Ent}\langle X,x\rangle}$$

is a norm on $\mathbf{R}^n$?

Is it true that if X is a symmetric log-concave random vector then

$$e^{\mathsf{Ent}\langle X,x+y\rangle} \le e^{\mathsf{Ent}\langle X,x\rangle} + e^{\mathsf{Ent}\langle X,y\rangle}?$$

Is it true that

$$e^{\mathsf{Ent}\langle X, x+y\rangle} \le e^{\mathsf{Ent}\langle X, x\rangle} + e^{\mathsf{Ent}\langle X, y\rangle}?$$

We can always apply a linear transformation to the plane so that x and y end up perpendicular. This means that we can rephrase the problem as follows.

Suppose U and V are random variables and the vector (U, V) is a symmetric log-concave random vector. Is it true that

$$e^{\mathsf{Ent}(U+V)} \le e^{\mathsf{Ent}U} + e^{\mathsf{Ent}V}?$$

Suppose U and V are random variables and the vector (U, V) is a symmetric log-concave random vector. Is it true that

$$e^{\mathsf{Ent}(U+V)} \leq e^{\mathsf{Ent}U} + e^{\mathsf{Ent}V}?$$

This looks a bit like a reverse Entropy Power Inequality (EPI). The EPI says that if U and V are independent random variables then

$$e^{2\,\mathsf{Ent}(U+V)} \geq e^{2\,\mathsf{Ent}U} + e^{2\,\mathsf{Ent}V}.$$

Bobkov and Madiman recently investigated reverse EPI inequalities for independent log-concave random variables.

If the inequality

$$e^{\mathsf{Ent}(U+V)} \le e^{\mathsf{Ent}U} + e^{\mathsf{Ent}V}$$

does hold then it is the best we can say. We can't expect a triangle inequality for any higher power

$$x \mapsto e^{\kappa \mathsf{Ent}\langle X,x\rangle}.$$

To see this choose U and V to be independent random variables uniformly distributed on intervals of length 1 and t say. Then we can evaluate all the entropies and the inequality becomes

$$e^{\kappa t/2} \le 1 + t^{\kappa}.$$

Now take a limit as $t \to 0$.

Using the observation mentioned above which relates entropy to value at 0 it is easy to get an inequality with some constant

$$e^{\mathsf{Ent}\langle X,x+y\rangle} \leq C\left(e^{\mathsf{Ent}\langle X,x\rangle} + e^{\mathsf{Ent}\langle X,y\rangle}\right).$$

Motivated by the Aoki-Rolewicz Theorem an obvious question is whether we can get a triangle inequality with constant 1 but for some power less than 1:

$$e^{\kappa\,\mathsf{Ent}(U+V)} \leq e^{\kappa\,\mathsf{Ent}U} + e^{\kappa\,\mathsf{Ent}V}$$

if (U,V) is a symmetric log-concave random vector.

The Entropy Power Inequality is proved by establishing a linearised version

$$\text{Ent}\left(\sqrt{1-\lambda}X + \sqrt{\lambda}Y\right) \geq (1-\lambda)\text{Ent}X + \lambda\text{Ent}Y.$$

We can find a linearised version of the Busemann problem

$$e^{\text{Ent}(U+V)} \leq e^{\text{Ent}U} + e^{\text{Ent}V}$$

namely:

Is it true that if (U, V) is a symmetric log-concave random vector and $\text{Ent}U = \text{Ent}V$ then

$$\text{Ent}\left((1-\lambda)U + \lambda V\right) \leq \text{Ent}U?$$

Is it true that if (U, V) is a symmetric log-concave random vector and $\text{Ent}U = \text{Ent}V$ then

$$\text{Ent}\left((1-\lambda)U + \lambda V\right) \leq \text{Ent}U?$$

In the case that U and V are independent we have two competing effects. Adding independent random variables increases entropy but taking convex combinations decreases the spread and hence the entropy.

There is a similar linearised version for our weaker inequality: if U and V have the same entropy

$$\mathrm{Ent}\left((1-\lambda)U+\lambda V\right) \leq \mathrm{Ent}U + \frac{1}{\kappa}\log\left((1-\lambda)^{\kappa}+\lambda^{\kappa}\right).$$

We shall prove this inequality (with $\kappa = 1/5$).

For λ in the middle of the range we can use a crude estimate of the type that we already wrote down. The issue is to check what happens as $\lambda \to 0$.

The problem then becomes: suppose (U, V) is a symmetric log-concave random vector and $\mathrm{Ent}U = \mathrm{Ent}V$. Then

$$\mathrm{Ent}((1-\lambda)U + \lambda V) \leq \mathrm{Ent}U + C\lambda.$$

Let $X_\lambda = (1-\lambda)U + \lambda V$ and $Y = V - U$. We want

$$\frac{d}{dt}\mathrm{Ent}\,(X_\lambda + tY)\,|_{t=0} \leq C.$$

The random vector (X_λ, Y) is log-concave.

The random vector (X_λ, Y) is log-concave. Suppose its density is $(x,y) \mapsto w(x,y)$ and

$$f(x) = \int w(x,y)\,dy$$

is the density of X_λ.

It is not too hard to calculate that

$$\frac{d}{dt}\mathrm{Ent}\,(X_\lambda + tY)\,|_{t=0} = -\int \frac{f'(x)}{f(x)}\,y\,w(x,y)\,dxdy.$$

We assume that $\mathrm{Ent}U = \mathrm{Ent}V$ and by the earlier remarks, the densities of X_λ and Y are roughly equal at 0. The problem doesn't change if we assume they are actually equal.

So we now have the following problem. Show that if w is a symmetric log-concave density on the plane,

$$f(x) = \int w(x,y)\, dy \quad \text{and} \quad g(y) = \int w(x,y)\, dx$$

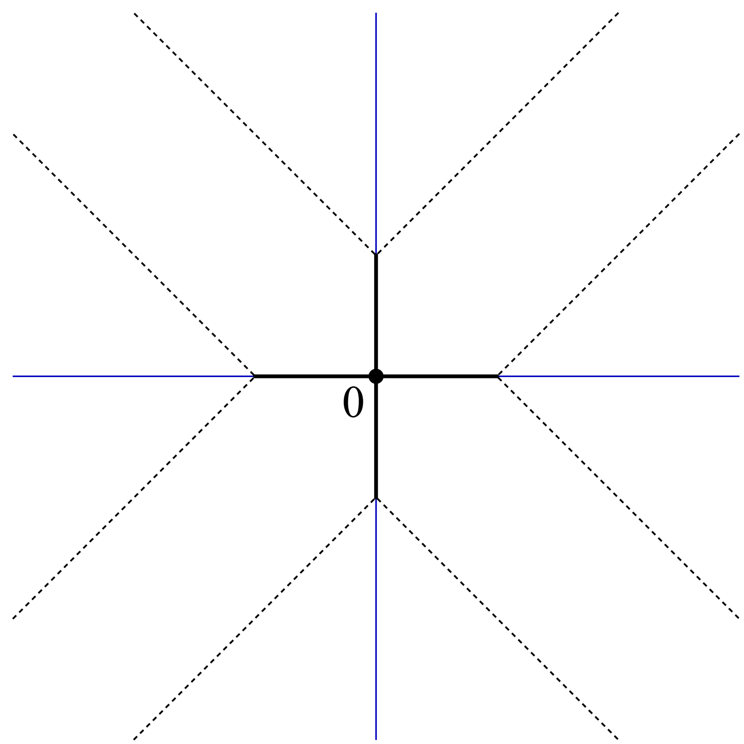
and $f(0) = g(0)$ then

$$-\int \frac{f'(x)}{f(x)}\, y\, w(x,y)\, dxdy \leq C.$$

Since f is log-concave and even, the ratio $-f'(x)/f(x)$ is increasing and odd. So it has the same sign as x.

So the problem is to show that if $x > 0$ then y cannot be too large in places where w is not too small.

Suppose that w is the indicator of a symmetric convex set in the plane. We know that its intersections with the axes have equal length. So it must be trapped in a sloping strip:



So wherever w is not 0, we have $y \leq x +$ (a bit) .

Estimate

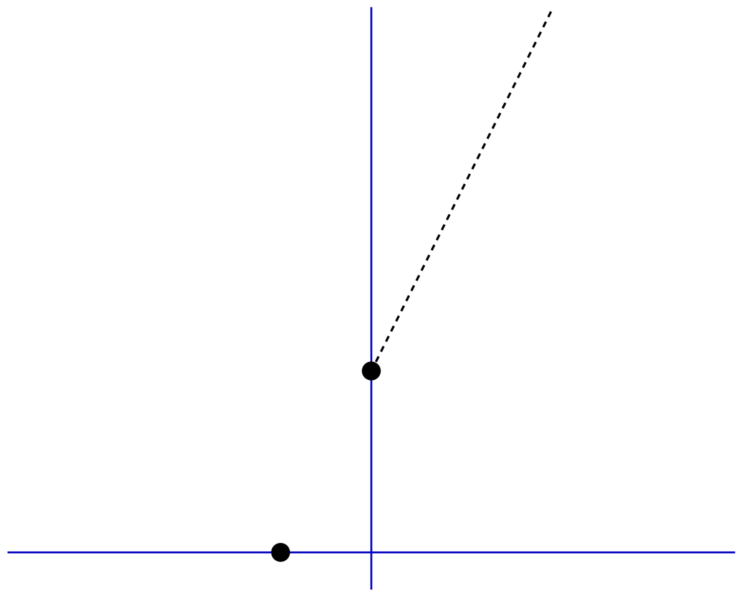
$$-\int_0^\infty \frac{f'(x)}{f(x)}\, y\, w(x,y)\, dxdy.$$

If w is the density of a convex set then wherever w is not 0, we have $y \le x + \text{(a bit)}$ and by log-concavity we have $-f'(x)/f(x) \ge 0$.

$$\begin{aligned} -\int_0^\infty \frac{f'(x)}{f(x)}\, y\, w(x,y)\, dxdy \quad &\le \quad -\int_0^\infty \frac{f'(x)}{f(x)}\, x\, w(x,y)\, dxdy \\ &= \quad -\int_0^\infty f'(x)\, x\, dx = 1. \end{aligned}$$

Now suppose we have a general log-concave symmetric w

$$-\int_0^\infty \frac{f'(x)}{f(x)}\, y\, w(x,y)\, dx dy \leq \text{?}$$



The assumption that $f(0)$ and $g(0)$ are equal tells us that w is small if y is much bigger than x. But this is not enough by itself because $f(x)$ might be tiny.

The assumption that $f(0)$ and $g(0)$ are equal tells us that w is small if y is much bigger than x. But this is not enough by itself because $f(x)$ might be tiny.

However it is possible to prove that (if $f(0) = 1$ say) then roughly

$$\frac{1}{f(x)} \int y\, w(x,y)\, dy \leq C(x - \log f(x) + 1).$$

So the quantity we want to estimate is controlled by

$$-\int_0^\infty f'(x)(x - \log f(x) + 1)\, dx = \frac{1}{2} + 2f(0) + f(0) \log f(0).$$

Is it true that if (U, V) is a symmetric log-concave random vector and $\text{Ent}U = \text{Ent}V$ then

$$\text{Ent}\left((1-\lambda)U + \lambda V\right) \leq \text{Ent}U?$$