Winter School
Combinatorial and algorithmic aspects of convexity
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Combinatorial properties of convex sets.

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In the first part of this lecture we shall recall some definitions as well as basic facts concerning convexity. Then we shall present a few classical theorems with combinatorial flavour, that is the theorems of Carathéodory, Radon and Tverberg respectively.

For two points \( a, b \in \mathbb{R}^d \) we define a segment \([a, b]\) joining \( a \) and \( b \) as the set \([a, b] = \{ \alpha a + \beta b, \alpha, \beta \geq 0, \alpha + \beta = 1 \}\). A set \( C \) in \( \mathbb{R}^d \) is called convex if for every two points \( a \) and \( b \) in \( C \), the segment \([a, b]\) is also contained in \( C \). Plainly, an intersection of convex sets is a convex set.

Given \( n \) points \( a_1, \ldots, a_n \) in \( \mathbb{R}^d \) and real coefficients \( \alpha_1, \ldots, \alpha_n \), the point \( a = \alpha_1 a_1 + \ldots + \alpha_n a_n \) is called their positive combination if all \( \alpha_i \)'s are nonnegative and if in addition \( \alpha_1 + \ldots + \alpha_n = 1 \), then the point \( a \) is called a convex combination (c.c. for short) of \( a_i \)'s. If the real coefficients \( \alpha_i \)'s only satisfy the condition \( \alpha_1 + \ldots + \alpha_n = 1 \), then the point \( a \) is called an affine combination of \( a_i \)'s. Note the following easy observation.

**Fact 1.** If \( C \) is a convex subset of \( \mathbb{R}^d \) and \( a_1, \ldots, a_n \in C \), then all convex combinations of \( a_1, \ldots, a_n \) belong to \( C \).

For a subset \( S \) in \( \mathbb{R}^d \) we define

\[
\text{conv } S = \{ \text{all convex combinations of elements of } S \},
\]
\[
\text{pos } S = \{ \text{all positive combinations of elements of } S \}.
\]

These are clearly convex sets. Moreover, \( \text{conv } S \) is the smallest convex set containing \( S \). It is called the convex hull of \( S \). The set \( \text{pos } S \) is called the positive hull of \( S \). For completeness, we also recall the definition of an affine hull,

\[
\text{aff } S = \{ \text{all affine combinations of elements of } S \}.
\]

Obviously, a segment in \( \mathbb{R}^d \) is a convex set. Another canonical example of a convex set is a (closed) half-space, \( H = \{ x \in \mathbb{R}^d, a \cdot x \geq \alpha \} \). The set of all positive semi-definite matrices of a fixed size is also convex. To see a nontrivial example, the interested reader is encouraged to look at the following exercise.

**Exercise 1.** Let \( f \) be a polynomial with complex coefficients which is not constant. Show that the roots of \( f' \) lie in the convex hull of the roots of \( f \).
Our first theorem basically says that being in a convex hull is a \textit{very finite} property.

\textbf{Theorem 1} (Carathéodory). \textit{Let }$A$\textit{ be a subset in }$\mathbb{R}^d$. \textit{Suppose that }$a \in \text{conv } A$\textit{. Then there exists a subset }$B$\textit{ of }$A$\textit{ with }$|B| \leq d+1$\textit{ such that }$a \in \text{conv } B$.

\textit{Proof.} For a vector $x$ in $\mathbb{R}^d$ and a real number $\alpha$ by $\left( \begin{array}{c} x \\ \alpha \end{array} \right)$ we mean a vector in $\mathbb{R}^{d+1}$ with the last coordinate $\alpha$. The fact that $a$ lies in the convex hull of $A$ can be written shortly as

$\left( \begin{array}{c} a \\ 1 \end{array} \right) = \sum_{i=1}^{n} \alpha_i \left( \begin{array}{c} a_i \\ 1 \end{array} \right),$

for some nonnegative reals $\alpha_i$’s (the last coordinate takes care of the condition that $\alpha_i$’s add up to 1). Without loss of generality we can assume that all $\alpha_i$’s are positive. Moreover, let $n$ be the smallest possible for which the above holds. We want to show that $n \leq d+1$. Suppose not; then the vectors $\left( \begin{array}{c} a_i \\ 1 \end{array} \right)$, $i = 1, \ldots, n$ cannot be linearly independent as they lie in a $d+1$ dimensional space. Therefore we get that

$\left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \sum_{i=1}^{n} \beta_i \left( \begin{array}{c} a_i \\ 1 \end{array} \right)$

for some $\beta_i$’s which are not all equal to zero. Hence,

$\left( \begin{array}{c} a \\ 1 \end{array} \right) = \sum_{i=1}^{n} (\alpha_i + t\beta_i) \left( \begin{array}{c} a_i \\ 1 \end{array} \right)$

for every $t \in \mathbb{R}$. Since for $t = 0$ all the coefficients $\alpha_i + t\beta_i$’s are positive, they remain positive for small $t$ and there is a choice for $t$, say $t_0$ for which at least one of the coefficients becomes 0 with the rest remaining positive. This contradicts the minimality of $n$, as

$\left( \begin{array}{c} a \\ 1 \end{array} \right) = \sum_{i=1}^{n} (\alpha_i + t_0\beta_i) \left( \begin{array}{c} a_i \\ 1 \end{array} \right)$

shows that the vector $\left( \begin{array}{c} a \\ 1 \end{array} \right)$ can be written as a positive combination of $\left( \begin{array}{c} a_i \\ 1 \end{array} \right)$’s with fewer than $n$ nonzero coefficients. \hfill \square
Geometrically speaking, Carathéodory’s theorem says that a convex set $A$ in $\mathbb{R}^d$ can be covered by simplices of $A$ (for us, a simplex in $\mathbb{R}^d$ is a convex hull of at most $d + 1$ points). With the same proof, we can also obtain an analogous result for cones (sets of all positive combinations).

**Theorem 2.** Let $A$ be a subset in $\mathbb{R}^d$. Suppose that $a \in \text{pos } A$. Then there exists a subset $B$ of $A$ with $|B| \leq d$ such that $a \in \text{pos } B$.

Sometimes, the following derivatives of Carathéodory’s theorem can be useful in applications.

**Theorem 3.** Let $A$ be a subset in $\mathbb{R}^d$ and $b \in \mathbb{R}^d$. Suppose that $a \in \text{conv } A$. Then there exists a subset $B$ of $A$ with $|B| \leq d$ such that $a \in \text{conv } (B \cup \{b\})$.

**Theorem 4.** Let $A$ be a subset in $\mathbb{R}^d$. Suppose that $a \in \text{int conv } A$. Then there exists a subset $B$ of $A$ with $|B| \leq 2d$ such that $a \in \text{int conv } B$.

Now we move on to Radon’s theorem and its further generalisation, Tverberg’s theorem, saying that there are some good partitions of sets having enough points.

**Theorem 5 (Radon).** Let $A$ be a subset in $\mathbb{R}^d$ with $|A| \geq d + 2$. Then there is a partition, $A = X \cup Y$ ($X \cap Y = \emptyset$), such that $\text{conv } X \cap \text{conv } Y \neq \emptyset$.

**Remark.** The constant $d + 2$ is the best possible as shown by an example of a simplex in $\mathbb{R}^d$.

**Remark.** When $d = 1$ the theorem is clear as considering three points on a line, there is always one, say $x$ between some two others, say $y, z$, so it suffices to take $X = \{x\}$ and $Y = A \setminus X \supset \{y, z\}$.

**Remark.** When $d = 2$, considering 4 points in the plane, there are two possibilities. Either certain three of them are the vertices of a triangle containing the fourth point, or the points are the vertices of a convex quadrilateral. In any case, it is clear what to take for the partition. (See the pictures.)
Proof. Since $|A| \geq d + 2$, the set $\left\{ \begin{pmatrix} a_1 \\ 1 \end{pmatrix}, a \in A \right\}$ of vectors in $\mathbb{R}^{d+1}$ is not linearly independent. Therefore there are $a_i \in A$ and nonzero coefficients $\alpha_i$ such that

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sum \alpha_i \begin{pmatrix} a_i \\ 1 \end{pmatrix}.
$$

Because the sum of $\alpha_i$’s is 0, some of them are positive, some are negative. Let $I$ be the set of all the indices $i$ for which $\alpha_i > 0$ and $J$ for which $\alpha_i < 0$ (neither $I$ nor $J$ is empty). Breaking the sum into two pieces yields

$$
\sum_{i \in I} \alpha_i \begin{pmatrix} a_i \\ 1 \end{pmatrix} = \sum_{i \in J} (-\alpha_i) \begin{pmatrix} a_i \\ 1 \end{pmatrix}.
$$

Dividing this by $t = \sum_{i \in I} \alpha_i = \sum_{i \in J} (-\alpha_i)$ shows that we can take $X = \{a_i, i \in I\}$ and $Y = A \setminus X$, as then

$$
\text{conv } X \ni \sum_{i \in I} \frac{\alpha_i}{t} a_i = \sum_{i \in J} \frac{-\alpha_i}{t} a_i \in \text{conv } Y.
$$

\[ \square \]

It is left as an exercise to show a combinatorial analogue of Radon’s theorem. We adopt the standard notation that $[m] = \{1, 2, \ldots, m\}$.

**Exercise 2.** Let $A_1, \ldots, A_{n+1}$ be nonempty subsets of $[n]$. Show that there are disjoint subsets $I$ and $J$ of $[n+1]$ such that $\bigcap_{k \in J} A_k = \bigcap_{k \in I} A_k$. Then do the same for the union instead of the intersection.

Now we would like to present a generalisation of Radon’s theorem, Tverberg’s theorem, along with a sketch of a proof. There are at least seven different proofs. The one we shall sketch comes from Roudneff.

**Theorem 6** (Tverberg). Let $r \geq 2$ be an integer. Let $X$ be a subset of $\mathbb{R}^d$ with $|X| = (r - 1)(d + 1) + 1$. Then there is a partition $X = X_1 \cup \ldots \cup X_r$ ($X_i$’s are pairwise disjoint) such that $\bigcap_{i=1}^r \text{conv } X_i \neq \emptyset$.

**Remark.** Taking here $r = 2$ recovers Radon’s theorem.

**Remark.** The constant $(r - 1)(d + 1) + 1$ is the best possible. To see this, consider $(r - 1)(d + 1)$ points in $\mathbb{R}^d$ in general position, meaning that no $d + 1$ of them lie in the same hyperplane.
Proof. We start off by stating a useful fact about intersections of affine subspaces.

Claim. Suppose that a finite subset $S$ of $\mathbb{R}^d$ of points lying in general position is partitioned into $r$ pairwise disjoint subsets $S = S_1 \cup \ldots \cup S_r$ with $|S_i| \leq d+1$ for every $i$. If $\bigcap_{i=1}^r \operatorname{aff} S_i = \emptyset$ then $|S| \leq (r-1)(d+1)$.

We shall not give a proof. Instead, we show a simple counting argument which we hope gives faith and indicates why the claim holds. Note that for a finite set $Y$ of points in $\mathbb{R}^d$ in general position we have that $\operatorname{aff} Y$ is the intersection of certain $d+1-|Y|$ hyperplanes. Therefore, $\bigcap_{i=1}^r \operatorname{aff} S_i$ is the intersection of at least $d+1$ hyperplanes, as

$$\sum_{i=1}^r (d+1 - |S_i|) = r(d+1) - |X| \geq r(d+1) - (r-1)(d+1) = d+1.$$ 

Hence, this intersection is empty.

Now for the proof of Tverberg’s theorem, we shall consider only partitions of $X$ into sets with at most $d+1$ elements each. Fix such a partition, $X = X_1 \cup \ldots \cup X_r$ with $|X_i| \leq d+1$, and consider the function

$$\mathbb{R}^d \ni x \mapsto \sum_{i=1}^r \left( \operatorname{dist}(x, \operatorname{conv} X_i) \right)^2.$$ 

As a sum of convex functions, it is convex, it goes to $\infty$ when $|x|$ does, hence it attains its minimum. Choose a partition for which this minimum is the smallest possible and suppose that the minimum equals $\mu$ and is attained at $x = z$. If $\mu = 0$ we are done, so suppose $\mu > 0$ and we will arrive at a contradiction. Choose $y_i$ in $\operatorname{conv} X_i$ such that $|z - y_i| = \operatorname{dist}(z, \operatorname{conv} X_i)$. Now look at the function

$$\mathbb{R}^d \ni x \mapsto \sum_{i=1}^r |x - y_i|^2.$$

Since its value at $x = z$ is $\mu$ and it is bounded from below (pointwise) by the function $f$ whose minimum is $\mu$, the point $x = z$ is also where $g$ attains its minimum. Thus, taking the gradient of $g$ at $x = z$ yields

$$\sum_{i=1}^r (z - y_i) = 0$$

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(i.e. $z$ is the centre of mass of $y_i$'s).

Notice that it is not possible that $z = y_i$ for all $i$'s because otherwise $\mu = 0$. Without loss of generality let us assume that $z \neq y_1$, so $z \notin \text{conv } X_1$.

For each $i$, let $S_i$ be a minimal subset of $X_i$ for which $y_i \in \text{conv } S_i$. We also have $z \notin \text{conv } S_1$.

First, we show that $z \notin \cap_{i=1}^r \text{aff } S_i$. If not, then in particular $z \in \text{aff } S_1$. But $y_1$ is the point in the polytope $\text{conv } S_1$ closest to $z$, which is outside the polytope as $z \neq y_1$, so $y_1$ has to lie on the boundary of the polytope which contradicts the minimality of $S_1$.

Second, we show that it is not possible for any other point $y \neq z$ to belong to $\cap_{i=1}^r \text{aff } S_i$. Indeed, if not, then for every $i$, by the minimality of $y_i$,

$$
(y - z) \cdot (y - z) \geq 0,
$$
as $y \in \text{aff } S_i$. Summing these inequalities yields

$$
0 = (y - z) \cdot \sum_{i=1}^r (y_i - z) \geq 0,
$$
and, as a result, $(y - z) \cdot (y_i - z) = 0$ for every $i$. Since $y \neq z$, this implies that $z = y_i$, for every $i$.

We have shown that $\cap_{i=1}^r \text{aff } S_i = \emptyset$. By the claim we get $\sum_{i=1}^r |S_i| \leq (r - 1)(d + 1)$. This means that we can choose a point $x$ in $X \setminus \bigcup_{i=1}^r S_i$.

Observe that

$$
\sum_{i=1}^r (z - y_i) \cdot (x - y_i) = \sum_{i=1}^r (z - y_i) \cdot ((z - y_i) + (x - z)) = \sum_{i=1}^r (z - y_i)^2 = \mu > 0.
$$

Hence, there is $j$ for which $(z - y_j) \cdot (x - y_j) > 0$. 

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Moving $x$ to the set $X_j$ leads to a partition with a smaller minimum of $f$ than $\mu$ (see the picture). This contradiction finishes the proof.

Now we shall discuss Helly’s theorem which also concerns intersections of convex hulls. A family of sets in $\mathbb{R}^d$ is said to have Helly’s property if every $d + 1$ of them have a nonempty intersection.

**Theorem 7** (Helly). Let $C_1, C_2, \ldots, C_n$, $n \geq d + 1$, be convex sets in $\mathbb{R}^d$ with Helly’s property. Then

$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$ 

In other words, having empty intersection has a finite reason.

**Proof.** By induction on $n$. Case $n = d + 1$ is trivial. Now suppose that $n \geq d + 2$ and the theorem holds for smaller $n$.

For each $j = 1, \ldots, n$ set

$$K_j = \bigcap_{i=1 \atop i \neq j}^{n} C_i.$$ 

By induction, $K_j \neq \emptyset$. Take arbitrary $z_j \in K_j$. By Radon’s Theorem, there is a partition of $z_1, \ldots, z_n$ into two sets $X, Y$ such that $\text{conv} \ X \cap \text{conv} \ Y \neq \emptyset$. Take $z$ in this intersection. We claim that $z \in C_i$ for all $i = 1, \ldots, n$.

Without loss of generality, focus on $C_1$ and suppose that $z_1 \in X$. Then all $z_j \in Y$ belong to $C_1$, hence $z \in C_1$ too.

Later on, we will see another proof related to Carathéodory’s Theorem.

In a special case $d = 1$ (intervals on a line), a different proof is possible. Sketch: Take the rightmost left-endpoint $x$ of these intervals. Then every interval starts to the left of $x$ and ends to the right of $x$.

**Exercise 3.** Extend the previous argument to trees, i.e. prove the following:

If $T_1, \ldots, T_n$ is a family of subtrees of a given tree $T$ such that $T_i \cap T_j \neq \emptyset$ for each $i, j = 1, \ldots, n$ then $\bigcap_{i=1}^{n} T_i \neq \emptyset$.

Note that the theorem poses no restriction on the nature of the sets $C_i$ apart from being convex. However, it can be seen that the case with all sets compact already captures all of the complexity. Indeed, for $I \subset$
We can take $Z_I = \bigcap_{i \in I} C_i \neq \emptyset$ and replace $C_i$ by $K_i = \text{conv} \{Z_I, i \in I\} \subset C_i$.

Note that Helly’s Theorem in general fails for infinite families. First example is the family of intervals of the form $I_n = [n, \infty)$, the other is the family of intervals of the form $J_n = (0, 1/n]$. However, if all the sets are compact, the theorem holds.

**Theorem 8** (Helly’s Theorem, infinite version). Let $C_1, C_2, \ldots$ be compact convex sets in $\mathbb{R}^d$ with Helly’s property. Then $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

**Proof.** Fix $n \geq 1$ and for $i = 1, \ldots, n$ set $K_i = C_1 \cap \cdots \cap C_i$. Then the family of the sets $K_1, \ldots, K_n$ has Helly’s property, hence, by usual Helly’s Theorem, there exists $z_n \in \bigcap_{i=1}^{n} K_i$. By compactness, take a convergent subsequence of $\{z_n\}$ and assume it tends to some $z_0$. Then $z_0 \in C_i$ for all $i$, thus $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

Helly’s Theorem has many applications. Here we state six of them.

1. Let $F$ be a finite family of convex sets in $\mathbb{R}^d$, $|F| = n \geq d + 1$ and let $C \subset \mathbb{R}^d$ be convex. Then there exists a translate of $C$ intersecting every $K \in F$ if and only if there exists a translate of $C$ intersecting every $(d + 1)$-tuple from $F$.

**Proof.** For $K \in F$, set $K^* = \{x \in \mathbb{R}^d, (x + C) \cap K \neq \emptyset\}$. If we prove that $K^*$ is convex, we are done by Helly’s Theorem. And indeed, take $x, y \in K^*$. Then there exist $x^* \in (x + C) \cap K \neq \emptyset$ and $y^* \in (y + C) \cap K \neq \emptyset$. Now for any $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ we have

$$\alpha x^* + \alpha y \in \alpha x + \alpha C,$$

$$\beta y^* \in \beta y + \beta C,$$

and summing:

$$\alpha x^* + \beta y^* \in \alpha x + \beta y + C.$$

Thus $\alpha x + \beta y + C$ is a translate of $C$ having nonempty intersection with $K$, i.e. $\alpha x + \beta y \in K^*$ and $K^*$ is convex.

2. Let $I_1, \ldots, I_n$, $n \geq 3$, be vertical intervals in $\mathbb{R}^2$ such that every three of them have a line transversal. Then all of them have a common line transversal.
Proof. A set of lines \( y = \alpha x + \beta \) intersecting some particular \( I_j = x_j \times [y_j^-, y_j^+] \) corresponds to a set
\[
S_j = \{ (\alpha, \beta) \in \mathbb{R}^2, y_j^- \leq \alpha x_j + \beta \leq y_j^+ \}.
\]
Being the intersections of two halfspaces, these \( S_j \) are convex, hence by Helly’s Theorem they have a nonempty intersection. Any point in it determines a line intersecting all the intervals \( I_j \).

3. Let \( \mathcal{H} \) be a finite family of open (closed) half-spaces in \( \mathbb{R}^d \) and let \( C \subset \mathbb{R}^d \), \( C \subset \bigcup_{H \in \mathcal{H}} H \) be convex. Then there exists a subfamily \( \mathcal{H}' \subset \mathcal{H}, |\mathcal{H}'| = d + 1 \) such that \( C \subset \bigcup_{H \in \mathcal{H}'} H \).

Proof. For all \( H \in \mathcal{H} \), set \( H^* = C \setminus H \). Then \( H^* \) is convex and \( \bigcap_{H \in \mathcal{H}} H^* = \emptyset \). By Helly’s Theorem, there exists a \((d + 1)\)-tuple \( H_1^*, \ldots, H_{d+1}^* \) with empty intersection. Then \( C \subset H_1 \cup \cdots \cup H_{d+1} \). \( \square \)

4. (Kirchberger’s Theorem) Sets \( R \) and \( B \) of red and blue points in \( \mathbb{R}^d \) are given. Then \( R \) and \( B \) can be strictly separated by a hyperplane if and only if for all \( Y \subset R \cup B \), \( |Y| \leq d + 2 \) one can separate the sets \( Y \cap R \) and \( Y \cap B \). (A hyperplane \( h \) \emph{strictly separates} sets \( A \) and \( B \) if \( A \) lies in one open half-space determined by \( h \) and \( B \) lies in the opposite closed half-space.)

Proof. With every \( r \in R \) we associate a half-space
\[
C_r = \left\{ \left( \begin{array}{c} a \\ \alpha \end{array} \right) \in \mathbb{R}^{d+1}, \left( \begin{array}{c} a \\ \alpha \end{array} \right) \cdot \left( \begin{array}{c} r \\ -1 \end{array} \right) > 0 \right\}.
\]
Likewise, with every $b \in B$ we associate

$$D_b = \left\{ \left( \frac{a}{\alpha} \right) \in \mathbb{R}^{d+1}, \left( \frac{a}{\alpha} \right) \cdot \left( \frac{b}{-1} \right) \leq 0 \right\}.$$

By assumption, every $d + 2$ half-spaces have a point in common, hence by Helly’s Theorem all the half-spaces have a point in common. This point determines a strictly separating hyperplane.

5. (Centrepoint Theorem) Let $X \subset \mathbb{R}^d$, $|X| \geq d + 1$. Then there exists a centrepoint of $X$, i.e. a point $z \in \mathbb{R}^d$ such that any closed half-space $H$ containing $z$ satisfies

$$|X \cap H| \geq \frac{1}{d+1} |X|.$$

Proof. Let

$$\mathcal{H} = \left\{ H, \ H \text{ is an open half-space such that } |X \cap H| < \frac{1}{d+1} |X| \right\}.$$

By summing the sizes we see that no $(d+1)$-tuple of sets in $\mathcal{H}$ covers the whole $X$. Hence the family of closed half-spaces satisfying $|X \cap H| \geq \frac{1}{d+1} |X|$ has Helly’s property. The compactness issues in the infinite Helly’s Theorem are avoided by focusing on a sufficiently large ball instead of the whole $\mathbb{R}^d$.

6. Let $X \subset \mathbb{R}^d$ be finite with diameter $\text{diam}(X) \leq 2$. Then there exists a ball $B$ of radius $r = \sqrt{\frac{2d}{d+1}}$ that contains $X$. Moreover, the cases requiring $r = \sqrt{\frac{2d}{d+1}}$ are precisely the regular simplices with side length 2.

Proof. For any $x \in X$, take the ball $B(x, r)$ with centre $x$ and radius $r = \sqrt{\frac{2d}{d+1}}$. We want to prove $\bigcap_{x \in X} B(x, r) \neq \emptyset$. By Helly’s Theorem, we only need to prove it for a $(d+1)$-tuple $x_0, x_1, \ldots, x_d \in X$.

Let $B(y, R)$ be the smallest ball containing $X$. By appropriate translation we may assume that $y = 0$. Suppose that $|x_i| = R$ for $i = 0, 1, \ldots, m \leq d$ and $|x_i| < R$ for the rest. We will only deal with $x_0, x_1, \ldots, x_m$. 

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If $y \not\in \text{conv}\{x_0, x_1, \ldots, x_m\}$, we could move $y$ closer, thereby reducing the radius. Assume otherwise. Then there exist $\alpha_0, \alpha_1, \ldots, \alpha_m \geq 0$ with
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_{i=0}^{m} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}.
\]
Now
\[
4 \geq |x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2x_i \cdot x_j = 2R^2 - 2x_i \cdot x_j.
\]
Multiplying by $\alpha_i/2$ and summing over $i = 0, 1, \ldots, m$, $i \neq j$, we get
\[
2(1 - a_j) \geq R^2 \cdot (1 - a_j) + (-\sum_{i \neq j} \alpha_i x_i) \cdot x_j = R^2 \cdot (1 - a_j) + \alpha_j \cdot R^2 = R^2.
\]
Summing over all $j = 0, 1, \ldots, m$ finally gives the desired
\[
2(m + 1 - 1) \geq (m + 1)R^2.
\]
\[\square\]
We continue with two more applications of Helly’s Theorem.

1. For any convex compact $C \subset \mathbb{R}^d$ there exists $z \in C$ so that for every chord $[u, v]$ of $C$ with $z \in [u, v]$ we have

$$\frac{1}{d+1} \leq \frac{\|u - z\|}{\|u - v\|} \leq \frac{d}{d+1}.$$ 

In other words, the ratio $\frac{\|u - z\|}{\|u - v\|}$ is separated from 0 and 1.

**Proof.** For every $x \in C$, we define $C_x := x + \frac{d}{d+1} (C - x)$. Note that $C_x \subset C$.

**Claim:** For any $x_1, \ldots, x_{d+1} \in C$, we have $\bigcap_{i=1}^{d+1} C_{x_i} \neq \emptyset$.

Set $y := \frac{1}{d+1} \sum_{i=1}^{d+1} x_i$. Let $j \in \{1, \ldots, d+1\}$ and rewrite $y$ as

$$y = x_j + \frac{d}{d+1} \left( \frac{1}{d} \sum_{i=1, i \neq j}^{d+1} x_i - x_j \right).$$

Note that $\frac{1}{d} \sum_{i=1, i \neq j}^{d+1} x_i$ is a convex combination of points in the convex set $C$ and hence belongs to $C$. We obtain that for any $j \in \{1, \ldots, d+1\}$, $y \in x_j + \frac{d}{d+1} (C - x_j) = C_{x_j}$. This yields the claim.

Helly’s theorem now implies that $\bigcap_{x \in C} C_x \neq \emptyset$. Let $z \in \bigcap_{x \in C} C_x \subset C$.

**Claim:** The point $z$ has the required property.

Let $u, v \in C$ so that $z \in [u, v]$. By definition $z \in C_u$. Hence

$$z \in u + \frac{d}{d+1} ([u, v] - u),$$

i.e. $z = u + \frac{d}{d+1} (\alpha u + (1 - \alpha)v - u) = u + \frac{d}{d+1} (1 - \alpha)(v - u)$ for some $\alpha \in [0, 1]$. Rearranging this equality, we obtain

$$\|z - u\| = \frac{d}{d+1} (1 - \alpha)\|v - u\| \leq \frac{d}{d+1}\|v - u\|.$$
This immediately gives the upper bound and after a simple computation the lower bound as well. Indeed, since \( z \in C_v \) as well, we also have 
\[
\|z - v\| \leq \frac{d}{d+1}\|u - v\|.
\]
The lower bound now follows from 
\[
\|u - v\| = \|u - z\| + \|z - v\| \leq \|u - z\| + \frac{d}{d+1}\|u - v\|.
\]
\hfill \square

2. Let \( a, b \in X \subset \mathbb{R}^d \). We say \( a \) sees \( b \) if \([a, b] \subset X\). We say \( X \) is visible from \( a \) if every \( c \in X \) is seen from \( a \), i.e. \( X \) is star-shaped.

**Krasnosel’skii Theorem:** Let \( X \subset \mathbb{R}^d \) compact. If any \( d+1 \) points in \( X \) are seen from some \( x \in X \), then \( X \) is star-shaped.

**Proof.** For \( x \in X \), define 
\[ V_x := \{\text{set of points visible from} \ x\} = \{y : [x, y] \subset X\} \]
Observe that for any \( x_1, \ldots, x_{d+1} \in X \), \( \bigcap_{i=1}^{d+1} \text{conv}V_{x_i} \supset \bigcap_{i=1}^{d+1} V_{x_i} \neq \emptyset \), i.e. the collection of sets \( \{\text{conv}V_x\}_{x \in X} \) enjoys the Helly property. Hence there is \( z \in \bigcap_{x \in X} \text{conv}V_x \).

**Exercise 1.** Show that \( z \in \bigcap_{x \in X} V_x \), i.e. \( z \) is visible from all \( x \in X \). 
\hfill \square

Sets \( C, D \subset \mathbb{R}^d \) are separated if there is a hyperplane \( H = \{x \mid ax = b\} \) with \( C \subset H^+ \) and \( D \subset H^- \), where \( H^+ = \{x \mid ax \geq b\} \) and \( H^- = \{x \mid ax \leq b\} \). The separation is strict if \( C \subset \text{int}H^+ \) and \( D \subset \text{int}H^- \). The sets are separated by a slab (or skip) if there are parallel hyperplanes \( H_1, H_2 \) with \( H_1 \neq H_2, C \subset H_1^+, D \subset H_2^- \) and \( H_1^+ \cap H_2^- = \emptyset \).

**Theorem 1** (Separation Theorem). Let \( C, D \subset \mathbb{R}^d \) convex, \( C \) compact, \( D \) closed. Then \( C \cap D = \emptyset \) if and only if \( C, D \) are separated by a slab.

**Proof.** Exercise. 
\hfill \square

**Lemma 1.** Let \( K_1, \ldots, K_n \) be closed convex sets and let \( K_1 \) be compact. Then \( \bigcap_{i=1}^{n} K_i = \emptyset \) if and only if there are closed halfspaces \( H_i \) with \( K_i \subset H_i \) and \( \bigcap_{i=1}^{n} H_i = \emptyset \).

“⇒”: The case \( n = 2 \) is clear (Separation theorem). Suppose \( n > 2 \). By assumption

\[
(K_1 \cap K_2 \cap \cdots \cap K_{n-1}) \cap K_n = \emptyset.
\]

Thus there are closed halfspaces \( H'_n \) and \( H_n \) with \( H'_n \cap H_n = \emptyset \) and \( K_1 \cap K_2 \cap \cdots \cap K_{n-1} \subset H'_n \), \( K_n \subset H_n \). Moreover, we have

\[
K_1 \cap K_2 \cap \cdots \cap K_{n-1} \cap H_n = \emptyset.
\]

Separate \( K_{n-1} \) from \( K_1 \cap K_2 \cap \cdots \cap K_{n-2} \cap H_n \) by \( H_{n-1} \) and so on. At the end we obtain \( H_1 \cap H_2 \cap \cdots \cap H_n = \emptyset \) with \( K_i \subset H_i \) for \( i \in [n] \).

Lemma 2. Let \( H_1, \ldots, H_m \) be closed halfspaces in \( \mathbb{R}^d \), \( H_i = \{ x | a_i x \leq \alpha_i \} \).

Then \( \bigcap_{i=1}^m H_i = \emptyset \) if and only if \( \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \in \text{pos} \left\{ \left( \begin{array}{c} a_1 \\ \alpha_1 \end{array} \right), \ldots, \left( \begin{array}{c} a_m \\ \alpha_m \end{array} \right) \right\} \).

Proof. \( \bigcap_{i=1}^m H_i = \emptyset \Leftrightarrow \) the system \( a_i x \leq \alpha_i, \ i = 1, \ldots, m \), has no solution.

“⇐”: \( \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \in \text{pos} \left\{ \ldots \right\} \) means there are \( \gamma_i \geq 0 \), not all zero, so that

\[
\left[ \begin{array}{c} 0 \\ -1 \end{array} \right] = \sum_{i=1}^m \gamma_i \left( \begin{array}{c} a_i \\ \alpha_i \end{array} \right),
\]

i.e. \( 0 = \sum_{i=1}^m \gamma_i a_i \) and \( -1 = \sum_{i=1}^m \gamma_i \alpha_i \). Suppose \( x \) is a solution of the system \( a_i x \leq \alpha_i, \ i = 1, \ldots, m \), then multiplying by \( \gamma_i \) and summing gives \( 0 = (\sum_{i=1}^m \gamma_i a_i) x \leq \sum_{i=1}^m \gamma_i \alpha_i = -1 \). A contradiction.

“⇒”: Suppose \( \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \notin \text{pos} \left\{ \ldots \right\} =: D \). Observe that \( D \) is closed and hence \( \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \) and \( D \) can be separated by a slab. It follows that there is \( \left( \begin{array}{c} b \\ \beta \end{array} \right) \in \mathbb{R}^{d+1} \)

so that \( \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \left( \begin{array}{c} b \\ \beta \end{array} \right) > 0 \) and \( \left( \begin{array}{c} a_i \\ \alpha_i \end{array} \right) \left( \begin{array}{c} b \\ \beta \end{array} \right) \leq 0 \) for \( i \in [n] \). This gives \( \beta < 0 \) and \( a_i b + \alpha_i \beta \leq 0 \). Thus \( \left[ \frac{b}{|\beta|} \right] \) is a solution of the system \( a_i x \leq \alpha_i \).

Remark. More is true. Let \( K_1, \ldots, K_n \) be closed convex sets in \( \mathbb{R}^d \) and let \( K_1 \)
be compact, \( n \geq d + 1 \). Then \( \bigcap_{i=1}^n K_i = \emptyset \) if and only if there is \( I \subset [n], |I| \leq d + 1 \) so that \( \cap_{i \in I} K_i = \emptyset \).
Another proof of Helly's Theorem

First, let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be convex sets, all closed and one compact. We have

$$\bigcap_{i=1}^n K_i = \emptyset \iff \exists H_i \supset K_i \text{ closed halfspaces with } \bigcap_{i=1}^n H_i = \emptyset$$

by the cone version of Carathéodory’s Theorem in $\mathbb{R}^{d+1}$ we have

$$\iff \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \in \text{pos} \left\{ \left( \begin{array}{c} a_1 \\ \alpha_1 \end{array} \right), \ldots, \left( \begin{array}{c} a_n \\ \alpha_n \end{array} \right) \right\}$$

$$\iff \bigcap_{j=1}^{d+1} H_{i_j} = \emptyset.$$  

This implies $\bigcap_{j=1}^{d+1} K_{i_j} = \emptyset$.

Now let $C_1, \ldots, C_n \subset \mathbb{R}^d$ be convex sets with Helly’s property. For every $J \subset [n]$ with $|J| = d + 1$ there exists $z(J) \in \mathbb{R}^d, z(J) \in \bigcap_{j \in J} C_j \neq \emptyset$. For $j \in [n]$, define $K_j := \text{conv} \{ z(J) \mid j \in J \}$. Each $K_j$ is a polytope, $K_j \subset C_j$ and they satisfy the Helly condition. Hence $\bigcap_{i=1}^n K_j \neq \emptyset$. This in turn implies that $\bigcap_{i=1}^n C_j \neq \emptyset$. □

Remark. We will refer to the method in second part of the above proof as reduction to polytopes.

**Theorem 2** (Fractional Helly, Katchalski-Lin).

Let $\alpha \in (0, 1]$ and $d \geq 2$ be fixed. If $\mathcal{C}$ is a finite family of convex sets, $n = |\mathcal{C}|$, with at least $\alpha \binom{n}{d+1}$ intersecting $(d+1)$-tuples, then there exists an intersecting subfamily $\mathcal{C}' \subset \mathcal{C}$, with $|\mathcal{C}'| \geq \beta n$, where $\beta$ is a constant that depends only on $d$ and $\alpha$.

Remark. The best known is $\beta = 1 - (1 - \alpha)^{\frac{1}{d+1}}$ (Kalai).

Proof. Let $\mathcal{C} = \{ C_1, C_2, \ldots, C_n \}$. First apply “reduction” so that we can consider the $C_j$’s to be polytopes.

For each $k \in [n]$, let $\mathcal{I}_k := \{ I \subset [n] : |I| = k, \bigcap_{i \in I} C_i \neq \emptyset \}$ be the set of indices of intersecting $k$-tuples. Given $I \in \mathcal{I}_k$, denote $C(I) := \bigcap_{i \in I} C_i$.

Since $C(I)$ is a polytope, and there are finitely many such polytopes, we can choose $a \in \mathbb{R}^d$ such that the functional $f(x) = ax \in \mathbb{R}$ takes different
values in the vertices of $C(I)$, for all possible $I$’s. In particular, this means that the set

$$\arg\min\{ ax : x \in C(I) \} = \{ z \in C(I) : az \leq ax, \ \forall x \in C(I) \}$$

consists of one single point for each $I$. Let $z(I) = \arg\min\{ ax : x \in C(I) \}$, then $ax > az(I)$ for all $x \in C(I) \setminus \{z(I)\}$.

**Claim.** For each $I \in \mathcal{I}_{d+1}$, there exists $J \in \mathcal{I}_d$, $J \subset I$, such that $z(J) = z(I)$.

**Proof.** Let $H_I := \{ x \in \mathbb{R}^d : ax < az(I) \}$ the open halfspace (orthogonal to $a$) that has $z(I)$ in the closed boundary. Then $\{C_i : i \in I\} \cup \{H_I\}$ is a family of $d+2$ convex sets in $\mathbb{R}^d$ such that $(\bigcap_{i \in I} C_i) \cap H_I = \emptyset$. By Helly, there exists a subfamily of $d+1$ sets with empty intersection. Since $\bigcap_{i \in I} C_i \neq \emptyset$, this implies that there exists $i_0 \in I$ such that $(\bigcap_{i \in I \setminus \{i_0\}} C_i) \cap H_I = \emptyset$. Let $J = I \setminus \{i_0\}$, since $z(I) \in C(J)$ we get that $z(I) = z(J) = \arg\min\{ ax : x \in C(J) \}$. \qed

There are at most \( \binom{n}{d} \) possible sets $J \in \mathcal{I}_d$ and exactly $|\mathcal{I}_{d+1}| = \alpha \binom{n}{d+1}$ sets $I \in \mathcal{I}_{d+1}$. Then there exists some $J_0 \in \mathcal{I}_d$ such that at least

$$\frac{\alpha \binom{n}{d+1}}{\binom{n}{d}}$$

distinct sets $I \in \mathcal{I}_{d+1}$ are mapped to $J_0$. That is, such that $z(I) = z(J_0)$.

For each of those $I$, $z(J_0) = z(I) \in C(I) \subset C_{i_0}$, where $\{i_0\} = I \setminus J_0$. On the other hand, $z(J_0) \in C(J_0) \subset C_j$, for all $j \in J_0$.

In total there are

$$\frac{\alpha \binom{n}{d+1}}{\binom{n}{d}} + d = \alpha \frac{n(n-1) \ldots (n-d)}{(d+1) \cdot n(n-1) \ldots (n-d+1)} + d = \alpha \frac{n-d}{d+1} + d \geq \frac{\alpha n}{d+1}$$

sets $C_i$ that contain the point $z(J_0)$ \( \square \)

**Theorem 3** (Colorful Carathéodory). Let $A_1, A_2, \ldots, A_{d+1} \subset \mathbb{R}^d$ and $a \in \bigcap_{i=1}^{d+1} \text{conv}(A_i)$. Then there exist $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ such that $a \in \text{conv}\{a_1, \ldots, a_{d+1}\}$.
Proof. By Carathéodory’s Theorem, without loss of generality $|A_i| \leq d + 1$. We can also assume $a = 0$.

Let $\{a_1, \ldots, a_{d+1}\}$ be the set, with $a_i \in A_i$ for each $i = 1, \ldots, d+1$, that minimizes the function $dist(0, \text{conv}\{a_1, \ldots, a_{d+1}\})$. Let $\mu$ be this minimum and $z \in \text{conv}\{a_1, \ldots, a_{d+1}\}$ the point that achieves it. That is,

$$\mu = ||z|| = dist(0, z) = dist(0, \text{conv}\{a_1, \ldots, a_{d+1}\})$$

If $\mu = 0$, then $z = 0 \in \text{conv}\{a_1, \ldots, a_{d+1}\}$. Suppose $\mu > 0$.

$z$ must be a point in the boundary of $\text{conv}\{a_1, \ldots, a_{d+1}\}$. That is, it must lie in a facet, which is spanned by $d$ vertices: without loss of generality, $z \in \text{conv}\{a_1, \ldots, a_d\}$. Let $H$ be the hyperplane containing this facet, and let $H^+$ be the open halfspace defined by $H$ and containing the origin. In order for $0 \in \text{conv}(A_{d+1})$, there must exist a point $a'_{d+1} \in A_{d+1} \cap H^+$. Then the segment $(z, a'_{d+1}] \subset \text{conv}\{a_1, \ldots, a_d, a'_{d+1}\} \cap H^+$, and there exists a point $z' \in (z, a'_{d+1}]$ such that $dist(0, z') < dist(0, z) = \mu$, which is a contradiction.

\[ \square \]

Remark.

- **General version:** Carathéodory for one single set: take the $A_i$ to be $d + 1$ copies of the same set.

- **Cone version:** Let $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$ and $a \in \bigcap_{i=1}^d \text{pos}(A_i)$.

Then there exist $a_1 \in A_1, \ldots, a_d \in A_d$ such that $a \in \text{pos}\{a_1, \ldots, a_d\}$.
• **Extra version:** Let $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$, $b \in \mathbb{R}^d$ and $a \in \bigcap_{i=1}^d \operatorname{conv}(A_i)$.

Then there exist $a_1 \in A_1, \ldots, a_d \in A_d$ such that $a \in \operatorname{conv}\{a_1, \ldots, a_d, b\}$.

**Proof.** Without loss of generality $a = 0$ and, by Carathéodory, $|A_i| \leq d + 1$.

If $0 \in \bigcap_{i=1}^d \operatorname{int}(\operatorname{conv}(A_i)) = \operatorname{int}(\bigcap_{i=1}^d \operatorname{conv}(A_i))$, then $-b \in \mathbb{R}^d = \bigcap_{i=1}^d \operatorname{pos}(A_i)$.

By the cone version of Colorful Carathéodory, there exist $\alpha_1, \ldots, \alpha_d \geq 0, a_i \in A_i$, such that $-b = \frac{b + \sum_{i=1}^d \alpha_i a_i}{\sum_{i=1}^d \alpha_i} \in \operatorname{conv}\{a_1, \ldots, a_d, b\}$.

If $0 \not\in \operatorname{int}(\bigcap_{i=1}^d \operatorname{conv}(A_i))$, then it lies in the interior of a lower dimensional face of the polytope $\bigcap_{i=1}^d \operatorname{conv}(A_i)$, and we can apply the previous for a lower dimension $d' < d$.

**Theorem 4** (Application 1: Colorful Helly). Let $C_1, C_2, \ldots, C_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} C_i \neq \emptyset$ for all $C_1 \in C_1, \ldots, C_{d+1} \in C_{d+1}$, then there exists $i \in [d+1]$ such that $\bigcap_{C \in C_i} C \neq \emptyset$.

**Proof 1:** First apply “reduction” so that $C$ is a polytope for all $C \in C$.

Suppose that $\bigcap_{C \in C_i} C = \emptyset$ for all $i = 1, \ldots, d+1$. Since the $C$ are polytopes, they are compact. Then there exist $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{d+1}$ finite families of closed halfspaces such that:

- $\bigcap_{H \in \mathcal{H}_i} H = \emptyset$ for all $i$.

- For each $i$ and for each $C \in C_i$, there exists $H_C \in \mathcal{H}_i$ such that $C \subset H_C$.

  Let $a_C \in \mathbb{R}^d$ and $\alpha_C \in \mathbb{R}$ such that $H_C = \{x \in \mathbb{R}^d : a_C x \leq \alpha_C\}$.

  For each $i$, since $\bigcap_{H \in \mathcal{H}_i} H = \emptyset$, we have that

$$
\left(\begin{array}{c}
0 \\
-1
\end{array}\right) \in \operatorname{pos}\{\left(\begin{array}{c}
a_C \\
\alpha_C
\end{array}\right) : C \in C_i\}
$$
By the cone version of Colorful Carathéodory, there exist \( (a_{C_1}, \alpha_{C_1}) \), \( \ldots \), \( (a_{C_{d+1}}, \alpha_{C_{d+1}}) \) \( \in \mathbb{R}^d \times \mathbb{R} \), where \( C_1 \in C_1, \ldots, C_{d+1} \in C_{d+1} \), such that
\[
\left( \begin{array}{c} 0 \\ -1 \end{array} \right) \in \text{pos}\left\{ \left( \begin{array}{c} a_{C_1} \\ \alpha_{C_1} \end{array} \right), \ldots, \left( \begin{array}{c} a_{C_{d+1}} \\ \alpha_{C_{d+1}} \end{array} \right) \right\}
\]
This implies that \( \bigcap_{i=1}^{d+1} H_{C_i} = \emptyset \), thus \( \bigcap_{i=1}^{d+1} C_i = \emptyset \), which is a contradiction.

Proof 2: First apply “reduction” so that we can consider each \( C \in C_i \) to be a polytope.

Fix \( i_0 \in \{1, \ldots, d+1\} \) and fix the set of indices \( I = \{j_1, \ldots, j_{i_0-1}, j_{i_0+1}, \ldots, j_d\} \) where each \( j_i \) represents the set \( C_{j_i} \in C_i \). That is, \( I \) represents a \( d \)-tuple where each set comes from a different family and there is no element from family \( C_{i_0} \). Denote the polytope
\[
C(I) := \bigcap_{i \in I} C_i \neq \emptyset
\]
There are finitely many such polytopes (as many as choices for \( i_0, j_1, \ldots, j_{d+1} \)) so there exists \( a \in \mathbb{R}^d \) such that the functional \( f(x) = ax \in \mathbb{R} \) takes different values in the vertices of \( C(I) \), for all possible \( I \)'s. In particular, this means that the set \( \arg\min \{ax : x \in C(I)\} \) consists of one single point for each \( I \).

Let \( z(I) = \arg\min \{ax : x \in C(I)\} \), then \( ax > az(I) \) for all \( x \in C(I) \setminus \{z(I)\} \).

Let now \( I_0 \) be the \( d \)-tuple such that the value \( az(I_0) \) is maximal and let \( i_0 \) be the index of the family \( C_{i_0} \) that is not represented. Without loss of generality, \( i_0 = d + 1 \), and \( I_0 = \{i_1, \ldots, i_d\} \).

Claim. \( z(I_0) \in C_i \), for all \( C_i \in C_{d+1} \).

Proof: Suppose that \( z(I_0) \notin C_i \), for some \( C_i = C_{i_{d+1}} \). Then \( z(I_0) \notin C(I_0) \cap C_i \). Let \( H := \{x \in \mathbb{R}^d : ax \leq az(I_0)\} \) the closed halfspace (orthogonal to \( a \)) that has \( z(I_0) \) in the boundary. Then \( \{C_i : i \in I_0\} \cup \{H\} \cup \{C_i\} \) is a family of \( d+2 \) convex sets in \( \mathbb{R}^d \) such that \( \bigcap_{i \in I_0} C_i \cap H \cap C_i = \emptyset \). By Helly, there exists a subfamily of \( d+1 \) sets with empty intersection. Since \( \bigcap_{i \in I_0} C_i \cap H = \{z(I_0)\} \neq \emptyset \), this implies that there exists a \( (d-1) \)-tuple
$J \subset I_0$ such that $(\bigcap_{i \in J} C_i) \cap H \cap C' = \emptyset$. Let the $d$-tuple $I_1 = J \cup \{i_{d+1}\}$, then $C(I_1) = (\bigcap_{i \in J} C_i) \cap C' \subset \mathbb{R}^d \setminus H$. That is, $ax > az(I_0)$ for all $x \in C(I_1)$. In particular $az(I_1) > az(I_0)$, and so $az(I_0)$ is not maximal, which is a contradiction. 

Hence $\bigcap_{C \in C_{d+1}} C \neq \emptyset$. 

**Theorem 5** (Application 2). Let $G = (V,A)$ be a directed graph with $n = |V|$ vertices, and let $C_1, \ldots, C_n$ be directed cycles in $G$. Then there exist arcs $a_i \in C_i$ such that $\{a_1, a_2, \ldots, a_n\}$ contains a directed cycle.

**Proof.** Label the vertices $\{v_1, \ldots, v_n\}$, and represent each arch $a \in A$ by a point $p_a = e_j - e_i \in \mathbb{R}^n$, where $v_i$ is the starting vertex of $a$ and $v_j$ is the ending vertex. In particular, all the points $p_a$ lie in the hyperplane $\sum_{i=1}^n x_i = 0$, which has dimension $d = n - 1$.

In this setting, a set of arcs $C \subset A$ contains a directed cycle if and only if $\sum_{a \in C_i} p_a = 0$ (there could be several pairwise disjoint cycles).

$C_1, \ldots, C_{d+1}$ are directed cycles in $G$. Then $\sum_{a \in C_i} p_a = 0$ for each $i \in [d+1]$

$\implies 0 = \frac{1}{n} \sum_{a \in C_i} p_a \in \text{conv}\{p_a : a \in C_i\}$ for all $i \implies 0 \in \bigcap_{i=1}^{d+1} \text{conv}\{p_a : a \in C_i\}$.

By Colorful Carathéodory, there exist $a_1 \in C_1$, $a_{d+1} \in C_{d+1}$ such that $0 \in \text{conv}\{p_{a_1}, \ldots, p_{a_{d+1}}\}$. Without loss of generality, $0 \in \text{conv}\{p_{a_1}, \ldots, p_{a_k}\}$, $a_i \neq a_j$, for $k$ minimal. Then there exist $\alpha_1, \ldots, \alpha_k > 0$, $\sum_{i=1}^k \alpha_i = 1$, such that

$0 = \sum_{i=1}^k \alpha_i p_{a_i}$.

It is easy to see that, under the previous conditions, all the (non-zero) coefficients must be equal: $\alpha_1 = \cdots = \alpha_k = \frac{1}{k}$, and $0 = \sum_{i=1}^k \frac{1}{k} p_{a_i} = \frac{k}{k} \sum_{i=1}^k p_{a_i}$, which implies that $\{a_1, \ldots, a_k\}$ contains a directed cycle in $G$. 

**Exercise 2.** Try to find a combinatorial proof of this last statement.

9
Theorem (Extended Colorful Carathéodory). Let $A_1, \ldots, A_d$ be nonempty sets in $\mathbb{R}^d$. Suppose that for any pair of distinct indices $i, j$ we have $0 \in \text{conv}(A_i \cup A_j)$. Then there exist $a_i \in A_i$, $i = 1, \ldots, d + 1$, such that $0 \in \text{conv}\{a_1, \ldots, a_{d+1}\}$.

Clearly, this is an extension of Colorful Carathéodory’s Theorem, since the condition $0 \in \bigcap_{i=1}^{d+1} \text{conv} A_i$ implies the above assumption.

Remark. This theorem does not have a conic analog (but it has some consequences for the spherical colorful version of Helly’s Theorem. However, we will not explain them here).

Proof. We start as in the proof of Colorful Carathéodory’s Theorem. Without loss of generality we may assume that all the sets $A_i$ are finite. Choose $a_i \in A_i$, $i = 1, \ldots, d + 1$, so that the distance $\text{dist}(0, \text{conv}\{a_1, \ldots, a_{d+1}\})$ is minimal.

If $\text{dist}(0, \text{conv}\{a_1, \ldots, a_{d+1}\}) = 0$ then we are done, so suppose further that $\text{dist}(0, \text{conv}\{a_1, \ldots, a_{d+1}\}) = ||z|| > 0$ (for some $z \in \text{conv}\{a_1, \ldots, a_{d+1}\}$). We may assume that the points $a_1, \ldots, a_{d+1}$ are in general position and that $z$ lies in the interior of the facet $\text{conv}\{a_1, \ldots, a_d\}$.

Denote $H = \text{aff}\{a_1, \ldots, a_d\}$. Because of the optimality of $z$ the set $A_{d+1}$ (and in particular the point $a_{d+1}$) has to lie “above” $H$ (see Figure 1). Consequently, because of the condition $0 \in \text{conv}(A_i \cup A_{d+1})$, there exist points $b_1 \in A_1, \ldots, b_d \in A_d$, which lie “below” $H$.

For $i = 1, \ldots, d$ we define $f(e_i) = a_i, f(-e_i) = b_i$. Then we extend $f$ to the mapping $f : \partial \text{conv}\{\pm e_1, \ldots, \pm e_d\} \rightarrow \mathbb{R}^d$ simply by setting $f$ to be affine on the facets.

Note that the facets of $\partial \text{conv}\{\pm e_1, \ldots, \pm e_d\}$ are mapped exactly to multicolor facets. The image of $f$ divides $\mathbb{R}^d$ into components, of which one is unbounded (we use topology here!). From the optimality of $z$ we have
The point $a_{d+1}$ lies "above" $H$.

$[0, z) \cap \text{Im } f = \emptyset$. Therefore 0 lies in a bounded component of $\mathbb{R}^d \setminus \text{Im } f$ (since the image of $f$ lies "below" $H$ and contains $z$). On the other hand the point $a_{d+1}$ lies in the unbounded component of $\mathbb{R}^d \setminus \text{Im } f$, since it lies "above" $H$. Therefore the half-ray starting from $a_{d+1}$ (in the unbounded component) after passing through 0 (in a bounded component) has to pierce a multicolor facet $\text{conv}\{c_1, \ldots, c_d\}$, for some $c_i \in \{a_i, b_i\}$ (see Figure 2).

The point $a_{d+1}$ lies "above" $H$. Therefore 0 lies in a bounded component of $\mathbb{R}^d \setminus \text{Im } f$ (since the image of $f$ lies "below" $H$ and contains $z$). On the other hand the point $a_{d+1}$ lies in the unbounded component of $\mathbb{R}^d \setminus \text{Im } f$, since it lies "above" $H$. Therefore the half-ray starting from $a_{d+1}$ (in the unbounded component) after passing through 0 (in a bounded component) has to pierce a multicolor facet $\text{conv}\{c_1, \ldots, c_d\}$, for some $c_i \in \{a_i, b_i\}$ (see Figure 2).

Figure 2: The half-ray starting at $a_{d+1}$ pierces a multicolor facet.
The simplex \( \conv\{c_1, \ldots, c_d, a_{d+1}\} \) is better than the originally chosen simplex \( \conv\{a_1, \ldots, a_{d+1}\} \) (note that we cannot have \( c_i = a_i \) for all \( i = 1, \ldots, d \) since our half-ray goes first through \( H \), then passes through 0 and only then pierces the facet \( \conv\{c_1, \ldots, c_d\} \)). This contradiction with the assumption \( \dist(0, \conv\{a_1, \ldots, a_{d+1}\}) > 0 \) and ends the proof. \qed

Remark (homework). Let \( \Gamma \) be a curve in \( \mathbb{R}^d \) (a continuous image of the interval \([0, 1]\)). Then every point in \( \conv \Gamma \) can be written as a convex combination of only \( d \) points from \( \Gamma \) (this is one point less than in Carathéodory’s Theorem, but our set is of a special form).

We will give now a second proof of Tverberg’s Theorem.

Sarkaria’s proof of Tverberg’s Theorem. Let \( X := \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R}^d \), where \( n = (r - 1)(d + 1) \). We want to prove that there exists a partition \( X = X_1 \cup \ldots \cup X_r \) such that \( \cap_{i=1}^r \conv X_i \neq \emptyset \).

We will use an “artificial” tool. Namely, let \( v_1, \ldots, v_r \) be vectors in \( \mathbb{R}^{r-1} \) such that every \( (r-1) \)-tuple of them is linearly independent and \( v_1 + \ldots + v_r = 0 \). For any \( i = 0, \ldots, n \) we define the set

\[
A_i := \left\{ v_j \otimes \begin{pmatrix} x_i \\ 1 \end{pmatrix} : j = 1, \ldots, r \right\} \subseteq \mathbb{R}^n.
\]

Since \( 0 = \sum_{i=1}^r \frac{1}{r} v_i \), the origin belongs to \( \bigcap_{i=0}^n \conv A_i \). By Colorful Carathéodory’s Theorem there exist \( a_i \in A_i \) such that \( 0 \in \conv\{a_0, \ldots, a_n\} \).

Hence there exist weights \( \alpha_i \geq 0, \sum_{i=0}^n \alpha_i = 1 \), such that

\[
0 = \sum_{i=0}^n \alpha_i a_i = \sum_{i=0}^n \alpha_i v_{j(i)} \otimes \begin{pmatrix} x_i \\ 1 \end{pmatrix}.
\] (1)

Let \( I_j := \{ i \in \{0, \ldots, n\} : j(i) = j \} \). These sets are a partition of \( \{0, \ldots, n\} \) and thus the sets \( X_j := \{x_i : i \in I_j\} \) are a partition of \( X \). We will find a common point to all the sets \( \conv X_i \), which will end the proof.

Note that for any distinct \( k, l \in \{1, \ldots, r\} \) there exists \( u \in \mathbb{R}^{r-1} \) such that \( u \cdot v_k = 1, u \cdot v_l = -1 \) and \( u \cdot v_i = 0 \) for all other \( i \). We multiply the equation (1) by \( u \) from the left side (in the sense of scalar product of vectors in \( \mathbb{R}^{r-1} \)) and get

\[
0 = \sum_{i \in I_k} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} - \sum_{i \in I_l} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}.
\]

In particular \( \sum_{i \in I_k} \alpha_i = \sum_{i \in I_l} \alpha_i \) for every pair of \( k, l \). Thus the sum \( \sum_{i \in I_k} \alpha_i \) does not depend on \( k \) and since the sum of all \( \alpha_i \) is 1, all the sums \( \sum_{i \in I_k} \alpha_i \) are equal to \( \frac{1}{r} \).
Similarly we get that $\sum_{i \in I_k} \alpha_i x_i$ does not depend on $k$. Denote this sum by $z$. Then

$$rz = \sum_{i \in I_1} (r\alpha_i)x_i = \sum_{i \in I_2} (r\alpha_i)x_i = \ldots = \sum_{i \in I_r} (r\alpha_i)x_i,$$

which (together with $\sum_{i \in I_k} (r\alpha_i) = 1$) means that $rz \in \text{conv } X_i$ for all $i = 1, \ldots, r$. \hfill $\square$

Remark. We can prove Tverberg’s Theorem in a different way. We will give only the main idea of the third proof: take $n + 1$ points in $\mathbb{R}^d$ ($n$ is as in the above proof), build a simplex in $\mathbb{R}^{d+1}$ by lifting the given points through fibres of an affine projection. Then it is enough to prove that there exist $r$ disjoint faces of this simplex such that their images under this projection on $\mathbb{R}^d$ have a common point.

Remark. The question what happens if we use a continuous map (rather than an affine one) in the formulation of Tverberg’s Theorem described in the previous Remark was long open. It is now known that the version with the continuous maps is true for $r = p^k$, where $p$ is a prime number (in the proof actions of groups are used).

The second proof of Tverberg’s Theorem is actually a copycat of the proof of Radon’s Theorem from the first lecture. We can formulate this method, invented by Sarkaria, as an easy lemma:

**Lemma (Sarkaria).** Assume $X$ is a finite set in $\mathbb{R}^d$ and $X = X_1 \cup \ldots \cup X_r$ is its partition. Let $n = (r - 1)(d + 1)$. With every $x \in X$, if $x \in X_j$, we associate a vector $\bar{x} := v_j \otimes (\frac{1}{r}) \in \mathbb{R}^n$, where $(v_i)_{i=1}^r$ is a sequence introduced in the previous proof. By $\bar{X}$ be denote a set of all $\bar{x}$. Then $\bigcap_{j=1}^r \text{conv } X_j = \emptyset$ if and only if $\bar{0} \notin \text{conv } \bar{X}$.

Remark. In Sarkaria’s Lemma we can skip the assumption that $X_i$ are pairwise disjoint. Then we have to consider multiset $X$.

Remark. This gives an “interior” condition equivalent to $\bigcap_{j=1}^r \text{conv } X_j = \emptyset$ (which means that $X$ is “separated along the colors” $1, \ldots, r$). Note that we found an “exterior” condition before, in two lemmas from the Tuesday lecture.

**Proof of Sarkaria’s Lemma.** We will show that $\bigcap_{j=1}^r \text{conv } X_j \neq \emptyset$ if and only if $0 \notin \text{conv } \bar{X}$. We will proceed in the same way as in the second proof of Tverberg’s Theorem.
Assume first that $0 \in \text{conv} \, \bar{X}$. Thus there exist weights $\alpha(x)$ such that

$$0 = \sum_{x \in X} \alpha(x) \bar{x} = \sum_{x \in X} \alpha(x)v_j(x) \otimes \left(\begin{array}{c} x \\ 1 \end{array}\right) = \sum_{j=1}^{r} v_j \otimes \left(\sum_{x \in X_j} \alpha(x) \left(\begin{array}{c} x \\ 1 \end{array}\right)\right).$$

Take $u$ such that $uv_k = 1$, $uv_l = -1$ and $uv_i = 0$ for all other $i$. We multiply the equation by $u$ and get, as in the previous proof, that both

$$\sum_{x \in X_k} \alpha(x) x$$

and

$$z := \sum_{x \in X_k} \alpha(x) x$$

do not depend on $k$. Thus $rz$ is a convex combination of points of $X_k$ for all $k = 1, \ldots, r$, which ends the proof of this implication.

Assume now that $z \in \bigcap_{j=1}^{r} \text{conv} \, X_j$. Then there exist nonnegative weights $\alpha(x)$ such that

$$z = \sum_{x \in X_1} \alpha(x)x = \ldots = \sum_{x \in X_r} \alpha(x)x,$$

thus

$$0 = 0 \otimes \left(\begin{array}{c} z \\ 1 \end{array}\right) = (v_1 + \ldots + v_r) \otimes \left(\begin{array}{c} z \\ 1 \end{array}\right) = \sum_{j=1}^{r} v_j \otimes \left(\sum_{x \in X_j} \alpha(x) \left(\begin{array}{c} x \\ 1 \end{array}\right)\right) = \sum_{x \in X} \alpha(x) \bar{x},$$

which means (since $\alpha(x)$ are nonnegative), that $0 \in \text{conv} \, \bar{X}$ and proves the second implication. \qed

We will also use Sarkaria’s Lemma to prove another theorem:

**Theorem** (Kirchberger). Assume $X = X_1 \cup \ldots \cup X_r \subset \mathbb{R}^d$ is a partition of a finite set $X$. Then $X$ is separated along the colors (i.e. $\bigcap_{j=1}^{r} \text{conv} \, X_j = \emptyset$) if and only if all $Y \subset X$ of cardinality at most $(r-1)(d+1)+1$ are separated along the colors (i.e. $\bigcap_{j=1}^{r} \text{conv} \, (Y \cap X_j) = \emptyset$).

This theorem will follow by a more general fact. Consider a partition of $X \subset \mathbb{R}^d$ as above and let $n = (r-1)(d+1)$. Let $X_{i,0}, \ldots, X_{i,n}$ be such that for any $i = 1, \ldots r$ we have

$$X_i = \bigcup_{k=0}^{n} X_{i,k}. $$

For $j = 0, \ldots n$ we introduce sets

$$G_j := \bigcup_{i=1}^{r} X_{i,j},$$

which we will call groups.
We call a set $Y = \{y_0, \ldots, y_n\}$ a transversal, if $y_j \in G_j$ for $j = 0, \ldots, n$. We say that a group $G_j$ is separated along the colors if $\bigcap_{i=1}^r \text{conv } X_{i,j} = \emptyset$.

Kirchberger’s Theorem is a special case of the following one (to recover it we have to set $X_{i,j} := X_i$).

**Theorem.** Under above notation and conditions, if every transversal is separated along the colors then there exists a group separated along the colors.

**Proof.** By Sarkaria’s Lemma if a transversal $Y$ is separated along the colors then $0 \notin \text{conv } \bar{Y}$, where in the construction of $\bar{Y}$ we use a partition of $Y$ induced by the partition $X_1, \ldots, X_r$ of $X$, namely $Y = (Y \cap X_1) \cap \ldots \cap (Y \cap X_r)$.

On the other hand, if every group $G_j$ was not separated along the colors, the origin would belong to all the sets $\text{conv } \bar{G}_j$, where again in the construction of $\bar{G}_j$ we use the partition given by the partition $X_1, \ldots, X_r$ of $X$. Then, by Colorful Carathéodory’s Theorem, there would exist a transversal $\bar{Y}$, such that $\text{conv } \bar{Y}$ would contain the origin, which would be a contradiction.

We will consider now another problem: by how many simplices can a single point be covered? The following theorem gives an exact answer in the case of the plane.

**Theorem (Boros-Füredi).** Assume $X \subset \mathbb{R}^2$ is a set of $n$ points. Then there exists a point covered by $\binom{n}{3} \left(\frac{2}{9} - o(1)\right)$ triangles with vertices in $X$.

**Remark.** We will prove later that the constant $\frac{2}{9}$ is optimal.

**Proof.** Consider a “nice” finite measure on the plane $\mathbb{R}^2$, say absolutely continuous with respect to the Lebesgue measure with a positive continuous density. By $M$ we denote the measure of the plane.

Fix any direction $v \in S^1$. Note that there exists a line $l = l(v)$ in this direction dividing the plane into two half-planes $H_1 = H_1(v)$ and $H_2 = H_2(v)$ of the same measure. We assume that $H_1$ is ‘on the left-hand side’ of $v$ (see Figure 3). Then fix any point $x$ on this line. It divides the line $l$ into two half-lines $l_1$ and $l_2$ (assume that $l_1$ goes in the direction of $v$ and $l_2$ in the direction of $-v$). Note that there exist two half-lines $k_1 \subset H_1$ and $k_2 \subset H_2$ originating at $x$, such that the sector between $l_1$ and $k_1$ has measure $\frac{M}{6}$ and so does the sector between $l_2$ and $k_2$. When we move the point $x$ along $l$ from ‘$-\infty$’ to ‘$\infty$’, the angle between $k_1$ and $k_2$ changes continuously from 0 to $2\pi$ and hence there exists a point $x = x(v)$, such that these two half-lines form a line (see Figure 3).

Moreover there exist two lines $m_1 \subset H_1$ and $m_2 \subset H_2$ originating at $x$ such that the sector between $k_1$ and $m_1$ is of measure $\frac{M}{6}$ and so is the
sector between $k_2$ and $m_2$ (see Figure 4). Let $\alpha = \alpha(v)$ be the angle between $m_1$ and $m_2$. Note that this angle is a continuous function of $v \in S^1$ (we do not assume this angle belongs to $[0, 2\pi]$). Moreover $H_1(v) = H_2(-v)$, $l_1(v) = l_2(-v)$, $m_1(v) = m_2(-v)$, etc. and thus $\alpha(v) = -\alpha(-v) \pmod{2\pi}$, hence there exists a direction $v_0$ such that $\alpha(v_0) = \pi \pmod{2\pi}$, which means that $m_1(v_0)$ and $m_2(v_0)$ are collinear (see Figure 4).

We have constructed three lines with a common point, dividing the plane into three sectors each of measure $\frac{M}{6}$. If we considered a counting measure (which is not absolutely continuous) on the set $X$, we could find (if $n = |X| is
large) three lines with a common point dividing the plane into three sectors each containing almost $\frac{n}{6}$ points of $X$. We skip this technical argument.

We will now prove that the point $z$ common to these three lines is covered by at least \( \binom{n}{3}(\frac{2}{3} - o(1)) \approx \frac{n^3}{27} - o(n^3) \) triangles. Note that if we pick three points, each from a different sector in such a way that any two points do not lie in neighbouring sectors, we get vertices of a triangle covering $z$ (see Figure 5). We can pick these points in \( \approx 2\left(\frac{n}{6}\right)^3 \) ways.

![Figure 5: Triangles covering $z$.](image)

Moreover, if we choose two nonneighbouring groups of neighbouring sectors and pick a point from each of them, we get vertices of possible four triangles of which at least two cover point $z$ (see Figure 5). This gives us a total number of triangles covering $z$ approximately equal to at least $\frac{n^3}{27}$.

**Theorem.** For any $d \geq 2$ there exists a positive constant $c(d)$ with the following property: for any set $X \subset \mathbb{R}^d$ of $n$ points in general position there is a point in at least $c(d)\binom{n}{d+1}$ simplices with vertices in $X$.

**First proof.** Let $r$ be the largest possible number such that $n = (r - 1)(d + 1) + k$ for some $k \geq 1$. Then $X$ has a Tverberg partition into $r$ pieces, i.e. $X = X_1 \cup \ldots \cup X_r$ and there exists a point $z \in \bigcap_{i=1}^r \text{conv} X_i$.

By Colorful Carathéodory’s Theorem for any $1 \leq i_1 \leq \ldots \leq i_{d+1} \leq r$ there exist $x_{i_1} \in X_{i_1}, \ldots, x_{i_{d+1}} \in X_{i_{d+1}}$ such that $z \in \text{conv}\{x_{i_1}, \ldots, x_{i_{d+1}}\}$.

Clearly the simplices $\text{conv}\{x_{i_1}, \ldots, x_{i_{d+1}}\}$ are pairwise distinct for different choices of the indices $1 \leq i_1 \leq \ldots \leq i_{d+1} \leq r$. Therefore the point $z$ is covered by at least

\[
\binom{r + d}{d + 1} = \left(\frac{n + k}{d + 1} + 1 + d\right) \geq \left(\frac{n}{d + 1}\right) \geq \frac{1}{(d + 1)^{d+1}} \left(\frac{n}{d + 1}\right)
\]

different simplices with vertices from $X$. \qed
Remark. We proved the theorem with \( c(d) = (d + 1)^{-d-1} \). We could improve this constant slightly by using the stronger version of Colorful Carathéodory’s Theorem, where one point is fixed.

Second proof. Let

\[ \mathcal{F} = \{ \text{conv } S : S \subset X, |S| = d + 1 \}. \]

Clearly \(|\mathcal{F}| = \binom{n}{d+1}\). We want to show that “many” of the elements of \( \mathcal{F} \) have a nonempty intersection. We will use Fractional Helly’s Theorem for this purpose. It is enough to show that a positive fraction of all

\[ \binom{\binom{n}{d+1}}{d+1} \]

\((d + 1)\)-tuples of elements from \( \mathcal{F} \) (or in other words, a positive fraction of all \((d + 1)\)-tuples of convex hulls of \( d + 1 \) elements from \( X \)) has a nonempty intersection.

Suppose we have a subset \( Y \subset X, |Y| = (d + 1)^2 \). Any \( d^2 + d + 1 = (d + 1)\) points from \( Y \) have a Tverberg’s partition, i.e. they can be split into mutually disjoint \( Y_1, \ldots, Y_{d+1} \) such that \( \bigcap_{i=1}^{d+1} \text{conv } Y_i \neq \emptyset \). By Carathéodory’s Theorem we may assume that each of the sets \( Y_i \) has no more than \( d + 1 \) elements. Then we distribute the points from \( Y \setminus \bigcup_{i=1}^{d+1} Y_i \) among the sets \( Y_i \) so that each of the \( d + 1 \) resulting sets \( \widetilde{Y}_i \) has exactly \( d + 1 \) elements. Summing up, we obtain a partition \( Y = \bigcup_{i=1}^{d+1} \widetilde{Y}_i, |\widetilde{Y}_i| = d + 1, \bigcap_{i=1}^{d+1} \text{conv } \widetilde{Y}_i \neq \emptyset \).

There are \( \binom{n}{(d+1)^2} \) possible choices of the subset \( Y \), each resulting in a different \((d + 1)\)-tuple of \( d + 1 \) elements from \( X \) such that the convex hulls of those elements intersect.

Since

\[ \binom{n}{(d+1)^2} \geq c(d) \binom{\binom{n}{d+1}}{d+1} \]

for some positive \( c(d) \) (both sides of the above inequality are polynomials of the variable \( n \), each of degree \((d + 1)^2\)), we can use Fractional Helly’s Theorem and the assertion of the theorem follows.

Remark. The constant form the first proof has not been improved for a long time. One can get a better constant considering a lifting of \( X \) into \( \mathbb{R}^n \). Gromov has shown that getting back we can find a point covered many times. His proof uses topology and gives the constant \( e^{\frac{n}{d}} \). It is still an open question if this is best possible.
We will end this lecture with a proof that the leading term $\frac{n^3}{27}$ in the Boros-Füredi Theorem is optimal. Our example will be the \textit{stretched grid}.

The $n \times n$ stretched grid consists of $n^2$ points in the plane. They form $n$ parallel rows ($n$ points in each), in each row the distance between two neighbouring points is one, and the points are aligned so that they also form $n$ parallel columns. The difference between the stretched grid and a normal $n \times n$ grid is the spacing between rows. The second row lies above the first (bottom) row, in distance 1 from it (no difference so far). The third row lies above the second row, on such a height that the segment connecting the rightmost point from the first row and the leftmost point from the third row passes between the two rightmost points in the second row (so the third row has to be in distance strictly greater than $n - 1$ from the bottom row; it is convenient to think of it as lying in distance $n$ from the bottom row). This construction is continued: the $j$-th row lies in such a distance above the first row, that the segment connecting the rightmost point from the first row and the leftmost point from the $j$-th row passes between the two rightmost points in the $(j - 1)$-th row (see Figure 6 for an example of the $4 \times 4$ stretched grid).

![Figure 6: The $4 \times 4$ stretched grid (stretched on the left, compressed on the right).](image)

We will be interested in the following question: how many points from the stretched grid lie in the interior of a triangle with vertices from stretched grid? The answer to this question will not change if we identify the $n \times n$ stretched grid with a normal $n \times n$ grid and the segments connecting two points with two perpendicular segments: we first go vertically from the lower
of the two points to the height of the higher of these points and then right or left to the second point (see Figure 6; more examples of segments and triangles are depicted in Figure 7).

![Segments and triangles](image)

We can now proceed to the proof of the optimality of the Boros-Füredi Theorem.

**Proof of the optimality of the Boros-Füredi Theorem.** Recall that the theorem states that for every set \( X \subset \mathbb{R}^2 \), \( |X| = n \), there exists a point covered by \( \binom{n}{3}\left(\frac{2}{9} - o(1)\right) \) triangles with vertices in \( X \). We need to show that (for every \( n \in \mathbb{N} \)) there exists a set \( X \subset \mathbb{R}^2 \), \( |X| = n \), such that every point in the plane is covered by at most \( \binom{n}{3}\left(\frac{2}{9} + o(1)\right) \) triangles with vertices in \( X \).

![The points on the diagonal are divided into three groups.](image)

Let \( X \) be the set of points from diagonal of the \( n \times n \) stretched grid (going from the bottom left-hand side corner to the top right-hand side corner). Using the above explications we can identify \( X \) with the set \( \{(1,1), \ldots, (n,n)\} \) of \( \mathbb{R}^2 \). If a point \((x,y) \in \mathbb{R}^2\) is covered by some triangles with vertices in \( X \),
$X$, then we must have $1 \leq x \leq y \leq n$. A horizontal and a vertical line through the point $(x,y)$ divide the points in the set $X$ into three groups, of cardinality $n_1, n_2, n_3$ respectively (see Figure 8).

Notice that if a triangle with vertices from $X$ covers the point $(x,y)$ then each of its vertices comes from a different group (and the covering triangle looks like the bottom right-hand side triangle in Figure 7). Therefore the point $(x,y)$ is covered by at most

$$n_1n_2n_3 \leq \left(\frac{n_1 + n_2 + n_3}{3}\right)^3 \leq \frac{(n + 2)^3}{3^3} = \frac{n^3}{27} + o(n^3)$$

triangles with vertices from the set $X$. \hfill \Box

*Remark.* A similar example can be constructed also in higher dimensions.
Weak $\varepsilon$-nets.

Let $X \subset \mathbb{R}^d$ be a set of $n$ points in general position (i.e. no $d + 1$ points of $X$ lie on the same hyperplane.) For a positive $0 < \varepsilon < 1$, we define

$$\mathcal{F}_\varepsilon = \{ \text{conv } Y : Y \subset X, |Y| \geq \varepsilon n \}.$$ 

A finite set $S \subset \mathbb{R}^d$ is called a (weak) $\varepsilon$-net of $X$ if $S \cap F \neq \emptyset$ for any $F \in \mathcal{F}_\varepsilon$. We note that $S$ is called a strong $\varepsilon$-net if, in addition, it also satisfies $S \subset X$. Since we are only going to discuss weak $\varepsilon$-nets, we will simply call these $\varepsilon$-nets.

Our goal is to find an $\varepsilon$-net of $X$ with small cardinality.

**Theorem 1.** For any $X \subset \mathbb{R}^d$ and any $0 < \varepsilon < 1$, there exists an $\varepsilon$-net $S$ of $X$ with cardinality

$$|S| \leq c_d \frac{1}{\varepsilon^{d+1}},$$

where $c_d$ is a constant depending only on the dimension $d$.

Note that this is a striking result: the cardinality of $S$ does not depend on the set $X!$

**Proof.** We may assume that $\varepsilon > 2^{d+1}n$ since, otherwise,

$$2^{d+1}(d+1) \frac{1}{\varepsilon^{d+1}} \geq 2^{d+1}(d+1) \frac{n^{d+1}}{2^{d+1}(d+1)^{d+1}} \geq n.$$

Therefore, $X$ is a suitable $\varepsilon$-net.

We construct $S$ by an iterative algorithm. We start with $S_0 = \emptyset$, and $\mathcal{H}_0 = \binom{X}{d+1}$. At the $i$th step, for $i \geq 1$, we check if there exists $Y \subset X$ with $|Y| \geq \varepsilon n$ and $S_i \cap Y = \emptyset$. If there is no such $Y$, then the algorithm terminates - we have found an $\varepsilon$-net. Otherwise, take such a $Y = Y_i$. By
the results of the previous lecture, there exists a highly covered point $z_i$ of $Y_i$; a point $z$, which is covered by at least

$$\frac{1}{(d+1)^d} \left( \frac{|Y|}{d+1} \right)$$

many different simplices with vertices from $Y_i$. Let $S_i = S_{i-1} \cup \{z_i\}$, and set $\mathcal{H}_i = \mathcal{H}_{i-1} \setminus \{T \in \binom{X}{d+1} : z \in \text{conv}(T)\}$.

Since $X$ is finite, the algorithm terminates. Therefore, we have to estimate the number of steps until this happens – this is the same as the cardinality of the constructed $\varepsilon$-net.

Note that, for every $i$, the convex hull of every element of $\binom{X}{d+1} \setminus \mathcal{H}_i$ contains a point from $S_i$. On the other hand, $\text{conv} Y_i \cap S_i = \emptyset$, which implies that every $(d+1)$-element subset of $Y_i$ is contained in $\mathcal{H}_{i-1}$. Therefore, since $z_i$ is a highly covered point of $Y_i$, we obtain that

$$|\mathcal{H}_i \setminus \mathcal{H}_{i-1}| \geq \frac{1}{(d+1)^d} \left( \frac{|Y|}{d+1} \right) \geq \frac{1}{(d+1)^d} \left( \frac{\varepsilon n}{d+1} \right).$$

Note that the condition $\varepsilon > \frac{2(d+1)}{n}$ implies that

$$\frac{n - (d + 1)}{\varepsilon n - (d + 1)} < \frac{2}{\varepsilon}.$$ 

Therefore, the number of iterations until termination is at most

$$\frac{\binom{n}{d+1} \varepsilon n}{(d+1)^d (\varepsilon n + 1)} \leq (d + 1)^d 2^{d+1} \frac{1}{\varepsilon^{d+1}}.$$ 

We note that a stronger upper bound of $c_d(1/\varepsilon^d)(\log(1/\varepsilon))^{d-1}$ has been proven.

What about lower bounds for the cardinality of an $\varepsilon$-net? The simplest construction is the following: take $1/\varepsilon + 1$ parallel lines on the plane, and let $X$ have $n\varepsilon$ points in each of the disjoint regions between the consecutive lines. An $\varepsilon$-net must contain a point in each of these components, therefore, its cardinality is at least $1/\varepsilon$.

A more elaborate example comes from the stretched grid and gives the stronger bound $(1/\varepsilon)(\log(1/\varepsilon))^{d-1}$. One has to investigate the sets that have a “negative staircase” form (see below).

Still, there is a huge gap between the lower and upper bounds: apart from logarithmic factors, it is $1/\varepsilon$ versus $1/\varepsilon^d$. The general belief is that the truth may be around $1/\varepsilon$. 

2
Next we are going to show that, for planar sets, a quadratic upper bound holds. To this end, let \( X \subset \mathbb{R}^2 \) be a set of \( n \) points in general position and, for any \( k > 0 \), introduce

\[
f(X, k) = \min \{ |S| : S \cap \text{int conv } Y \neq \emptyset, \forall Y \subset X, |Y| = k \}.
\]

Note that the definition is slightly stronger than in the original version; we require that the interior of the convex hulls be covered. Furthermore, let

\[
f(n, k) = \max_{X \subset \mathbb{R}^2, |X| = n} f(X, k).
\]

**Theorem 2.** \( f(n, k) \leq 7 \left( \frac{n}{k} \right)^2 \).

For \( k = 3 \), a smaller covering set may be constructed:

**Lemma 1.** \( f(n, 3) \leq 2n - 5 \).

**Proof.** We may assume that no two points of \( X \) determine a vertical line. Let \( v \) be the vertical vector of length 1. The set \( S \) consists of the points of the form \( x \pm \delta v \), for every \( x \in X \), where \( \delta \) is a very small positive number. It is easy to see that every triangle with 3 vertices from \( X \) contains at least one point of \( S \). In fact, it suffices to take only \( S \cap \text{conv } X \); elementary arguments show that by taking this intersection, we drop at least 5 points out of the \( 2n \).

**Proof of Theorem 2.** We assume that \( n \geq k \geq 6 \); the remaining cases can be checked by hand, using the above Lemma. Choose a line \( L \) in general position (i.e. no segment determined by \( X \) is parallel to \( L \)), which halves \( L \). That is, there are no points of \( X \) on \( L \), and if \( m_1 \) and \( m_2 \) denote the number of points in the halfplanes \( L^+ \) and \( L^- \) determined by \( L \), then \( |m_1 - m_2| \leq 1 \).

We are going to construct \( S \) recursively, and we proceed by induction on \( n \). Let \( l < k/2 \) be a number whose value we are going to choose later in order to optimize the bound. Introduce

\[
\mathcal{Y} = \{ Y \subset X, |Y| = k, |Y \cap L^+| \geq l, |Y \cap L^-| \geq l \}.
\]
Our first, intermediate target is to construct a set $S_0 \subset L$, for which $S_0 \cap \text{int conv} Y \neq \emptyset$ for every $Y \in \mathcal{Y}$. The remaining sets, which are not covered by $S_0$, have many points on one side of $L$ and we are going to handle them with the recursive argument.

Take all the segments $xy$, where $x \in X \cap L^+$ and $y \in X \cap L^-$. Altogether, they determine $m_1m_2$ intersection points with $L$. (No two of these coincide because $X$ is in general position). Order the intersection points linearly on $L$: they are $p_1, p_2, \ldots, p_m$, where $m = m_1m_2$. Let $h = (l+1)(k-l-1)$. To any set $Y \in \mathcal{Y}$ there correspond at least $h$ intersection points (of $L$ with segments both of whose endpoints are in $Y$). Construct $S_0$ by taking one point in each of the open intervals $(p_1, p_2), (p_{h+1}, p_{h+2}), \ldots, (p_{m-1}, p_m)$, so that there are at most $h-1$ intersection points between any two consecutive points of $S_0$. Then, by convexity, $S_0$ has a point in $\text{int conv}Y$ for each $Y \in \mathcal{Y}$. Therefore, $S_0 = \left\lceil \frac{m_1m_2}{h} \right\rceil$. On the other hand, any $Y \subset X$ with $|Y| = k$ that is not contained in $Y$ has at least $k-l$ points in $L^+$ or $L^-$. By the recursion, there exists a $(k-l)$–covering set $S^+$ for $X \cap L^+$ with at most $f(m_1, k-l)$ points and a $(k-l)$–net $S^-$ for $X \cap L^-$ with at most $f(m_2, k-l)$. Then, $S = S_0 \cup S^+ \cup S^-$ is a suitable covering set for $X$ whose cardinality can bounded using the inductive hypothesis:

$$|S| \leq |S_0| + f(m_1, k-l) + f(m_2, k-l)$$

$$\leq \left\lceil \frac{m_1m_2}{h} \right\rceil + 7 \left( \frac{m_1}{k-l} \right)^2 + 7 \left( \frac{m_2}{k-l} \right)^2$$

$$\approx \frac{n^2}{4l(k-l)} + 14 \frac{n^2}{4(k-l)^2}$$

$$= \frac{n^2}{4} \left( \frac{13l+k}{l(k-l)^2} \right).$$

A simple numerical argument shows that this is minimized when $l \approx 0.1467k$, yielding the upper bound

$$f(n, k) < \frac{27.2161}{4} \left( \frac{n}{k} \right)^2 < 7 \left( \frac{n}{k} \right)^2.$$

\hfill \Box

**Halving Lines**

Let $X \subset \mathbb{R}^2$ be a set of $n$ points in general position in the plane, where $n$ is even. A pair of distinct points $a, b \in X$ determine a halving line (and $ab$ is called a halving segment) if, on both (open) sides of $\text{aff}\{a, b\}$, there are exactly $n/2 - 1$ points of $X$. By a simple continuity argument, there is
at least one halving line through every point of $X$. Let $f(X)$ denote the number of halving lines of $X$. Then, by the previous observation,

$$\frac{n}{2} \leq f(X) \leq \binom{n}{2}.$$ 

Furthermore, let $f(n) = \max_{|X|=n} f(X)$. In the 1970’s, Erdős asked for upper and lower bounds on $f(n)$.

**Theorem 3.** $f(n) > cn \log n$. 

**Proof.** We are going to construct $X$ iteratively. Let $P_1$ be the set shown in the figure below (three points distributed on a circle, plus the center of the circle). Then $P_1$ has 3 halving lines. 

Assume that $P_i$ has been constructed. Note that an affine transformation does not change the halving property. Let $L$ be a general direction with respect to $P_i$. Now compress $P_i$ in the direction $L$ so that it is contained in a very thin slab perpendicular to $L$, call the resulting set $P'_i$. Then, place three rotated copies of $P'_i$ in a “Mercedes frame” (see the figure below).

![Diagram](image)

Each of the original halving lines remains a halving line plus we obtain $|P_i|/2$ new halving lines. This leads to the lower bound $\frac{1}{2} n \log_3 n$. 

Next, consider the graph of the halving segments of $X$: take the points of $X$ as the vertices and we all of the halving segments as edges. By our previous observation, every vertex has degree $\geq 1$. We define the wedge $\overline{xyz}$ of three distinct points $x, y, z$ on the plane to be the smaller region between the two half-lines containing $x$ and $z$, respectively, with common endpoint $y$.

**Lemma 2** (Odd-Star Lemma, Lovász). Take the graph of the halving segments. For any point $x \in X$, and for any two halving segments $xa$ and $xb$ emanating from $X$, there exists another halving segment with endpoint $x$ in the wedge opposite to $axb$.
Proof. Assume that $x$ is the origin, and the wedge $\hat{ab}$ is in counterclockwise orientation. Take a rotating line $\ell$ with center $x$, which rotates from $a$ to $b$ along the wedge - at its starting position, it contains $x$ and $a$, and at its final position, it contains $x$ and $b$. Therefore, the ray $\overrightarrow{xa}$ scans through the wedge $axb$, and the opposite half-line scans the opposite wedge.

Consider the number of points of $X$ on the “lower” side – i.e. the (open) half-plane which, at the beginning, does not contain $b$. At the starting position, the number of points is $n/2 - 1$. As $\ell$ leaves $a$, it changes to $n/2$. When terminating (when $\ell$ contains $b$), it is $n/2 - 1$ again. How does this number change? Well, it only happens when the line meets a point; if the rotation of the half-line $\overrightarrow{xa}$ hits a point, then the number will increase; if the opposite ray meets a point, the number will decrease. Moreover, because of general position, the number always changes by at most 1. Therefore, there must be a position somewhere along the way where the number drops down to $n/2 - 1$, and this belongs to a halving line determined by $x$ and a further point $c$, which necessarily lies in the opposite wedge.

\textbf{Theorem 4.} $f(n) < cn^{\frac{3}{2}}$.

\textit{Proof.} We may assume that no two points of $X$ lie on a vertical line. Consider a vertical line which traverses through $X$ and, at each position, count how many halving segments it intersects. The resulting number is 0 before the line hits $X$ or after it leaves $X$. In between, the number only changes when hitting points of $X$. The odd-star lemma gives us that each change is exactly $\pm 1$. Since $|X| = n$, the line intersects at most $n/2$ halving segments at every position.

Let $L$ be a horizontal line and consider the orthogonal projection $P_L$ onto $L$. The projection of the halving segments of $X$ is a system $I$ of intervals

\begin{align*}
\text{Page 41}
\end{align*}
on $L$ whose endpoints are among the $n$ points of $P_L(X)$. Moreover, by the
previous argument, we deduce that the system of intervals covers each point
of $L$ at most $n/2$ times. How many intervals can $\mathcal{I}$ contain?

Order the points of $P_L(X)$ linearly on $L$: $p_1, p_2, \ldots, p_n$. Choose $k < n$
(that we will specify later) and take every $k$th point of $P_L(X)$ as such:
$p_1, p_{k+1}, p_{2k+1}, \ldots, p_n$. We are going to call these dividing points. These cut
the segment $p_1p_n$ into $\lfloor n/k \rfloor$ blocks. There are two types of intervals in $\mathcal{I}$
depending on if the two endpoints are in the same block (“short intervals”) or in different blocks (“long intervals”). The number of long intervals is at
most $\frac{n}{k} \cdot \frac{n}{2}$, since each dividing point is covered by at most $n/2$ intervals
(which are necessarily long). On the other hand, in each block, there are at
most $\binom{k}{2}$ short intervals (the total number of intervals with two endpoints
in a block). Therefore,

$$|\mathcal{I}| \leq \frac{n}{k} \cdot \frac{n}{2} + \binom{k}{2} \left( \frac{n}{k} + 1 \right).$$

Setting $k = \sqrt[3]{n}$ yields the upper bound $cn^{\frac{3}{2}}$.

We note that a stronger upper bound can also be given: $f(n) < cn^4$. The
probabilistic proof uses the crossing lemma which states that, in a
plane drawing of a graph with $n$ vertices and $m$ edges, there are at least
$\frac{1}{50} \frac{m^3}{n^2}$ crossings between the edges.

**Halving Planes**

We are going to consider the 3–dimensional analogue of halving lines.
Let $X \subset \mathbb{R}^3$ be a set of $n$ points in general position, where $n$ is odd. The
points $a, b, c \in X$ determine a halving plane defined by their affine hull if the
plane dissects the set $X$ into two equal parts, that is, if each (open) side
of $\text{aff}\{a, b, c\}$ contains exactly $\frac{n-3}{2}$ points of $X$. We denote $f(X)$ to be the
number of halving planes of the set $X$. The aim is to find bounds on the
maximum number of halving planes of a set of $n$ points of $\mathbb{R}^3$ in general
position. That is, to bound

$$f_3(n) = \max\{ f(X) : X \subset \mathbb{R}^3, |X| = n, X \text{ is in general position} \}.$$

It is clear that $\frac{1}{3} \binom{n}{2} \leq f(n) \leq \binom{n}{3}$. The upper bound is simply the
total number of possible planes determined by points of $X$. For the lower bound,
given any two points of $X$, if we rotate a plane through the line
they determine, we see that there exists a third point of $X$ that generates

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a halving plane of $X$ together with the first two points. Each such plane is counted three times; one for each side of the triangle.

A stronger lower bound has been given: $n^2 \log n < f_3(n)$. What can we say about an upper bound?

The planar result depended on the odd-star lemma. In fact, considering the 2–dimensional picture as a projection of the 3–dimensional situation, the same proof yields the following analogue:

**Lemma 3** (Odd-Star Lemma for $\mathbb{R}^3$). Let $a, b \in X$. If there exist points $c$ and $d$ such that $\text{aff}\{a, b, c\}$ and $\text{aff}\{a, b, d\}$ are halving planes with corresponding closed halfspaces $\mathcal{H}_c$ and $\mathcal{H}_d$, respectively, such that $\{a, b, c, d\} \subset \mathcal{H}_c \cap \mathcal{H}_d$, then there exists a point $x \notin \mathcal{H}_c \cup \mathcal{H}_d$ such that $\text{aff}\{a, b, x\}$ is also a halving plane of $X$.

The trivial upper bound for the number of halving planes is $\binom{n}{3}$, which is of order $n^3$. We are able to provide a polynomial strengthening (although, with small exponent) to this bound.

**Theorem 5.** $f_3(n) < n^{3-\varepsilon}$ for some small $\varepsilon > 0$.

_Proof._ Consider a plane $H$ in general position with respect to $X$ and the orthogonal projection $P_H$ of $X$ onto $H$ (shown on the following figure). Let $\mathcal{H}$ denote the system of the projections of the halving triangles of $X$.

![Diagram](image)

First, we state that every point of $H$ is covered by at most $\binom{n}{2}$ members of $\mathcal{H}$. Indeed, this covering function (the number of triangles covering a specific point) is constant on all the components of the plane determined by the segments between points of $P_H(X)$ (this is the same as moving a vertical line $L$ throughout $H$, and counting how many halving triangles it hits). Moreover, the odd-star lemma and general position guarantee that the value differs by exactly 1 on two sides of a segment. Since there are altogether $\binom{n}{2}$ segments, we obtain the bound.
Therefore, we conclude that \( P_H(\mathcal{H}) \subset \binom{P_H(X)}{3} \) is a system of triangles which covers each point of \( H \) at most \( O(n^2) \) times. How large can \( |\mathcal{H}| = |P_H(\mathcal{H})| \) be? This is the content of the following result. Combining this result with the \( O(n^2) \) estimate for the covering number concludes the proof of Theorem 5.

**Theorem 6.** Given a set \( X \subset \mathbb{R}^2 \) of \( n \) points in general position, let \( \mathcal{H} \subset \binom{X}{3} \) such that \( |\mathcal{H}| = p \binom{n}{3} \) where \( p = n^{-\alpha} \) for some \( \alpha > 0 \). Then there exists a point common to at least \( cp^s \binom{n}{3} \) triangles of \( \mathcal{H} \) where \( s \) and \( c \) are (large) universal constants.

**Proof.** Define \( Q(X) := \{\text{aff}(u, v) \cap \text{aff}(x, y) : u, v, x, y \in X \text{ distinct}\} \) to be the crossings in \( X \). Then \( Q(X) \) contains \( \frac{1}{2} \binom{n}{2} \binom{n-2}{2} \approx n^4/8 \) crossings.

The main idea of the proof is the following. We are going to show that there exists a crossing \( q \), which is contained in many distinct triangles of \( \mathcal{H} \). To this end, we will consider the \((q, \Delta)\) pairs, where \( q \) is a crossing, and \( \Delta \) is a triangle of \( \mathcal{H} \), so that \( q \in \text{int} \mathcal{H} \).

For any points \( a, b, c \in X \), denote the number of crossings in \( \text{conv}\{a, b, c\} \) by \( N(a, b, c) \). That is, \( N(a, b, c) = |Q(X) \cap \text{conv}\{a, b, c\}| \).

As an illustration of the technique that we are going to use, we now show that the average of \( N(a, b, c) \) is of order \( n^4 \). Because of Tverberg’s theorem, any 9 points of \( X \) can be partitioned into 3 intersecting triangles. This shows that for of any 9 points, there exists a triangle \( \Delta \) determined by them, and two further segments determined by them, whose crossing point \( q \) lies in the triangle. This configuration \((q, \Delta)\) is determined by 7 points (out of the 9). We assign to each 9-tuple such a \((q, \Delta)\)-configuration.

We want to count the number of distinct \((q, \Delta)\)-configurations. In the above assignment, each such \((q, \Delta)\)-configuration appears at most \( \frac{n}{2}(n-2) \) times (since we can add two arbitrary points in order to obtain a 9-tuple). Therefore, the number of distinct \((q, \Delta)\)-configurations is at least

\[
\binom{n}{9} / \binom{n-7}{2} \sim n^7.
\]

Since there are \( \binom{n}{3} \sim n^3 \) triangles, on average, they must contain at least \( \sim n^7/n^3 \) crossings, which is the desired bound.

Next, we transform this argument so that only the triangles of \( \mathcal{H} \) are counted.

We will use the following result of Erdős and Simonovits. For preparation, we need to introduce some terminology. A \((d+1)\)-uniform hypergraph

on a base set \( V \) is a system of \((d+1)\)-tuples of \( V \). Elements of \( V \) are called
the vertices of the hypergraph, while the selected \((d+1)\)-tuples are the edges. The complete \((d+1)\)-partite hypergraph, denoted by \(K(t, \ldots, t)\), is obtained as follows: let \(V_1, \ldots, V_{d+1}\) be sets of cardinality \(t\), and let \(K(t, \ldots, t)\) contain all the transversals (i.e. all the sets containing one point from each \(V_i\)). \(K(t, \ldots, t)\) is illustrated in the following figure.

Here comes the result of Erdős and Simonovits.

**Theorem 7.** For every \(d\) and every \(t\), there exists a \(b(d,t) > 0\) with the following property. Any \((d+1)\)-uniform hypergraph \(\mathcal{H}\) defined on \(n\) vertices with \(p\binom{n}{d+1}\) edges, where \(n^{-t^2} \leq p < 1\), contains at least

\[
bp^{d+1} \binom{n}{t, \ldots, t}
\]

copies of \(K(t, \ldots, t)\).

We provide no proof here, only an intuitive explanation why this bound holds. Look at a random \((d+1)\)-uniform hypergraph \(\mathcal{H}\) on \(n\) vertices with \((d+1)\)-tuples chosen randomly with probability \(x\). Then the expected value of \(|\mathcal{H}|\) is \(E|\mathcal{H}| = x\binom{n}{d+1}\). Furthermore, the expected value of the number of copies of \(K(t, \ldots, t)\) is \(x^{d+1} \binom{n}{t, \ldots, t}\). The proof is based on this observation, and it uses an averaging argument.

Let us return to the proof of Theorem 6. We consider \(\mathcal{H}\) as a 3-uniform hypergraph with vertex set \(X\). Applying Theorem 7, we obtain that there are at least \(cp^{d+1} \binom{n}{t,t,t}\) copies of \(K(t, t, t)\) in \(\mathcal{H}\). For each such copy, we consider the three vertex classes of cardinality \(t\) as color classes. Next, we show that there exists a \(t\) so, for every geometric realization of \(K_{t,t,t}\), there exist three distinct multi-coloured triangles which intersect (illustrated in the following figure.)
This is exactly the content of the colorful version of Tverberg’s theorem, which is cited from the next lecture:

**Theorem 8 (Colorful Tverberg Theorem).** Given $d$ and $k$, there exists a $t(d,k)$ large enough such that for any sets $C_1, C_2, \ldots, C_{d+1} \subset \mathbb{R}^d$, where $|C_i| = t$, there are disjoint sets $S_1, S_2, \ldots, S_k \subset \bigcup_{i=1}^{d+1} C_i$ such that

1. $\bigcap_{i=1}^{k} \text{conv } S_i \neq \emptyset$
2. $|S_i \cap C_j| = 1$.

In fact, in the above setting $(d = 2, k = 3)$, one can show that the statement holds with $t = 4$.

Therefore, as before, we can assign a $(q, \Delta)$-pair for each copy of $K(t, t, t)$ in $\mathcal{H}$ where $q$ is a crossing. Thus, $\Delta$ is a triangle of $\mathcal{H}$ with $q \in \text{int } \delta$. How many distinct such crossing pairs are there? Well, the number of distinct copies of $K(t, t, t)$ in $\mathcal{H}$ is at least $cp^3 \binom{n}{t,t,t}$. On the other hand, each crossing pair may be assigned to at most $(n - \frac{7}{3t - 7})$ copies of $K(t, t, t)$. Therefore, the number of distinct crossing pairs is at least the quotient of these two quantities:

$$\frac{cp^3 \binom{n}{t,t,t}}{(n - \frac{7}{3t - 7})} = c' p^3 n^{\frac{7}{3t - 7}}.$$

Yet, there are only $O(n^4)$ crossing points. Therefore, there must be a crossing point which is contained by at least $cp^3 n^3$ triangles of $\mathcal{H}$. \hfill \Box

Analyzing the above argument one obtains that Theorem 6 holds with $s = 64$, and, in turn, Theorem 5 is valid with $\varepsilon = 1/64$. There are stronger bounds; the current best estimate is around $n^{2.5}$. 

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Theorem 1 (Colored Tverberg). For every $d, k \geq 1$, there exists $t = t(d, k)$ such that if $C_1, \ldots, C_{d+1} \subset \mathbb{R}^d$ with \((\forall i)\left| C_i \right| = t\) then there exist $S_1, \ldots, S_k \subset \bigcup_{i=1}^{d+1} C_i$ which are pairwise disjoint, \((\forall j)(\forall i)\left| S_j \cap C_i \right| = 1\) and $\bigcap_{j=1}^k \text{conv} S_j \neq \emptyset$.

Proof. Let $C_i = \{a_i, b_i\}$ for every $i = 1, \ldots, d+1$. We consider the mapping $f(e_i) = a_i$ and $f(-e_i) = b_i$ for $i = 1, \ldots, d+1$. Let $X$ be the cross polytope defined by $-e_i, e_i$ in $\mathbb{R}^{d+1}$. Furthermore, we can extend $f$ s.t. it maps points from $\partial X$ to $\mathbb{R}^d$ as follows:

$$f(x) = \sum_{i=1}^{d+1} \alpha_i f(v_i),$$

where $x = \sum_{i=1}^{d+1} \alpha_i v_i \in \partial X$ and for $i = 1, \ldots, d+1$ $\alpha_i \geq 0$ and $\sum_{i=1}^{d+1} \alpha_i = 1$. For every $i = 1, \ldots, d+1$, $v_i \in \{-e_i, e_i\}$.

By the Borsuk-Ulam theorem, there exists $x \in \partial X$ s.t. $f(x) = f(-x)$. Hence, there exist $\alpha_1, \ldots, \alpha_{d+1}, v_1, \ldots, v_{d+1}$ s.t. $\sum_{i=1}^{d+1} \alpha_i f(v_i) = \sum_{i=1}^{d+1} \alpha_i f(-v_i)$. By definition the two simplices $\text{conv}(f(v_1), \ldots, f(v_{d+1}))$ and $\text{conv}(f(-v_1), \ldots, f(-v_{d+1}))$ cover each color exactly once and their intersection contains at least $x$.

We also present an alternative proof of Theorem 1.

Alternative Proof. We consider the vectors $a_1 - b_1, \ldots, a_{d+1} - b_{d+1}$. Since the number of such vectors is $d+1$ and they are in $\mathbb{R}^d$, they are linearly dependent which implies that there exist $\alpha_1, \ldots, \alpha_{d+1}$ s.t. $\sum_{i=1}^{d+1} \alpha_i(a_i - b_i) = 0$. Now we define $f$ s.t. for $i = 1, \ldots, d+1$, if $\alpha_i \geq 0$, $f(i) = a_i$ and if $\alpha_i < 0$ then $f(i) = b_i$ and $g$ s.t. for $i = 1, \ldots, d+1$ if $f(i) = a_i$ then $g(i) = b_i$ and if $f(i) = b_i$ then $g(i) = a_i$. The two simplices $\text{conv}(f(1), \ldots, f(d+1))$ and $\text{conv}(g(1), \ldots, g(d+1))$ cover each color exactly once and their intersection contains the point $x = \sum_{i=1}^{d+1} \alpha_i f(i) = \sum_{i=1}^{d+1} \alpha_i g(i)$.

Theorem 3. Let $\mathcal{H} \subset \left( \binom{X}{d+1} \right)$ where points in $X \subset \mathbb{R}^d$ are in general position, $|X| = n$ and $|\mathcal{H}| = p\binom{n}{d+1}$. Then, there exists point which is common to $\beta p^{d+1}\binom{n}{d+1}$ simplices in $\mathcal{H}$.

We prove the above theorem for the case $d = 2$. The proof for the general case can be found in [Mat02, Sec. 9].

Proof. In Fig. 1 we can see three triangles in $\mathbb{R}^2$ with non empty intersection. The vertices are colored and each triangle contains each color exactly once.

We can view $\mathcal{H}$ as a hypergraph $H$ where the vertices correspond to points in $X$ and the edges correspond to simplices in $\mathcal{H}$. Let $K(t, t, t)$ be the complete $3$-partite graph where each of the three.
classes consists of \( t \) vertices. Intuitively, if \( H \) contains many edges then many copies of \( K(t, t, t) \) also appear. Each copy of \( K(t, t, t) \) contributes with a pair \((p, \Delta) \in (Q(X), H)\) s.t. \( q \in \text{int}\Delta \) where \( Q(X) \) denotes the crossings in \( X \).

\( H \) contains \( cp^3 \binom{n}{t, t, t} \) copies of \( K(t, t, t) \) for some constant \( c \). The number of copies that we need to take into consideration, that is the number of pairs \((p, \Delta)\) equals \( \frac{cp^3 n^7}{n^d + 1} \) \( \sim p^3 n^7 \).

Hence,
\[
cp^3 n^7 \leq \#\{(p, \Delta), \ldots\} = \sum_{p \in Q(x)} \#\{\Delta \in H : p \in \Delta\} \Rightarrow \Rightarrow \#\{\Delta : q^* \in \Delta\} \geq cp^3 n^3.
\]

In the number of halving triangles, if you project them down no point is covered by \( n^2 \) triangles.

\[
\beta p^{d+1} \left( \frac{n}{d+1} \right) \Rightarrow p^3 \left( \frac{n}{3} \right) < n^2 \Rightarrow \# \text{ halving triangles} < n^3 - \frac{1}{n^2}
\]

However this is not the best outcome for \( d = 3 \). A better bound of \( n^{2.5} \) has been proved.

**Colorful Tverberg ⇔ Point Selection Theorem:** We have seen that the colorful Tverberg implies the Point Selection theorem. We will now also prove the opposite direction.

**Proof.** Let \( \mathcal{H} \) denote the hypergraph depicted in figure 2 and \(|C_i| = t, 1 \leq i \leq d + 1\). This means that each edge of \( \mathcal{H} \) is a multicolor simplex. For the number of edges in \( \mathcal{H} \) it holds that:
\[
|\mathcal{H}| = t^{d+1} \gg \left( \frac{n}{d+1} \right), \text{ where } n = (d+1)t
\]

If we apply the point selection theorem we can choose a \( \mathcal{H}' \subset \mathcal{H} \), such that \(|\mathcal{H}'| \gg \left( \frac{n}{d+1} \right)\). Consider the largest \( k \) such that there are \( S_1, S_2, \ldots, S_k \) disjoint multicolored simplices in \( \mathcal{H} \). Then any other \( S \subset \mathcal{H}' \) intersects \( \bigcup_i S_i \). The number of such \( S \) is at most \( k(d+1)t^d + k \geq |\mathcal{H}'| \gg t^{d+1} \), which implies that \( k \gg t \).
Figure 2: Transversals between the color classes $C_1, C_2, \ldots, C_{d+1}$ in the hypergraph $H$.

$(p, q)$-property

**Definition 4.** A family of sets $C$ has the $(p, q)$-property if among any $p$ sets in $C$, there are $q$ intersecting ones.

Of course this property is only interesting if $p \geq q \geq d + 1$.

Let $C$ be a finite family of $n$ convex sets in $\mathbb{R}^d$ with the $(p, q)$-property. We want to conclude that under these conditions there exists a small set $S$ of points that covers all sets in the family. So, our goal is to find a set $S \subset \mathbb{R}^d$ of small size such that $S \cap C \neq \emptyset$, $\forall C \in C$ and $|S| \leq f(p, q, d)$. Without loss of generality we will consider that $q = d + 1$.

**Lemma 5.** If a family of sets $C$ has the $(p, q)$-property, then there exists a point common to a linear size subset of $C$.

**Proof.** The number of tuples in the set is:

$$\binom{n}{p} \left( \begin{array}{c} n \\ p-(d+1) \end{array} \right) = \binom{n}{d+1} \left( \frac{1}{d^p + 1} \right)$$

Let $\alpha = \frac{1}{d^p + 1}$. Then we fulfill the requirements of the fractional Helly theorem and so there is a point common to $\alpha d + 1$ sets.

**Lemma 6.** Consider a finite family of $n$ convex sets $C = C_1, C_2, \ldots, C_n$ with the $(p, q)$-property. Then the blown up copy $C^*$, where $C^* = k_1$ copies of $C_1, k_2$ copies of $C_2, \ldots, k_n$ copies of $C_n, k_i \in \mathbb{N}$, has the $(p', d+1)$-property, where $p' = d(p-1) + 1$.

**Proof.** Pick $B_1, B_2, \ldots, B_{p'} \in C^*$ and count the number of intersecting sets.

If some set is repeated $d + 1$ times we are done.

If no set is repeated $d + 1$ times, then every set is repeated at most $d$ times and so we have that $(p-1)d + 1$ of them will contain $d + 1$ intersecting.

Lemma 5 implies that $O(\log n)$ points can capture any set with the $(p, d+1)$-property.

**Theorem 7** (Alon-Kleitman). There is a set $S$ intersecting every $C \in C$ of size $|S| \leq f(p, d)$.

**Proof.** We are looking for a finite set $X \subset \mathbb{R}^d$ such that $|X \cap C| \geq \epsilon |X|, \forall C \in C$, with $\epsilon = f(p, d)$. If $S$ is a weak $\epsilon$-net for $X$, then we are done, because if $|X \cap C| \geq \epsilon$, then $S \cap \text{conv}(X \cap C) \neq \emptyset$.

**Target:** Find such set $X$ (and $\epsilon$)
Choose $I \subset [n]$ s.t. $C(I) := \bigcap_{i \in I} C_i \neq \emptyset$ and pick $z \in C(I)$. Next, let $m(I)$ be a number in $\{0, 1, 2, \ldots \}$ and define $X = \{m(I) \text{ copies of } z(I) \mid \forall I \subset [n] \text{ s.t } C(I) \neq \emptyset \}$. Obviously, $|X| = \sum_{I \ni \gamma} m(I)$.

**Target:** choose $\gamma > 0$ and $m(I)$ such that: $\sum_{I \ni \gamma} m(I) \geq \gamma |X|, \forall i \Rightarrow |C_i \cap X| \geq \gamma |X|$. 

**Target:** Find $x(I)$ such that: 

$$\frac{1}{\gamma} = \min \sum_{I \ni \gamma} x(I), \text{ s.t. } \sum_{I \ni \gamma} x(I) \geq 1, \forall i \& x(I) \geq 0$$

Equivalently, its dual l.p. is the following: 

$$\frac{1}{\gamma} = \max \sum_{I \ni \gamma} y_i, \text{ s.t. } \sum_{i \ni \gamma} y_i \leq 1, \forall I \& y_i \geq 0$$

**Target:** Find $y_i \geq 0$ such that: 

$$\sum_{I \ni \gamma} y_i \geq \frac{1}{\gamma} \Rightarrow \sum_{I \ni \gamma} y_i > 1 \text{ for some } I$$

We replace $y_i = \frac{k_i}{D}$, in (1) where $D$ is the common denominator: 

$$\text{Target: } \sum_{I \ni \gamma} k_i > \frac{D}{\gamma} \Rightarrow \sum_{I \ni \gamma} k_i > D$$

and now we set $\frac{D}{\gamma} = M$: 

$$\sum_{I \ni \gamma} k_i > M \Rightarrow \sum_{I \ni \gamma} k_i > \gamma M$$

Create $C^*$ by taking each set $C_i$, $k_i$ times. From the last inequality, $C^*$ has an intersecting subfamily of size $\gamma M$. Lemma 6 completes the proof. 

Alternative algorithmic proof:

**Proof.** Start with: 

$S = \emptyset$. $C_0 = C$.

Pick $z_1$ common to $\beta |C_0|$ sets from $C_0$. 

$S_1 = S_0 \cup \{z_1\}$. 

$C_1 = C_0 + \{C_0 \setminus (\beta (C_0))\}$ --repeat sets not hit by $z_1$

In the $(i+1)$-th step: 

Pick $z_{i+1}$ common to $\beta |C_i|$ sets from $C_i$. 

$S_{i+1} = S_i \cup \{z_{i+1}\}$ 

$C_{i+1} = C_i + \{C_i \setminus (\beta (C_i))\}$ --repeat sets not hit by $z_{i+1}$

In the $t$-th iteration we will have: $|S_t| = t, |C_t| = (2 - \beta)^t \rho$, because $|C_{i+1}| \leq (2 - \beta)|C_i|$. 

Claim: After $t$ steps, with $t$ large enough, $|S_t \cap C| \geq \gamma t, \forall C \in C$. 

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Proof of claim: Assume $C \in \mathcal{C}$ is hit $k$ times. Then it was doubled $t - k$ times. That means that $2^{t-k}$ copies of $C$ are present.

$$2^{t-k} \leq |\mathcal{C}| \leq (2 - \beta)^k n \Rightarrow$$

$$t - k \leq t \log(2 - \beta) + \log n \Rightarrow$$

$$k \geq t(1 - \log(2 - \beta)) - \log n$$

We solve for

$$t(1 - \log(2 - \beta)) - \log n = \log n \Rightarrow t = \frac{2 \log n}{1 - \log(2 - \beta)}$$

and so:

$$k \geq \log n \Rightarrow \frac{k}{|S \cap \mathcal{C}|} \geq \frac{(1 - \log(2 - \beta))}{2} \gamma t$$

\□

**Order types and the Same-Type lemma.** For completeness we reproduce the following definition which can also be found in [Mat02, Sec. 9].

**Definition 8.** Let $p = (p_1, ..., p_n)$ a sequence of points in $\mathbb{R}^d$. The order type of $p$ is defined as the mapping assigning to each $(d + 1)$-tuple $(i_1, i_2, ..., i_{d+1})$ of indices, $1 < i_1 < i_2 < ... < i_{d+1} < n$, the orientation of the $(d + 1)$-tuple $(p_{i_1}, p_{i_2}, ..., p_{i_{d+1}})$. Thus, the order type of $p$ can be described by a sequence of $+1$’s, $-1$’s, and $0$’s with $\binom{n}{d+1}$ terms.

**Theorem 9** (Same-Type lemma). For any integers $d$, $m$ there exists $c = c(d, m) > 0$ s.t. the following holds. Let $X_1, ..., X_m \subset \mathbb{R}^d$ finite sets in general position. There exist $Y_1 \subset X_1, ..., Y_m \subset X_m$ s.t. for $i = 1, ..., m$ $|Y_i| \geq c|X_i|$ and for $a_i, b_i \in Y_i$ the order type of $a_1, ..., a_m$ and $b_1, ..., b_m$ is the same.

**Proof.** One can observe that it is sufficient to prove the theorem for $m = d + 1$. Now list all non trivial partitions ($\emptyset$ excluded) of $[d+1]$:

$$(I_1, J_1), (I_2, J_2), ..., (I_{2^d-1}, J_{2^d-1}).$$

For every $i$ we define a chain of sets $X_i = X_1^0 \supset X_1^1 \supset ... \supset X_1^{2^d-1} = Y_i$. For any $\alpha = 0, ..., 2^d - 1$ it holds $|X_i^\alpha| \geq \frac{1}{2}|X_i^{\alpha-1}|$.

For some $\alpha$ take a hyperplane $H_\alpha$ which halves $X_1^0, ..., X_1^\alpha$ (ham-sandwich theorem) and define halfspace $h$ by choosing the larger from $H_\alpha^+ \cap X_1^{\alpha+1}, H_\alpha^- \cap X_1^{\alpha+1}$. Wlog, we assume that $d+1 \in J_{\alpha+1}$. For all $i \in J_{\alpha+1}$ we discard all points of $X_i$ not lying in $h$ and for all $i \in I_{\alpha+1}$ we discard all points of $X_i$ lying in $h$. This results in subsets of $X_i$’s with at most half their elements. We follow the same procedure $2^d - 1$ times and we obtain $Y_1 = X_1^{2^d-1}, ..., Y_m = X_m^{2^d-1}$. Notice that $|Y_i| \geq 2^{2^d-1}|X_i|$.

We need to show that $a_1, ..., a_{d+1}$ and $b_1, ..., b_{d+1}$ have the same order type. If they do not have the same order type then $z_i = t x_i + (1 - t)x_i$ for all $i \in [d+1]$ all lie in a hyperplane $H$ for some $t$. Now, $z_0, z_1, ..., z_{d+1}$ have a Radon partition $\{z_i | i \in I_\alpha\}, \{z_i | i \in J_\alpha\}$ ($d+1 \in J_\alpha$). Obviously, $z_i \in \text{conv}(X_i^\alpha)$ and,

$$\text{conv}(\{z_i | i \in I_\alpha\}) \subset \text{conv}(\bigcup_{i \in I_\alpha} X_i^\alpha),$$
conv(\{z_j \mid j \in J_\alpha\}) \subset \text{conv}(\bigcup_{j \in J_\alpha} X^\alpha_j).

However, \bigcup_{i \in I_\alpha} X^\alpha_i and \bigcup_{j \in J_\alpha} X^\alpha_j are separated by a hyperplane \(H_\alpha\).

We also present two related results without proving them.

**Theorem 10** (Erdős-Szekeres theorem). Let \(X \subset \mathbb{R}^2\) a finite set of points in general position. If \(|X|\) is large enough then it contains the vertices of a convex \(n\)-gon.

**Theorem 11.** Let \(X_1, X_2, \ldots, X_{d+1} \subset \mathbb{R}^d\) sets of points in general position and \((\forall i)|X_i| = N\). Then there exists \(Y_i \subset X_i\) such that \(|Y_i| \geq c(d)N\) and

\[
\bigcap_{\text{all transversals of } Y_i's} \text{conv}(y_1, \ldots, y_{d+1}) \neq \emptyset.
\]

**References**

