I. Bárány, Lecture 2

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We continue with two more applications of Helly's Theorem.

1. For any convex compact $C \subset \mathbb{R}^d$ there exists $z \in C$ so that for every chord $[u, v]$ of C with $z \in [u, v]$ we have
$$\frac{1}{d+1} \leq \frac{\|u-z\|}{\|u-v\|} \leq \frac{d}{d+1}\,.$$
In other words, the ratio $\frac{\|u-z\|}{\|u-v\|}$ is separated from 0 and 1.

Proof. For every $x \in C$, we define $C_x := x + \frac{d}{d+1}(C - x)$. Note that $C_x \subset C$.

<u>Claim:</u> For any $x_1, \ldots, x_{d+1} \in C$, we have $\bigcap_{i=1}^{d+1} C_{x_i} \neq \emptyset$.

Set $y := \frac{1}{d+1} \sum_{i=1}^{d+1} x_i$. Let $j \in \{1, \ldots, d+1\}$ and rewrite y as
$$y = x_j + \frac{d}{d+1}\left(\frac{1}{d}\sum_{i=1, i\neq j}^{d+1} x_i - x_j\right).$$
Note that $\frac{1}{d}\sum_{i=1, i\neq j}^{d+1} x_i$ is a convex combination of points in the convex set C and hence belongs to C. We obtain that for any $j \in \{1, \ldots, d+1\}$, $y \in x_j + \frac{d}{d+1}(C - x_j) = C_{x_j}$. This yields the claim.
Helly's theorem now implies that $\bigcap_{x\in C} C_x \neq \emptyset$. Let $z \in \bigcap_{x \in C} C_x \subset C$.

<u>Claim:</u> The point z has the required property.

Let $u, v \in C$ so that $z \in [u, v]$. By definition $z \in C_u$. Hence
$$z \in u + \frac{d}{d+1}([u, v] - u)\,,$$
i.e. $z = u + \frac{d}{d+1}(\alpha u + (1-\alpha)v - u) = u + \frac{d}{d+1}(1-\alpha)(v-u)$ for some $\alpha \in [0, 1]$. Rearranging this equality, we obtain
$$\|z - u\| = \frac{d}{d+1}(1-\alpha)\|v-u\| \leq \frac{d}{d+1}\|v-u\|\,.$$

This immediately gives the upper bound and after a simple computation the lower bound as well. Indeed, since $z \in C_v$ as well, we also have $\|z - v\| \le \frac{d}{d+1}\|u - v\|$. The lower bound now follows from

$$\|u - v\| = \|u - z\| + \|z - v\| \le \|u - z\| + \frac{d}{d+1}\|u - v\| .$$

□

2. Let $a, b \in X \subset \mathbb{R}^d$. We say *a sees b* if $[a, b] \subset X$. We say X is *visible from a* if every $c \in X$ is seen from a, i.e. X is star-shaped.

   Krasnosel'skii Theorem: Let $X \subset \mathbb{R}^d$ compact. If any $d + 1$ points in X are seen from some $x \in X$, then X is star-shaped.

   Proof. For $x \in X$, define $V_x :=$ {set of points visible from x} $= \{y : [x, y] \subset X\}$. Observe that for any $x_1, \dots, x_{d+1} \in X$, $\bigcap_{i=1}^{d+1} \mathrm{conv} V_{x_i} \supset \bigcap_{i=1}^{d+1} V_{x_i} \neq \emptyset$, i.e. the collection of sets $\{\mathrm{conv} V_x\}_{x \in X}$ enjoys the Helly property. Hence there is $z \in \bigcap_{x \in X} \mathrm{conv} V_x$.

   Exercise 1. Show that $z \in \bigcap_{x \in X} V_x$, i.e. z is visible from all $x \in X$.

   □

Sets $C, D \subset \mathbb{R}^d$ are *separated* if there is a hyperplane $H = \{x \,|\, ax = b\}$ with $C \subset H^+$ and $D \subset H^-$, where $H^+ = \{x \,|\, ax \ge b\}$ and $H^- = \{x \,|\, ax \le b\}$. The separation is *strict* if $C \subset \mathrm{int} H^+$ and $D \subset \mathrm{int} H^-$. The sets are separated by a *slab (or skip)* if there are parallel hyperplanes H_1, H_2 with $H_1 \neq H_2, C \subset H_1^+, D \subset H_2^-$ and $H_1^+ \cap H_2^- = \emptyset$.

Theorem 1 (Separation Theorem)**.** *Let $C, D \subset \mathbb{R}^d$ convex, C compact, D closed. Then $C \cap D = \emptyset$ if and only if C, D are separated by a slab.*

Proof. Exercise. □

Lemma 1. *Let $K_1, \dots, K_n$ be closed convex sets and let K_1 be compact. Then $\bigcap_1^n K_i = \emptyset$ if and only if there are closed halfspaces H_i with $K_i \subset H_i$ and $\cap_1^n H_i = \emptyset$.*

Proof. " $\Leftarrow$ ": trivial.
" $\Rightarrow$ ": The case $n = 2$ is clear (Separation theorem). Suppose $n > 2$. By assumption

$$(K_1 \cap K_2 \cap \cdots \cap K_{n-1}) \cap K_n = \emptyset\,.$$

Thus there are closed halfspaces H'_n and H_n with $H'_n \cap H_n = \emptyset$ and $K_1 \cap K_2 \cap \cdots \cap K_{n-1} \subset H'_n, K_n \subset H_n$. Moreover, we have

$$K_1 \cap K_2 \cap \cdots \cap K_{n-1} \cap H_n = \emptyset\,.$$

Separate K_{n-1} from $K_1 \cap K_2 \cap \cdots \cap K_{n-2} \cap H_n$ by H_{n-1} and so on. At the end we obtain $H_1 \cap H_2 \cap \cdots \cap H_n = \emptyset$ with $K_i \subset H_i$ for $i \in [n]$. $\square$

Lemma 2. *Let $H_1, \ldots, H_m$ be closed halfspaces in $\mathbb{R}^d$, $H_i = \{x \,|\, a_i x \leq \alpha_i\}$. Then $\bigcap_1^m H_i = \emptyset$ if and only if $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \operatorname{pos}\left\{ \begin{pmatrix} a_1 \\ \alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_m \\ \alpha_m \end{pmatrix} \right\}$.*

Proof. $\cap_1^m H_i = \emptyset \Leftrightarrow$ the system $a_i x \leq \alpha_i$, $i = 1, \ldots, m$, has no solution.
" $\Leftarrow$ ": $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \operatorname{pos}\{\ldots\}$ means there are $\gamma_i \geq 0$, not all zero, so that

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \sum_1^m \gamma_i \begin{pmatrix} a_i \\ \alpha_i \end{pmatrix}\,,$$

i.e. $0 = \sum_1^m \gamma_i a_i$ and $-1 = \sum_1^m \gamma_i \alpha_i$. Suppose x is a solution of the system $a_i x \leq \alpha_i$, then multiplying by γ_i and summing gives $0 = \left(\sum_1^m \gamma_i a_i\right) x \leq \sum_1^m \gamma_i \alpha_i = -1$. A contradiction.
" $\Rightarrow$ ": Suppose $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \notin \operatorname{pos}\{\ldots\} =: D$. Observe that D is closed and hence $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and D can be separated by a slab. It follows that there is $\begin{pmatrix} b \\ \beta \end{pmatrix} \in \mathbb{R}^{d+1}$ so that $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix} > 0$ and $\begin{pmatrix} a_i \\ \alpha_i \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix} \leq 0$ for $i \in [n]$. This gives $\beta < 0$ and $a_i b + \alpha_i \beta \leq 0$. Thus $\frac{b}{|\beta|}$ is a solution of the system $a_i x \leq \alpha_i$. $\square$

Remark. More is true. Let $K_1, \ldots, K_n$ be closed convex sets in $\mathbb{R}^d$ and let K_1 be compact, $n \geq d+1$. Then $\bigcap_1^n K_i = \emptyset$ if and only if there is $I \subset [n], |I| \leq d+1$ so that $\cap_{i \in I} K_i = \emptyset$.

Another proof of Helly's Theorem First, let $K_1, \dots, K_n \subset \mathbb{R}^d$ be convex sets, all closed and one compact. We have

$$\begin{aligned} \cap_1^n K_i = \emptyset &\Leftrightarrow \exists H_i \supset K_i \text{ closed halfspaces with } \cap_{i=1}^n H_i = \emptyset \\ &\Leftrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \operatorname{pos}\left\{ \begin{pmatrix} a_1 \\ \alpha_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ \alpha_n \end{pmatrix} \right\} \end{aligned}$$

by the cone version of Carathéodory's Theorem in $\mathbb{R}^{d+1}$ we have

$$\begin{aligned} &\Leftrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \operatorname{pos}\left\{ \begin{pmatrix} a_{i_1} \\ \alpha_{i_1} \end{pmatrix}, \dots, \begin{pmatrix} a_{i_{d+1}} \\ \alpha_{i_{d+1}} \end{pmatrix} \right\} \\ &\Leftrightarrow \cap_{j=1}^{d+1} H_{i_j} = \emptyset\,. \end{aligned}$$

This implies $\cap_{j=1}^{d+1} K_{i_j} = \emptyset$.
Now let $C_1, \dots, C_n \subset \mathbb{R}^d$ be convex sets with Helly's property. For every $J \subset [n]$ with $|J| = d+1$ there exists $z(J) \in \mathbb{R}^d, z(J) \in \cap_{j \in J} C_j \neq \emptyset$. For $j \in [n]$, define $K_j := \operatorname{conv}\{z(J) \,|\, j \in J\}$. Each K_j is a polytope, $K_j \subset C_j$ and they satisfy the Helly condition. Hence $\cap_1^n K_j \neq \emptyset$. This in turn implies that $\cap_1^n C_j \neq \emptyset$. $\square$

Remark. We will refer to the method in second part of the above proof as *reduction to polytopes.*

Theorem 2 (Fractional Helly, Katchalski-Lin)**.**
Let $\alpha \in (0,1]$ and $d \geq 2$ be fixed. If $\mathcal{C}$ is a finite family of convex sets, $n = |\mathcal{C}|$, with at least $\alpha\binom{n}{d+1}$ intersecting $(d+1)$-tuples, then there exists an intersecting subfamily $\mathcal{C}' \subset \mathcal{C}$, with $|\mathcal{C}'| \geq \beta n$, where β is a constant that depends only on d and α.

Remark. The best known is $\beta = 1 - (1-\alpha)^{\frac{1}{d+1}}$ (Kalai).

Proof. Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$. First apply "reduction" so that we can consider the C_i's to be polytopes.

For each $k \in [n]$, let $\mathcal{I}_k := \{I \subset [n] : |I| = k,\ \bigcap_{i \in I} C_i \neq \emptyset\}$ be the set of indices of intersecting k-tuples. Given $I \in \mathcal{I}_k$, denote $C(I) := \bigcap_{i \in I} C_i$.

Since $C(I)$ is a polytope, and there are finitely many such polytopes, we can choose $a \in \mathbb{R}^d$ such that the functional $f(x) = ax \in \mathbb{R}$ takes different values in the vertices of $C(I)$, for all possible I's. In particular, this means that the set

$$\operatorname{argmin}\{ax : x \in C(I)\} = \{z \in C(I) : az \leq ax,\ \forall x \in C(I)\}$$

consists of one single point for each I. Let $z(I) = \operatorname{argmin}\{ax : x \in C(I)\}$, then $ax > az(I)$ for all $x \in C(I) \setminus \{z(I)\}$.

Claim. *For each $I \in \mathcal{I}_{d+1}$, there exists $J \in \mathcal{I}_d$, $J \subset I$, such that $z(J) = z(I)$.*

Proof*: Let $H_I := \{x \in \mathbb{R}^d : ax < az(I)\}$ the open halfspace (orthogonal to a) that has $z(I)$ in the closed boundary. Then $\{C_i : i \in I\} \cup \{H_I\}$ is a family of $d+2$ convex sets in $\mathbb{R}^d$ such that $(\bigcap_{i\in I} C_i) \cap H_I = \emptyset$. By Helly, there exists a subfamily of $d+1$ sets with empty intersection. Since $\bigcap_{i\in I} C_i \neq \emptyset$, this implies that there exists $i_0 \in I$ such that $(\bigcap_{i\in I\setminus\{i_0\}} C_i) \cap H_I = \emptyset$. Let $J = I \setminus \{i_0\}$, since $z(I) \in C(J)$ we get that $z(I) = z(J) = \operatorname{argmin}\{ax : x \in C(J)\}$.* □

There are at most $\binom{n}{d}$ possible sets $J \in \mathcal{I}_d$ and exactly $|\mathcal{I}_{d+1}| = \alpha\binom{n}{d+1}$ sets $I \in \mathcal{I}_{d+1}$. Then there exists some $J_0 \in \mathcal{I}_d$ such that at least

$$\frac{\alpha\binom{n}{d+1}}{\binom{n}{d}}$$

distinct sets $I \in \mathcal{I}_{d+1}$ are mapped to J_0. That is, such that $z(I) = z(J_0)$.

For each of those I, $z(J_0) = z(I) \in C(I) \subset C_{i_0}$, where $\{i_0\} = I \setminus J_0$. On the other hand, $z(J_0) \in C(J_0) \subset C_j$, for all $j \in J_0$.

In total there are

$$\frac{\alpha\binom{n}{d+1}}{\binom{n}{d}} + d = \alpha\frac{n(n-1)\dots(n-d)}{(d+1)\cdot n(n-1)\dots(n-d+1)} + d = \alpha\frac{n-d}{d+1} + d \geq \frac{\alpha}{d+1}n$$

sets C_i that contain the point $z(J_0)$. □

Theorem 3 (Colorful Carathéodory)**.** *Let $A_1, A_2, \dots, A_{d+1} \subset \mathbb{R}^d$ and $a \in \bigcap_{i=1}^{d+1} \operatorname{conv}(A_i)$. Then there exist $a_1 \in A_1, \dots, a_{d+1} \in A_{d+1}$ such that $a \in \operatorname{conv}\{a_1, \dots, a_{d+1}\}$.*

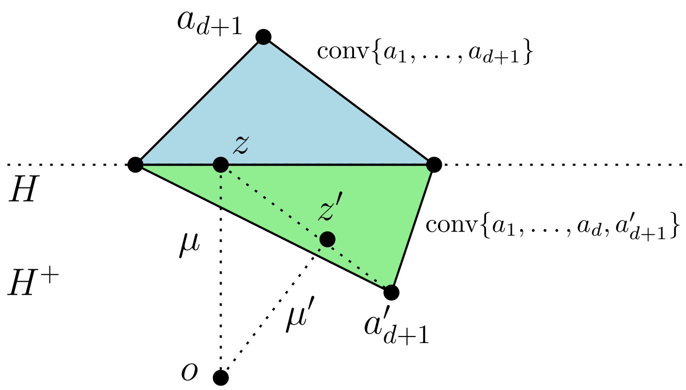
Proof. By Carathéodory's Theorem, without loss of generality $|A_i| \leq d+1$. We can also assume $a = 0$.

Let $\{a_1, \dots, a_{d+1}\}$ be the set, with $a_i \in A_i$ for each $i = 1, \dots, d+1$, that minimizes the function $dist(0, \text{conv}\{a_1, \dots, a_{d+1}\})$. Let μ be this minimum and $z \in \text{conv}\{a_1, \dots, a_{d+1}\}$ the point that achieves it. That is,

$$\mu = ||z|| = dist(0, z) = dist(0, \text{conv}\{a_1, \dots, a_{d+1}\})$$

If $\mu = 0$, then $z = 0 \in \text{conv}\{a_1, \dots, a_{d+1}\}$. Suppose $\mu > 0$.

z must be a point in the boundary of $\text{conv}\{a_1, \dots, a_{d+1}\}$. That is, it must lie in a facet, which is spanned by d vertices: without loss of generality, $z \in \text{conv}\{a_1, \dots, a_d\}$. Let H be the hyperplane containing this facet, and let H^+ be the open halfspace defined by H and containing the origin. In order for $0 \in \text{conv}(A_{d+1})$, there must exist a point $a'_{d+1} \in A_{d+1} \cap H^+$. Then the segment $(z, a'_{d+1}] \subset \text{conv}\{a_1, \dots, a_d, a'_{d+1}\} \cap H^+$, and there exists a point $z' \in (z, a'_{d+1}]$ such that $dist(0, z') < dist(0, z) = \mu$, which is a contradiction.



□

Remark.

- <u>General version:</u> Carathéodory for one single set: take the A_i to be $d+1$ copies of the same set.

- <u>Cone version:</u> Let $A_1, A_2, \dots, A_d \subset \mathbb{R}^d$ and $a \in \bigcap_{i=1}^{d} \text{pos}(A_i)$.

  Then there exist $a_1 \in A_1, \dots, a_d \in A_d$ such that $a \in \text{pos}\{a_1, \dots, a_d\}$.

- Extra version: Let $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$, $b \in \mathbb{R}^d$ and $a \in \bigcap_{i=1}^{d} \operatorname{conv}(A_i)$.

  Then there exist $a_1 \in A_1, \ldots, a_d \in A_d$ such that $a \in \operatorname{conv}\{a_1, \ldots, a_d, b\}$.

  Proof. Without loss of generality $a = 0$ and, by Carathéodory, $|A_i| \leq d + 1$.
  If $0 \in \bigcap_{i=1}^{d} \operatorname{int}(\operatorname{conv}(A_i)) = \operatorname{int}(\bigcap_{i=1}^{d} \operatorname{conv}(A_i))$, then $-b \in \mathbb{R}^d = \bigcap_{i=1}^{d} \operatorname{pos}(A_i)$. By the cone version of Colorful Carathéodory, there exist $\alpha_1, \ldots, \alpha_d \geq 0$, $a_i \in A_i$, such that $-b = \sum_{i=1}^{d} \alpha_i a_i$. Hence $0 = \frac{b + \sum_{i=1}^{d} \alpha_i a_i}{1 + \sum_{i=1}^{d} \alpha_i} \in \operatorname{conv}\{a_1, \ldots, a_d, b\}$.

  If $0 \notin \operatorname{int}(\bigcap_{i=1}^{d} \operatorname{conv}(A_i))$, then it lies in the interior of a lower dimensional face of the polytope $\bigcap_{i=1}^{d} \operatorname{conv}(A_i)$, and we can apply the previous for a lower dimension $d' < d$. □

Theorem 4 (Application 1: Colorful Helly)**.** *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} C_i \neq \emptyset$ for all $C_1 \in \mathcal{C}_1$, ..., $C_{d+1} \in \mathcal{C}_{d+1}$, then there exists $i \in [d+1]$ such that $\bigcap_{C \in \mathcal{C}_i} C \neq \emptyset$.*

Proof 1: First apply "reduction" so that C is a polytope for all $C \in \mathcal{C}$.

Suppose that $\bigcap_{C \in \mathcal{C}_i} C = \emptyset$ for all $i = 1, \ldots, d+1$. Since the C are polytopes, they are compact. Then there exist $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{d+1}$ finite families of closed halfspaces such that:

- $\bigcap_{H \in \mathcal{H}_i} H = \emptyset$ for all i.
- For each i and for each $C \in \mathcal{C}_i$, there exists $H_C \in \mathcal{H}_i$ such that $C \subset H_C$. Let $a_C \in \mathbb{R}^d$ and $\alpha_C \in \mathbb{R}$ such that $H_C = \{x \in \mathbb{R}^d : a_C x \leq \alpha_C\}$.

For each i, since $\bigcap_{H \in \mathcal{H}_i} H = \emptyset$, we have that

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \operatorname{pos}\{\begin{pmatrix} a_C \\ \alpha_C \end{pmatrix} : C \in \mathcal{C}_i\}$$

By the cone version of Colorful Carathéodory, there exist $\begin{pmatrix} a_{C_1} \\ \alpha_{C_1} \end{pmatrix}, \dots, \begin{pmatrix} a_{C_{d+1}} \\ \alpha_{C_{d+1}} \end{pmatrix} \in$ $\mathbb{R}^d \times \mathbb{R}$, where $C_1 \in \mathcal{C}_1$, ..., $C_{d+1} \in \mathcal{C}_{d+1}$, such that

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \text{pos}\{\begin{pmatrix} a_{C_1} \\ \alpha_{C_1} \end{pmatrix}, \dots, \begin{pmatrix} a_{C_{d+1}} \\ \alpha_{C_{d+1}} \end{pmatrix}\}$$

This implies that $\bigcap_{i=1}^{d+1} H_{C_i} = \emptyset$, thus $\bigcap_{i=1}^{d+1} C_i = \emptyset$, which is a contradiction.

□

Proof 2: First apply "reduction" so that we can consider each $C \in \mathcal{C}_i$ to be a polytope.

Fix $i_0 \in \{1, \dots, d{+}1\}$ and fix the set of indices $I = \{j_1, \dots, j_{i_0-1}, j_{i_0+1}, \dots, j_d\}$ where each j_i represents the set $C_{j_i} \in \mathcal{C}_i$. That is, I represents a d-tuple where each set comes from a different family and there is no element from family $\mathcal{C}_{i_0}$. Denote the polytope

$$C(I) := \bigcap_{i \in I} C_i \neq \emptyset$$

There are finitely many such polytopes (as many as choices for $i_0, j_1, \dots, j_{d+1}$) so there exists $a \in \mathbb{R}^d$ such that the functional $f(x) = ax \in \mathbb{R}$ takes different values in the vertices of $C(I)$, for all possible I's. In particular, this means that the set $\text{argmin}\{ax : x \in C(I)\}$ consists of one single point for each I. Let $z(I) = \text{argmin}\{ax : x \in C(I)\}$, then $ax > az(I)$ for all $x \in C(I) \setminus \{z(I)\}$.

Let now I_0 be the d-tuple such that the value $az(I_0)$ is maximal and let i_0 be the index of the family C_{i_0} that is not represented. Without loss of generality, $i_0 = d + 1$, and $I_0 = \{i_1, \dots, i_d\}$.

Claim. *$z(I_0) \in C$, for all $C \in \mathcal{C}_{d+1}$.*

Proof*: Suppose that $z(I_0) \notin C'$, for some $C' = C_{i_{d+1}} \in \mathcal{C}_{d+1}$. Then $z(I_0) \notin C(I_0) \cap C'$. Let $H := \{x \in \mathbb{R}^d : ax \leq az(I_0)\}$ the closed halfspace (orthogonal to a) that has $z(I_0)$ in the boundary. Then $\{C_i : i \in I_0\} \cup \{H\} \cup \{C'\}$ is a family of $d + 2$ convex sets in $\mathbb{R}^d$ such that $(\bigcap_{i \in I_0} C_i) \cap H \cap C' = \emptyset$. By Helly, there exists a subfamily of $d + 1$ sets with empty intersection. Since $(\bigcap_{i \in I_0} C_i) \cap H = \{z(I_0)\} \neq \emptyset$, this implies that there exists a $(d - 1)$-tuple*

$J \subset I_0$ such that $(\bigcap_{i\in J} C_i) \cap H \cap C' = \emptyset$. Let the d-tuple $I_1 = J \cup \{i_{d+1}\}$, then $C(I_1) = (\bigcap_{i\in J} C_i) \cap C' \subset \mathbb{R}^d \setminus H$. That is, $ax > az(I_0)$ for all $x \in C(I_1)$. In particular $az(I_1) > az(I_0)$, and so $az(I_0)$ is not maximal, which is a contradiction. □

Hence $\bigcap_{C\in\mathcal{C}_{d+1}} C \neq \emptyset$. □

Theorem 5 (Application 2)**.** *Let $G = (V, A)$ be a directed graph with $n = |V|$ vertices, and let $C_1, \ldots, C_n$ be directed cycles in G. Then there exist arcs $a_i \in C_i$ such that $\{a_1, a_2, \ldots, a_n\}$ contains a directed cycle.*

Proof. Label the vertices $\{v_1 \ldots, v_n\}$, and represent each arch $a \in A$ by a point $p_a = e_j - e_i \in \mathbb{R}^n$, where v_i is the starting vertex of a and v_j is the ending vertex. In particular, all the points p_a lie in the hyperplane $\sum_{i=1}^{n} x_i = 0$, which has dimension $d = n - 1$.

In this setting, a set of arcs $C \subset A$ contains a directed cycle if and only if $\sum_{a\in C} p_a = 0$ (there could be several pairwise disjoint cycles).

$C_1, \ldots, C_{d+1}$ are directed cycles in G. Then $\sum_{a\in C_i} p_a = 0$ for each $i \in [d+1]$ $\Longrightarrow 0 = \frac{1}{n}\sum_{a\in C_i} p_a \in \text{conv}\{p_a : a \in C_i\}$ for all $i \Longrightarrow 0 \in \bigcap_{i=1}^{d+1} \text{conv}\{p_a : a \in C_i\}$. By Colorful Carathéodory, there exist $a_1 \in C_1$, ..., $a_{d+1} \in C_{d+1}$ such that $0 \in \text{conv}\{p_{a_1}, \ldots, p_{a_{d+1}}\}$. Without loss of generality, $0 \in \text{conv}\{p_{a_1}, \ldots, p_{a_k}\}$, $a_i \neq a_j$, for k minimal. Then there exist $\alpha_1, \ldots, \alpha_k > 0$, $\sum_{i=1}^{k} \alpha_i = 1$, such that $0 = \sum_{i=1}^{k} \alpha_i p_{a_i}$.

It is easy to see that, under the previous conditions, all the (non-zero) coefficients must be equal: $\alpha_1 = \cdots = \alpha_k = \frac{1}{k}$, and $0 = \sum_{i=1}^{k} \frac{1}{k} p_{a_i} = \frac{1}{k}\sum_{i=1}^{k} p_{a_i}$, which implies that $\{a_1, ..., a_k\}$ contains a directed cycle in G. □

Exercise 2. Try to find a combinatorial proof of this last statement.