Imre Bárány: Combinatorial properties of convex sets – lecture 3

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Theorem (Extended Colorful Carathéodory)**.** *Let* $A_1, \dots, A_d$ *be nonempty sets in* $\mathbb{R}^d$*. Suppose that for any pair of distinct indices* i, j *we have* $0 \in \operatorname{conv}(A_i \cup A_j)$*. Then there exist* $a_i \in A_i$*,* $i = 1, \dots, d+1$*, such that* $0 \in \operatorname{conv}\{a_1, \dots, a_{d+1}\}$*.*

Clearly, this is an extension of Colorful Carathéodory's Theorem, since the condition $0 \in \bigcap_{i=1}^{d+1} \operatorname{conv} A_i$ implies the above assumption.

Remark. This theorem does not have a conic analog (but it has some consequences for the spherical colorful version of Helly's Theorem. However, we will not explain them here).

Proof. We start as in the proof of Colorful Carathéodory's Theorem. Without loss of generality we may assume that all the sets A_i are finite. Choose $a_i \in A_i$, $i = 1, \dots, d+1$, so that the distance $\operatorname{dist}(0, \operatorname{conv}\{a_1, \dots, a_{d+1}\})$ is minimal.

If $\operatorname{dist}(0, \operatorname{conv}\{a_1, \dots, a_{d+1}\}) = 0$ then we are done, so suppose further that $\operatorname{dist}(0, \operatorname{conv}\{a_1, \dots, a_{d+1}\}) = \|z\| > 0$ (for some $z \in \operatorname{conv}\{a_1, \dots, a_{d+1}\}$). We may assume that the points $a_1, \dots, a_{d+1}$ are in general position and that z lies in the interior of the facet $\operatorname{conv}\{a_1, \dots, a_d\}$.

Denote $H = \operatorname{aff}\{a_1, \dots, a_d\}$. Because of the optimality of z the set A_{d+1} (and in particular the point a_{d+1}) has to lie "above" H (see Figure 1). Consequently, because of the condition $0 \in \operatorname{conv}(A_i \cup A_{d+1})$, there exist points $b_1 \in A_1, \dots, b_d \in A_d$, which lie "below" H.

For $i = 1, \dots, d$ we define $f(e_i) = a_i, f(-e_i) = b_i$. Then we extend f to the mapping $f : \partial \operatorname{conv}\{\pm e_1, \dots, \pm e_d\} \to \mathbb{R}^d$ simply by setting f to be affine on the facets.

Note that the facets of $\partial \operatorname{conv}\{\pm e_1, \dots, \pm e_d\}$ are mapped exactly to multicolor facets. The image of f divides $\mathbb{R}^d$ into components, of which one is unbounded (we use topology here!). From the optimality of z we have

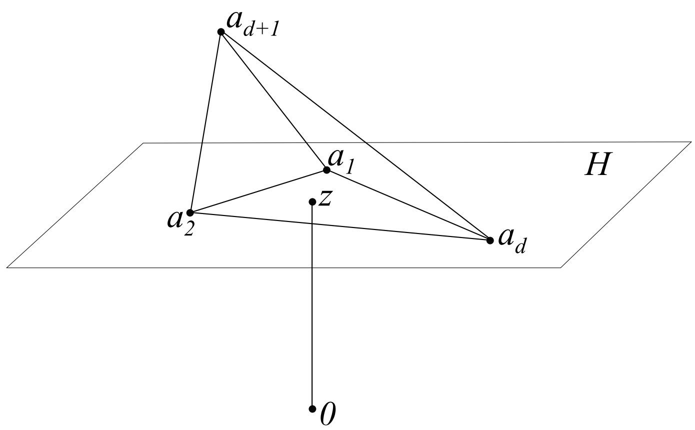

Figure 1: The point a_{d+1} lies "above" H.

$[0, z) \cap \operatorname{Im} f = \emptyset$. Therefore 0 lies in a bounded component of $\mathbb{R}^d \setminus \operatorname{Im} f$ (since the image of f lies "below" H and contains z). On the other hand the point a_{d+1} lies in the unbounded component of $\mathbb{R}^d \setminus \operatorname{Im} f$, since it lies "above" H. Therefore the half-ray starting from a_{d+1} (in the unbounded component) after passing through 0 (in a bounded component) has to pierce a multicolor facet $\operatorname{conv}\{c_1, \ldots, c_d\}$, for some $c_i \in \{a_i, b_i\}$ (see Figure 2).

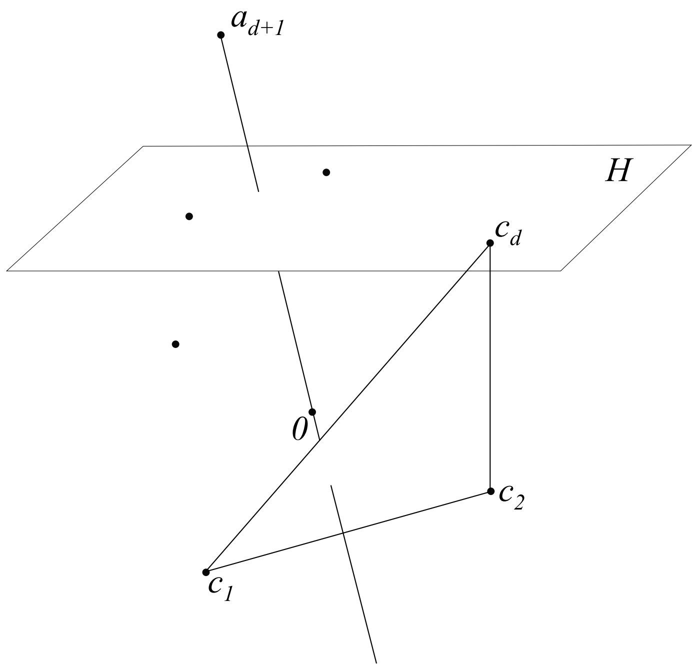

Figure 2: The half-ray starting at a_{d+1} pierces a multicolor facet.

The simplex $\operatorname{conv}\{c_1, \ldots, c_d, a_{d+1}\}$ is better than the originally chosen simplex $\operatorname{conv}\{a_1, \ldots, a_{d+1}\}$ (note that we cannot have $c_i = a_i$ for all $i = 1, \ldots, d$ since our half-ray goes first through H, then passes through 0 and only then pierces the facet $\operatorname{conv}\{c_1, \ldots, c_d\}$). This contradiction with the assumption $\operatorname{dist}(0, \operatorname{conv}\{a_1, \ldots, a_{d+1}\}) > 0$ and ends the proof. $\square$

Remark (homework). Let Γ be a curve in $\mathbb{R}^d$ (a continuous image of the interval $[0, 1]$). Then every point in $\operatorname{conv}\Gamma$ can be written as a convex combination of only d points from Γ (this is one point less than in Carathéodory's Theorem, but our set is of a special form).

We will give now a second proof of Tverberg's Theorem.

Sarkaria's proof of Tverberg's Theorem. Let $X := \{x_0, x_1, \ldots, x_n\} \subset \mathbb{R}^d$, where $n = (r-1)(d+1)$. We want to prove that there exists a partition $X = X_1 \cup \ldots \cup X_r$ such that $\bigcap_{i=1}^r \operatorname{conv} X_i \neq \emptyset$.

We will use an "artificial" tool. Namely, let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^{r-1}$ such that every $(r-1)$-tuple of them is linearly independent and $v_1 + \ldots + v_r = 0$. For any $i = 0, \ldots n$ we define the set

$$A_i := \left\{ v_j \otimes \begin{pmatrix} x_i \\ 1 \end{pmatrix} : j = 1, \ldots, r \right\} \subset \mathbb{R}^n.$$

Since $0 = \sum_{i=1}^r \frac{1}{r} v_i$, the origin belongs to $\bigcap_{i=0}^n \operatorname{conv} A_i$. By Colorful Carathéodory's Theorem there exist $a_i \in A_i$ such that $0 \in \operatorname{conv}\{a_0, \ldots, a_n\}$. Hence there exist weights $\alpha_i \geqslant 0$, $\sum_{i=0}^n \alpha_i = 1$, such that

$$0 = \sum_{i=0}^n \alpha_i a_i = \sum_{i=0}^n \alpha_i v_{j(i)} \otimes \begin{pmatrix} x_i \\ 1 \end{pmatrix}. \tag{1}$$

Let $I_j := \{i \in \{0, \ldots, n\} : j(i) = j\}$. These sets are a partition of $\{0, \ldots, n\}$ and thus the sets $X_j := \{x_i : i \in I_j\}$ are a partition of X. We will find a common point to all the sets $\operatorname{conv} X_i$, which will end the proof.

Note that for any distinct $k, l \in \{1, \ldots r\}$ there exists $u \in \mathbb{R}^{r-1}$ such that $u \cdot v_k = 1$, $u \cdot v_l = -1$ and $u \cdot v_i = 0$ for all other i. We multiply the equation (1) by u from the left side (in the sense of scalar product of vectors in $\mathbb{R}^{r-1}$) and get

$$0 = \sum_{i \in I_k} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} - \sum_{i \in I_l} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}.$$

In particular $\sum_{i \in I_k} \alpha_i = \sum_{i \in I_l} \alpha_i$ for every pair of k, l. Thus the sum $\sum_{i \in I_k} \alpha_i$ does not depend on k and since the sum of all α_i is 1, all the sums $\sum_{i \in I_k} \alpha_i$ are equal to $\frac{1}{r}$.

Similarly we get that $\sum_{i \in I_k} \alpha_i x_i$ does not depend on k. Denote this sum by z. Then

$$rz = \sum_{i \in I_1} (r\alpha_i)x_i = \sum_{i \in I_2} (r\alpha_i)x_i = \ldots = \sum_{i \in I_r} (r\alpha_i)x_i,$$

which (together with $\sum_{i \in I_k} (r\alpha_i) = 1$) means that $rz \in \operatorname{conv} X_i$ for all $i = 1, \ldots, r$. □

Remark. We can prove Tverberg's Theorem in a different way. We will give only the main idea of the third proof: take $n+1$ points in $\mathbb{R}^d$ (n is as in the above proof), build a simplex in $\mathbb{R}^{d+1}$ by lifting the given points through fibres of an affine projection. Then it is enough to prove that there exist r disjoint faces of this simplex such that their images under this projection on $\mathbb{R}^d$ have a common point.

Remark. The question what happens if we use a continuous map (rather than an affine one) in the formulation of Tverberg's Theorem described in the previous Remark was long open. It is now known that the version with the continuous maps is true for $r = p^k$, where p is a prime number (in the proof actions of groups are used).

The second proof of Tverberg's Theorem is actually a copycat of the proof of Radon's Theorem from the first lecture. We can formulate this method, invented by Sarkaria, as an easy lemma:

Lemma (Sarkaria)**.** *Assume X is a finite set in $\mathbb{R}^d$ and $X = X_1 \cup \ldots \cup X_r$ is its partition. Let $n = (r-1)(d+1)$. With every $x \in X$, if $x \in X_j$, we associate a vector $\bar{x} := v_j \otimes \binom{x}{1} \in \mathbb{R}^n$, where $(v_i)_{i=1}^r$ is a sequence introduced in the previous proof. By $\bar{X}$ be denote a set of all $\bar{x}$. Then $\bigcap_{j=1}^r \operatorname{conv} X_j = \emptyset$ if and only if $0 \notin \operatorname{conv} \bar{X}$.*

Remark. In Sarkaria's Lemma we can skip the assumption that X_i are pairwise disjoint. Then we have to consider multiset X.

Remark. This gives an "interior" condition equivalent to $\bigcap_{j=1}^r \operatorname{conv} X_j = \emptyset$ (which means that X is "separated along the colors" $1, \ldots, r$). Note that we found an "exterior" condition before, in two lemmas from the Tuesday lecture.

Proof of Sarkaria's Lemma. We will show that $\bigcap_{j=1}^r \operatorname{conv} X_j \neq \emptyset$ if and only if $0 \in \operatorname{conv} \bar{X}$. We will proceed in the same way as in the second proof of Tverberg's Theorem.

Assume first that $0 \in \operatorname{conv} \bar{X}$. Thus there exist weights $\alpha(x)$ such that

$$0 = \sum_{x \in X} \alpha(x)\bar{x} = \sum_{x \in X} \alpha(x) v_{j(x)} \otimes \begin{pmatrix} x \\ 1 \end{pmatrix} = \sum_{j=1}^{r} v_j \otimes \left(\sum_{x \in X_j} \alpha(x) \begin{pmatrix} x \\ 1 \end{pmatrix} \right).$$

Take u such that $uv_k = 1$, $uv_l = -1$ and $uv_i = 0$ for all other i. We multiply the equation by u and get, as in the previous proof, that both $\sum_{x \in X_k} \alpha(x)$ and $z := \sum_{x \in X_k} \alpha(x)x$ do not depend on k. Thus rz is a convex combination of points of X_k for all $k = 1, \ldots, r$, which ends the proof of this implication.

Assume now that $z \in \bigcap_{j=1}^{r} \operatorname{conv} X_j$. Then there exist nonegative weights $\alpha(x)$ such that

$$z = \sum_{x \in X_1} \alpha(x)x = \ldots = \sum_{x \in X_r} \alpha(x)x,$$

thus

$$0 = 0 \otimes \begin{pmatrix} z \\ 1 \end{pmatrix} = (v_1 + \ldots + v_r) \otimes \begin{pmatrix} z \\ 1 \end{pmatrix} = \sum_{j=1}^{r} v_j \otimes \left(\sum_{x \in X_j} \alpha(x) \begin{pmatrix} x \\ 1 \end{pmatrix} \right) = \sum_{x \in X} \alpha(x)\bar{x},$$

which means (since $\alpha(x)$ are nonegative), that $0 \in \operatorname{conv} \bar{X}$ and proves the second implication. □

We will also use Sarkaria's Lemma to prove another theorem:

Theorem (Kirchberger)**.** *Assume $X = X_1 \cup \ldots \cup X_r \subset \mathbb{R}^d$ is a partition of a finite set X. Then X is separated along the colors (i.e. $\bigcap_{j=1}^{r} \operatorname{conv} X_j = \emptyset$) if and only if all $Y \subset X$ of cardinality at most $(r-1)(d+1)+1$ are separated along the colors (i.e. $\bigcap_{j=1}^{r} \operatorname{conv}(Y \cap X_j) = \emptyset$).*

This theorem will follow by a more general fact. Consider a partition of $X \subset \mathbb{R}^d$ as above and let $n = (r-1)(d+1)$. Let $X_{i,0}, \ldots, X_{i,n}$ be such that for any $i = 1, \ldots r$ we have

$$X_i = \bigcup_{k=0}^{n} X_{i,k}.$$

For $j = 0, \ldots n$ we introduce sets

$$G_j := \bigcup_{l=1}^{r} X_{l,j},$$

which we will call groups.

We call a set $Y = \{y_0, \dots, y_n\}$ a transversal, if $y_j \in G_j$ for $j = 0, \dots, n$. We say that a group G_j is separated along the colors if $\bigcap_{i=1}^{r} \operatorname{conv} X_{i,j} = \emptyset$.

Kirchberger's Theorem is a special case of the following one (to recover it we have to set $X_{i,j} := X_i$).

Theorem. *Under above notation and conditions, if every transversal is separated along the colors then there exists a group separated along the colors.*

Proof. By Sarkaria's Lemma if a transversal Y is separated along the colors then $0 \notin \operatorname{conv} \bar{Y}$, where in the construction of $\bar{Y}$ we use a partition of Y induced by the partition $X_1, \dots, X_r$ of X, namely $Y = (Y \cap X_1) \cap \dots \cap (Y \cap X_r)$.

On the other hand, if every group G_j was not separated along the colors, the origin would belong to all the sets $\operatorname{conv} \bar{G}_j$, where again in the construction of $\bar{G}_j$ we use the partition given by the partition $X_1, \dots X_r$ of X. Then, by Colorful Carathéodory's Theorem, there would exist a transversal Y, such that $\operatorname{conv} \bar{Y}$ would contain the origin, which would be a contradiction. $\square$

We will consider now another problem: by how many simplices can a single point be covered? The following theorem gives an exact answer in the case of the plane.

Theorem (Boros-Füredi)**.** *Assume $X \subset \mathbb{R}^2$ is a set of n points. Then there exists a point covered by $\binom{n}{3}(\frac{2}{9} - o(1))$ triangles with vertices in X.*

Remark. We will prove later that the constant $\frac{2}{9}$ is optimal.

Proof. Consider a "nice" finite measure on the plane $\mathbb{R}^2$, say absolutely continuous with respect to the Lebesgue measure with a positive continuous density. By M we denote the measure of the plane.

Fix any direction $v \in S^1$. Note that there exists a line $l = l(v)$ in this direction dividing the plane into two half-planes $H_1 = H_1(v)$ and $H_2 = H_2(v)$ of the same measure. We assume that H_1 is 'on the left-hand side' of v (see Figure 3). Then fix any point x on this line. It divides the line l into two half-lines l_1 and l_2 (assume that l_1 goes in the direction of v and l_2 in the direction of $-v$). Note that there exist two half-lines $k_1 \subset H_1$ and $k_2 \subset H_2$ originating at x, such that the sector between l_1 and k_1 has measure $\frac{M}{6}$ and so does the sector between l_2 and k_2. When we move the point x along l from '$-\infty$' to '∞', the angle between k_1 and k_2 changes continuously from 0 to 2π and hence there exists a point $x = x(v)$, such that these two half-lines form a line (see Figure 3).

Moreover there exist two lines $m_1 \subset H_1$ and $m_2 \subset H_2$ originating at x such that the sector between k_1 and m_1 is of measure $\frac{M}{6}$ and so is the

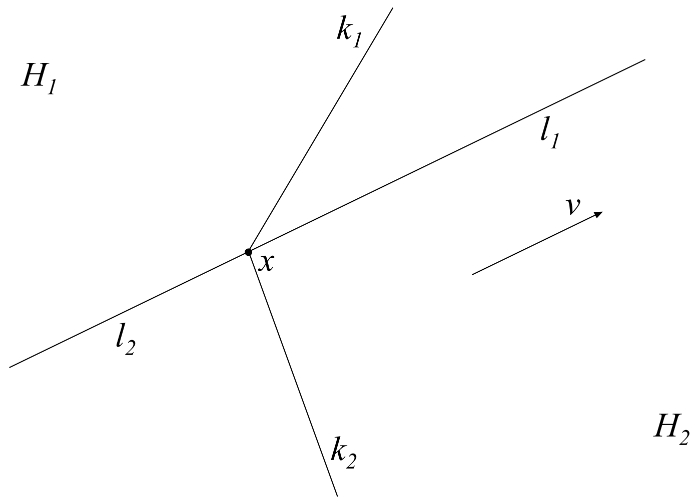

Figure 3: We move x so that k_1 and k_2 form a line.

sector between k_2 and m_2 (see Figure 4). Let $\alpha = \alpha(v)$ be the angle between m_1 and m_2. Note that this angle is a continuous function of $v \in S^1$ (we do **not** assume this angle belongs to $[0, 2\pi)$). Moreover $H_1(v) = H_2(-v)$, $l_1(v) = l_2(-v)$, $m_1(v) = m_2(-v)$, etc. and thus $\alpha(v) = -\alpha(-v) \pmod{2\pi}$, hence there exists a direction v_0 such that $\alpha(v_0) = \pi \pmod{2\pi}$, which means that $m_1(v_0)$ and $m_2(v_0)$ are collinear (see Figure 4).

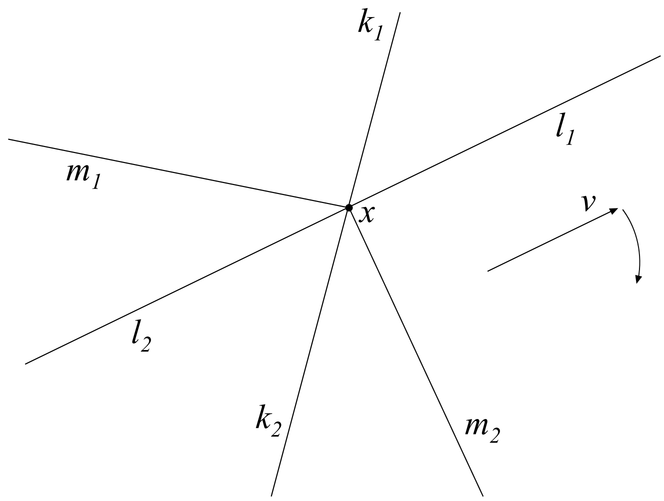

Figure 4: We rotate v so that m_1 and m_2 form a line.

We have constructed three lines with a common point, dividing the plane into three sectors each of measure $\frac{M}{6}$. If we considered a counting measure (which is not absolutely continuous) on the set X, we could find (if $n = |X|$ is large) three lines with a common point dividing the plane into three sectors each containing almost $\frac{n}{6}$ points of X. We skip this technical argument.

We will now prove that the point z common to these three lines is covered by at least $\binom{n}{3}(\frac{2}{9} - o(1)) \approx \frac{n^3}{27} - o(n^3)$ triangles. Note that if we pick three points, each from a different sector in such a way, that any two points do not lie in neighbouring sectors, we get vertices of a triangle covering z (see Figure 5). We can pick these points in $\approx 2(\frac{n}{6})^3$ ways.

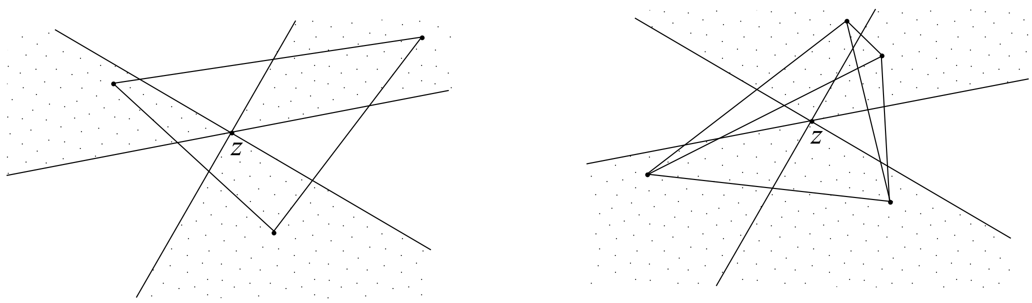

Figure 5: Triangles covering z.

Moreover, if we choose two nonneighbouring groups of neighbouring sectors and pick a point from each of them, we get vertices of possible four triangles of which at least two cover point the z (see Figure 5). We can pick these triangles in $\approx 3 \cdot 2 \cdot (\frac{n}{6})^4 \cdot \frac{1}{\frac{n}{6}}$ ways (we divide by $\frac{n}{6}$ because we count every triangle $\frac{n}{6}$ times). This gives us a total number of triangles covering z approximately equal to at least $\frac{n^3}{27}$. □

Theorem. *For any $d \geq 2$ there exists a positive constant $c(d)$ with the following property: for any set $X \subset \mathbb{R}^d$ of n points in general position there is a point in at least $c(d)\binom{n}{d+1}$ simplices with vertices in X.*

First proof. Let r be the largest possible number such that $n = (r-1)(d+1) + k$ for some $k \geq 1$. Then X has a Tverberg partition into r pieces, i.e. $X = X_1 \cup \ldots \cup X_r$ and there exists a point $z \in \bigcap_{i=1}^r \operatorname{conv} X_i$.

By Colorful Carathédory's Theorem for any $1 \leq i_1 \leq \ldots \leq i_{d+1} \leq r$ there exist $x_{i_1} \in X_{i_1}, \ldots, x_{i_{d+1}} \in X_{i_{d+1}}$ such that $z \in \operatorname{conv}\{x_{i_1}, \ldots, x_{i_{d+1}}\}$.

Clearly the simplices $\operatorname{conv}\{x_{i_1}, \ldots, x_{i_{d+1}}\}$ are pairwise distinct for different choices of the indices $1 \leq i_1 \leq \ldots \leq i_{d+1} \leq r$. Therefore the point z is covered by at least

$$\binom{r+d}{d+1} = \binom{\frac{n-k}{d+1} + 1 + d}{d+1} \geq \binom{\frac{n}{d+1}}{d+1} \geq \frac{1}{(d+1)^{d+1}}\binom{n}{d+1}$$

different simplices with vertices from X. □

Remark. We proved the theorem with $c(d) = (d+1)^{-d-1}$. We could improve this constant slightly by using the stronger version of Colorful Carathéodory's Theorem, where one point is fixed.

Second proof. Let

$$\mathcal{F} = \{\operatorname{conv} S : S \subset X, |S| = d+1\}.$$

Clearly $|\mathcal{F}| = \binom{n}{d+1}$. We want to show that "many" of the elements of $\mathcal{F}$ have a nonempty intersection. We will use Fractional Helly's Theorem for this purpose. It is enough to show that a positive fraction of all

$$\binom{\binom{n}{d+1}}{d+1}$$

$(d+1)$-tuples of elements from $\mathcal{F}$ (or in other words, a positive fraction of all $(d+1)$-tuples of convex hulls of $d+1$ elements from X) has a nonempty intersection.

Suppose we have a subset $Y \subset X$, $|Y| = (d+1)^2$. Any $d^2+d+1 = ((d+1)-1)(d+1)+1$ points from Y have a Tverberg's partition, i.e. they can be splited into mutually disjoint $Y_1, \ldots, Y_{d+1}$ such that $\bigcap_{i=1}^{d+1} \operatorname{conv} Y_i \neq \emptyset$. By Carathédory's Theorem we may assume that each of the sets Y_i has no more than $d+1$ elements. Then we distribute the points from $Y \setminus \bigcup_{i=1}^{d+1} Y_i$ among the sets Y_i so that each of the $d+1$ resulting sets $\widetilde{Y_i}$ has exactly $d+1$ elements. Summing up, we obtain a partition $Y = \bigcup_{i=1}^{d+1} \widetilde{Y_i}$, $|\widetilde{Y_i}| = d+1$, $\bigcap_{i=1}^{d+1} \operatorname{conv} \widetilde{Y_i} \neq \emptyset$.

There are $\binom{n}{(d+1)^2}$ possible choices of the subset Y, each resulting in a different $(d+1)$-tuple of $d+1$ elements from X such that the convex hulls of those elements intersect.

Since

$$\binom{n}{(d+1)^2} \geq c(d) \binom{\binom{n}{d+1}}{d+1}$$

for some positive $c(d)$ (both sides of the above inequality are polynomials of the variable n, each of degree $(d+1)^2$), we can use Fractional Helly's Theorem and the assertion of the theorem follows. $\square$

Remark. The constant form the first proof has not been improved for a long time. One can get a better constant considering a lifting of X into $\mathbb{R}^n$. Gromov has shown that getting back we can find a point covered many times. His proof uses topology and gives the constant $\frac{e^d}{d^d}$. It is still an open question if this is best possible.

We will end this lecture with a proof that the leading term $\frac{n^3}{27}$ in the Boros-Füredi Theorem is optimal. Our example will be the *stretched grid*.

The $n \times n$ stretched grid consists of n^2 points in the plane. They form n parallel rows (n points in each), in each row the distance between two neighbouring points is one, and the points are aligned so that they also form n parallel columns. The difference between the stretched grid and a normal $n \times n$ grid is the spacing between rows. The second row lies above the first (bottom) row, in distance 1 from it (no difference so far). The third row lies above the second row, on such a height that the segment connecting the rightmost point from the first row and the leftmost point from the third row passes between the two rightmost points in the second row (so the third row has to be in distance strictly greater than $n - 1$ from the bottom row; it is convenient to think of it as lying in distance n from the bottom row). This construcion is continued: the j-th row lies in such a distance above the first row, that the segment connecting the rightmost point from the first row and the leftmost point from the j-th row passes between the two rightmost points in the $(j-1)$-th row (see Figure 6 for an example of the 4×4 stretched grid).

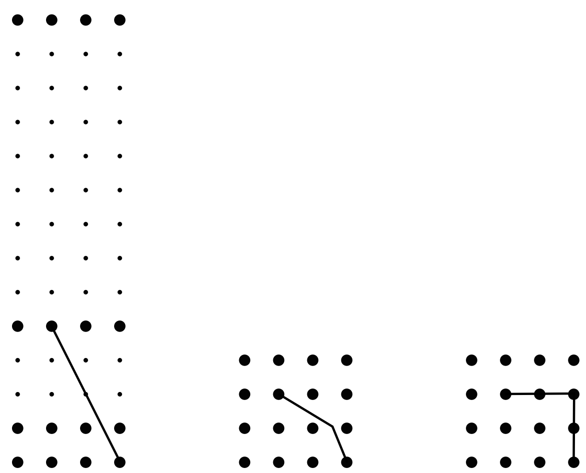

Figure 6: The 4×4 stretched grid (stretched on the left, compressed on the right).

We will be interested in the following question: how many points from the stretched grid lie in the interior of a triangle with vertices from stretched grid? The answer to this question will not change if we identify the $n \times n$ stretched grid with a normal $n \times n$ grid and the segments connecting two points with two perpendicular segments: we first go vertically from the lower of the two points to the height of the higher of these points and then right or left to the second point (see Figure 6; more examples of segments and triangles are depicted in Figure 7).

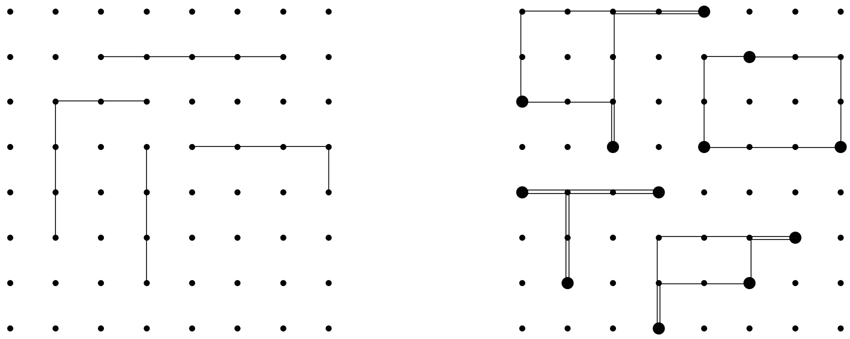

Figure 7: Segments and triangles.

We can now proceed to the proof of the optimality of the Boros-Füredi Theorem.

Proof of the optimality of the Boros-Füredi Theorem. Recall that the theorem states that for every set $X \subset \mathbb{R}^2$, $|X| = n$, there exists a point covered by $\binom{n}{3}(\frac{2}{9} - o(1))$ triangles with vertices in X. We need to show that (for every $n \in \mathbb{N}$) there exists a set $X \subset \mathbb{R}^2$, $|X| = n$, such that every point in the plane is covered by at most $\binom{n}{3}(\frac{2}{9} + o(1))$ triangles with vertices in X.

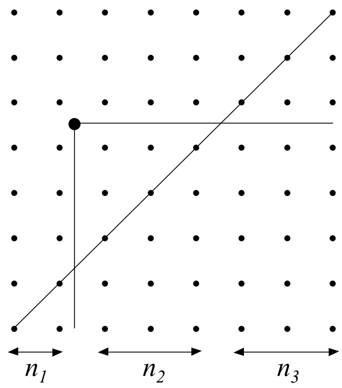

Figure 8: The points on the diagonal are divided into three groups.

Let X be the set of points from diagonal of the $n \times n$ stretched grid (going from the bottom left-hand side corner to the top right-hand side corner). Using the above explications we can identify X with the set $\{(1, 1), \ldots, (n, n)\}$ of $\mathbb{R}^2$. If a point $(x, y) \in \mathbb{R}^2$ is covered by some triangles with vertices in

X, then we must have $1 \leq x \leq y \leq n$. A horizontal and a vertical line through the point (x, y) divide the points in the set X into three groups, of cardinality n_1, n_2, n_3 respectively (see Figure 8).

Notice that if a triangle with vertices from X coveres the point (x, y) then each of its vertices comes from a different group (and the covering triangle looks like the bottom right-hand side triangle in Figure 7). Therefore the point (x, y) is covered by at most

$$n_1 n_2 n_3 \leq \left(\frac{n_1 + n_2 + n_3}{3} \right)^3 \leq \frac{(n+2)^3}{27} = \frac{n^3}{27} + o(n^3)$$

triangles with vertices from the set X. □

Remark. A similar example can be constructed also in higher dimensions.