Imre Bárány, Lecture 4

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Weak ε-nets.

Let $X \subset \mathbb{R}^d$ be a set of n points in general position (i.e. no $d+1$ points of X lie on the same hyperplane.) For a positive $0 < \varepsilon < 1$, we define

$$\mathcal{F}_\varepsilon = \{\operatorname{conv} Y : \ Y \subset X, \ |Y| \geq \varepsilon n\}.$$

A finite set $S \subset \mathbb{R}^d$ is called a *(weak) ε-net of X* if $S \cap F \neq \emptyset$ for any $F \in \mathcal{F}_\varepsilon$. We note that S is called a strong ε-net if, in addition, it also satisfies $S \subset X$. Since we are only going to discuss weak ε-nets, we will simply call these ε-nets.

Our goal is to find an ε-net of X with small cardinality.

Theorem 1. *For any $X \subset \mathbb{R}^d$ and any $0 < \varepsilon < 1$, there exists an ε-net S of X with cardinality*

$$|S| \leq c_d \frac{1}{\varepsilon^{d+1}},$$

where c_d is a constant depending only on the dimension d.

Note that this is a striking result: the cardinality of S does not depend on the set X!

Proof. We may assume that $\varepsilon > 2\frac{d+1}{n}$ since, otherwise,

$$2^{d+1}(d+1)\frac{1}{\varepsilon^{d+1}} \geq 2^{d+1}(d+1)\frac{n^{d+1}}{2^{d+1}(d+1)^{d+1}} \geq n.$$

Therefore, X is a suitable ε-net.

We construct S by an iterative algorithm. We start with $S_0 = \emptyset$, and $\mathcal{H}_0 = \binom{X}{d+1}$. At the ith step, for $i \geq 1$, we check if there exists $Y \subset X$ with $|Y| \geq \varepsilon n$ and $S_i \cap Y = \emptyset$. If there is no such Y, then the algorithm terminates - we have found an ε-net. Otherwise, take such a $Y = Y_i$. By the results of the previous lecture, there exists a highly covered point z_i of Y_i; a point z, which is covered by at least

$$\frac{1}{(d+1)^d}\binom{|Y|}{d+1}$$

many different simplices with vertices from Y_i. Let $S_i = S_{i-1} \cup \{z_i\}$, and set $\mathcal{H}_i = \mathcal{H}_{i-1} \setminus \left\{T \in \binom{X}{d+1} \,:\; z \in \operatorname{conv}(T)\right\}$.

Since X is finite, the algorithm terminates. Therefore, we have to estimate the number of steps until this happens – this is the same as the cardinality of the constructed ε-net.

Note that, for every i, the convex hull of every element of $\binom{X}{d+1} \setminus \mathcal{H}_i$ contains a point from S_i. On the other hand, $\operatorname{conv} Y_i \cap S_i = \emptyset$, which implies that every $(d+1)$-element subset of Y_i is contained in $\mathcal{H}_{i-1}$. Therefore, since z_i is a highly covered point of Y_i, we obtain that

$$|\mathcal{H}_i \setminus \mathcal{H}_{i-1}| \geq \frac{1}{(d+1)^d}\binom{|Y|}{d+1} \geq \frac{1}{(d+1)^d}\binom{\varepsilon n}{d+1}.$$

Note that the condition $\varepsilon > \frac{2(d+1)}{n}$ implies that

$$\frac{n-(d+1)}{\varepsilon n-(d+1)} < \frac{2}{\varepsilon}.$$

Therefore, the number of iterations until termination is at most

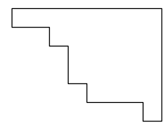
$$\frac{\binom{n}{d+1}}{\frac{1}{(d+1)^d}\binom{\varepsilon n}{d+1}} \leq (d+1)^d\, 2^{d+1}\frac{1}{\varepsilon^{d+1}}. \qquad \square$$

We note that a stronger upper bound of $c_d(1/\varepsilon^d)(\log(1/\varepsilon))^{d-1}$ has been proven.

What about lower bounds for the cardinality of an ε-net? The simplest construction is the following: take $1/\varepsilon+1$ parallel lines on the plane, and let X have $n\,\varepsilon$ points in each of the disjoint regions between the consecutive lines. An ε-net must contain a point in each of these components, therefore, its cardinality is at least $1/\varepsilon$.

A more elaborate example comes from the stretched grid and gives the stronger bound $(1/\varepsilon)\,(\log(1/\varepsilon))^{d-1}$. One has to investigate the sets that have a “negative staircase” form (see below).

Still, there is a huge gap between the lower and upper bounds: apart from logarithmic factors, it is $1/\varepsilon$ versus $1/\varepsilon^d$. The general belief is that the truth may be around $1/\varepsilon$.



Next we are going to show that, for planar sets, a quadratic upper bound holds. To this end, let $X \subset \mathbb{R}^2$ be a set of n points in general position and, for any $k > 0$, introduce

$$f(X,k) = \min\{|S| : \ S \cap \operatorname{int} \operatorname{conv} Y \neq \emptyset, \ \forall Y \subset X, \ |Y| = k\}.$$

Note that the definition is slightly stronger than in the original version; we require that the interior of the convex hulls be covered. Furthermore, let

$$f(n,k) = \max_{X \subset \mathbb{R}^2, |X|=n} f(X,k).$$

Theorem 2. $f(n,k) \leq 7\left(\frac{n}{k}\right)^2$.

For $k = 3$, a smaller covering set may be constructed:

Lemma 1. $f(n,3) \leq 2n - 5$.

Proof. We may assume that no two points of X determine a vertical line. Let v be the vertical vector of length 1. The set S consists of the points of the form $x \pm \delta v$, for every $x \in X$, where δ is a very small positive number. It is easy to see that every triangle with 3 vertices from X contains at least one point of S. In fact, it suffices to take only $S \cap \operatorname{conv} X$; elementary arguments show that by taking this intersection, we drop at least 5 points out of the $2n$. □

Proof of Theorem 2. We assume that $n \geq k \geq 6$; the remaining cases can be checked by hand, using the above Lemma. Choose a line L in general position (i.e. no segment determined by X is parallel to L), which halves L. That is, there are no points of X on L, and if m_1 and m_2 denote the number of points in the halfplanes L^+ and L^- determined by L, then $|m_1 - m_2| \leq 1$.

We are going to construct S recursively, and we proceed by induction on n. Let $l < k/2$ be a number whose value we are going to choose later in order to optimize the bound. Introduce

$$\mathcal{Y} = \{Y \subset X, \ |Y| = k, \ |Y \cap L^+| \geq l, \ |Y \cap L^-| \geq l\}.$$

Our first, intermediate target is to construct a set $S_0 \subset L$, for which $S_0 \cap \operatorname{int}\operatorname{conv} Y \neq \emptyset$ for every $Y \in \mathcal{Y}$. The remaining sets, which are not covered by S_0, have many points on one side of L and we are going to handle them with the recursive argument.

Take all the segments xy, where $x \in X \cap L^+$ and $y \in X \cap L^-$. Altogether, they determine $m_1 m_2$ intersection points with L . (No two of these coincide because X is in general position). Order the intersection points linearly on L: they are $p_1, p_2, \ldots, p_m$, where $m = m_1 m_2$. Let $h = (l+1)(k-l-1)$. To any set $Y \in \mathcal{Y}$ there correspond at least h intersection points (of L with segments both of whose endpoints are in Y). Construct S_0 by taking one point in each of the open intervals $(p_1, p_2), (p_{h+1}, p_{h+2}), \ldots, (p_{m-1}, p_m)$, so that there are at most $h-1$ intersection points between any two consecutive points of S_0. Then, by convexity, S_0 has a point in $\operatorname{int}\operatorname{conv} Y$ for each $Y \in \mathcal{Y}$. Therefore, $S_0 = \lceil \frac{m}{h} \rceil$. On the other hand, any $Y \subset X$ with $|Y| = k$ that is not contained in $\mathcal{Y}$ has at least $k - l$ points in L^+ or L^-. By the recursion, there exists a $(k-l)$–covering set S^+ for $X \cap L^+$ with at most $f(m_1, k-l)$ points and a $(k-l)$–net S^- for $X \cap L^-$ with at most $f(m_2, k-l)$. Then, $S = S_0 \cup S^+ \cup S^-$ is a suitable covering set for X whose cardinality can bounded using the inductive hypothesis:

$$\begin{aligned} |S| &\leq |S_0| + f(m_1, k-l) + f(m_2, k-l) \\ &\leq \left\lceil \frac{m_1 m_2}{h} \right\rceil + 7\left(\frac{m_1}{k-l}\right)^2 + 7\left(\frac{m_2}{k-l}\right)^2 \\ &\approx \frac{n^2}{4l(k-l)} + 14\frac{n^2}{4(k-l)^2} \\ &= \frac{n^2}{4}\,\frac{13l+k}{l(k-l)^2}\,. \end{aligned}$$

A simple numerical argument shows that this is minimized when $l \approx 0.1467k$ yielding the upper bound

$$f(n,k) < \frac{27.2161}{4}\left(\frac{n}{k}\right)^2 < 7\left(\frac{n}{k}\right)^2. \qquad \square$$

Halving Lines

Let $X \subset \mathbb{R}^2$ be a set of n points in general position in the plane, where n is even. A pair of distinct points $a, b \in X$ determine a *halving line* (and ab is called a halving segment) if, on both (open) sides of $\operatorname{aff}\{a,b\}$, there are exactly $n/2 - 1$ points of X. By a simple continuity argument, there is at least one halving line through every point of X,. Let $f(X)$ denote the number of halving lines of X. Then, by the previous observation,

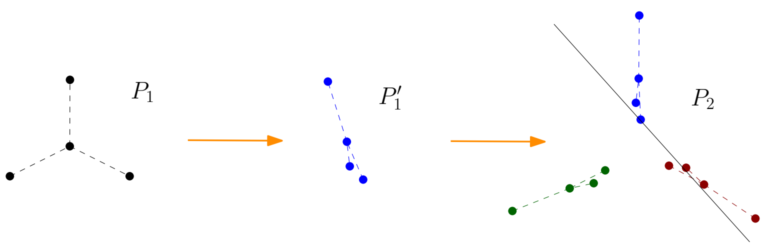
$$\frac{n}{2} \leq f(X) \leq \binom{n}{2}.$$

Furthermore, let $f(n) = \max_{|X|=n} f(X)$. In the 1970's, Erdős asked for upper and lower bounds on $f(n)$.

Theorem 3. *$f(n) > cn \log n$.*

Proof. We are going to construct X iteratively. Let P_1 be the set shown in the figure below (three points distributed on a circle, plus the center of the circle). Then P_1 has 3 halving lines.

Assume that P_i has been constructed. Note that an affine transformation does not change the halving property. Let L be a general direction with respect to P_i. Now compress P_i in the direction L so that it is contained in a very thin slab perpendicular to L, call the resulting set P_i'. Then, place three rotated copies of P_i' in a "Mercedes frame" (see the figure below).

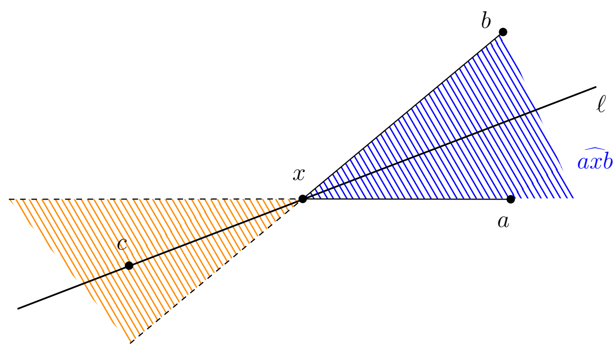


Each of the original halving lines remains a halving line plus we obtain $|P_i|/2$ new halving lines. This leads to the lower bound $\frac{1}{2} n \log_3 n$. □

Next, consider the graph of the halving segments of X: take the points of X as the vertices and we all of the halving segments as edges. By our previous observation, every vertex has degree ≥ 1. We define the *wedge* $\widehat{xyz}$ of three distinct points x, y, z on the plane to be the smaller region between the two half-lines containing x and z, respectively, with common endpoint y.

Lemma 2 (Odd-Star Lemma, Lovász)**.** *Take the graph of the halving segments. For any point $x \in X$, and for any two halving segments xa and xb emanating from X, there exists another halving segment with endpoint x in the wedge opposite to $\widehat{axb}$.*

Proof. Assume that x is the origin, and the wedge $\widehat{axb}$ is in counterclockwise orientation. Take a rotating line ℓ with center x, which rotates from a to b along the wedge - at its starting position, it contains x and a, and at its final position, it contains x and b. Therefore, the ray $\overrightarrow{xa}$ scans through the wedge $\widehat{axb}$, and the opposite half-line scans the opposite wedge.



Consider the number of points of X on the 'lower" side – i. e. the (open) half-plane which, at the beginning, does not contain b. At the starting position, the number of points is $n/2-1$. As ℓ leaves a, it changes to $n/2$. When terminating (when ℓ contains b), it is $n/2-1$ again. How does this number change? Well, it only happens when the line meets a point; if the rotation of the half-line $\overrightarrow{xa}$ hits a point, then the number will increase; if the opposite ray meets a point, the number will decrease. Moreover, because of general position, the number always changes by at most 1. Therefore, there must be a position somewhere along the way where the number drops down to $n/2-1$, and this belongs to a halving line determined by x and a further point c, which necessarily lies in the opposite wedge. □

Theorem 4. $f(n) < cn^{\frac{3}{2}}$.

Proof. We may assume that no two points of X lie on a vertical line. Consider a vertical line which traverses through X and, at each position, count how many halving segments it intersects. The resulting number is 0 before the line hits X or after it leaves X. In between, the number only changes when hitting points of X. The odd-star lemma gives us that each change is exactly ± 1. Since $|X| = n$, the line intersects at most $n/2$ halving segments at every position.

Let L be a horizontal line and consider the orthogonal projection P_L onto L. The projection of the halving segments of X is a system $\mathcal{I}$ of intervals on L whose endpoints are among the n points of $P_L(X)$. Moreover, by the previous argument, we deduce that the system of intervals covers each point of L at most $n/2$ times. How many intervals can $\mathcal{I}$ contain?

Order the points of $P_L(X)$ linearly on L: $p_1, p_2, \ldots, p_n$. Choose $k < n$ (that we will specify later) and take every kth point of $P_L(X)$ as such: $p_1, p_{k+1}, p_{2k+1}, \ldots, p_n$. We are going to call these dividing points. These cut the segment $p_1 p_n$ into $\lfloor n/k \rfloor + 1$ blocks. There are two types of intervals in $\mathcal{I}$ depending on if the two endpoints are in the same block ("short intervals") or in different blocks ("long intervals"). The number of long intervals is at most $\frac{n}{k} \cdot \frac{n}{2}$, since each dividing point is covered by at most $n/2$ intervals (which are necessarily long). On the other hand, in each block, there are at most $\binom{k}{2}$ short intervals (the total number of intervals with two endpoints in a block). Therefore,

$$|\mathcal{I}| \leq \frac{n}{k}\frac{n}{2} + \binom{k}{2}\left(\frac{n}{k} + 1\right).$$

Setting $k = \sqrt{n}$ yields the upper bound $cn^{\frac{3}{2}}$. □

We note that a stronger upper bound can also be given: $f(n) < cn^{\frac{4}{3}}$. The probabilistic proof uses the crossing lemma which states that, in a plane drawing of a graph with n vertices and m edges, there are at least $\frac{1}{50}\frac{m^3}{n^2}$ crossings between the edges.

Halving Planes

We are going to consider the 3–dimensional analogue of halving lines. Let $X \subset \mathbb{R}^3$ be a set of n points in general position, where n is odd. The points $a, b, c \in X$ determine a *halving plane* defined by their affine hull if the plane dissects the set X into two equal parts, that is, if each (open) side of $\text{aff}\{a, b, c\}$ contains exactly $\frac{n-3}{2}$ points of X. We denote $f(X)$ to be the number of halving planes of the set X. The aim is to find bounds on the maximum number of halving planes of a set of n points of $\mathbb{R}^3$ in general position. That is, to bound

$$f_3(n) = \max\{f(X) : X \subset \mathbb{R}^3,\ |X| = n,\ X \text{ is in general position}\}.$$

It is clear that $\frac{1}{3}\binom{n}{2} \leq f(n) \leq \binom{n}{3}$. The upper bound is simply the total number of possible planes determined by points of X. For the lower bound, given any two points of X, if we rotate a plane through the line they determine, we see that there exists a third point of X that generates a halving plane of X together with the first two points. Each such plane is counted three times; one for each side of the triangle.

A stronger lower bound has been given: $n^2 \log n < f_3(n)$. What can we say about an upper bound?

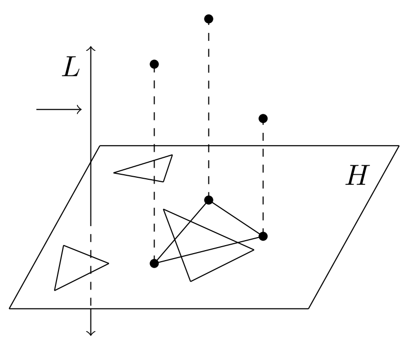
The planar result depended on the odd-star lemma. In fact, considering the 2–dimensional picture as a projection of the 3–dimensional situation, the same proof yields the following analogue:

Lemma 3 (Odd-Star Lemma for $\mathbb{R}^3$)**.** *Let $a, b \in X$. If there exist points c and d such that aff$\{a, b, c\}$ and aff$\{a, b, d\}$ are halving planes with corresponding closed halfspaces $\mathcal{H}_c$ and $\mathcal{H}_d$, respectively, such that $\{a, b, c, d\} \subset \mathcal{H}_c \cap \mathcal{H}_d$, then there exists a point $x \notin \mathcal{H}_c \cup \mathcal{H}_d$ such that aff$\{a, b, x\}$ is also a halving plane of X.*

The trivial upper bound for the number of halving planes is $\binom{n}{3}$, which is of order n^3. We are able to provide a polynomial strenghtening (although, with small exponent) to this bound.

Theorem 5. *$f_3(n) < n^{3-\epsilon}$ for some small $\varepsilon > 0$.*

Proof. Consider a plane H in general position with respect to X and the orthogonal projection P_H of X onto H (shown on the following figure). Let $\mathcal{H}$ denote the system of the projections of the halving triangles of X.



First, we state that every point of H is covered by at most $\binom{n}{2}$ members of $\mathcal{H}$. Indeed, this covering function (the number of triangles covering a specific point) is constant on all the components of the plane determined by the segments between points of $P_H(X)$ (this is the same as moving a vertical line L throughout H, and counting how many halving triangles it hits). Moreover, the odd-star lemma and general position guarantee that the value differs by exactly 1 on two sides of a segment. Since there are altogether $\binom{n}{2}$ segments, we obtain the bound.

Therefore, we conclude that $P_H(\mathcal{H}) \subset \binom{P_H(X)}{3}$ is a system of triangles which covers each point of H at most $O(n^2)$ times. How large can $|\mathcal{H}| = |P_H(\mathcal{H})|$ be? This is the content of the following result. Combining this result with the $O(n^2)$ estimate for the covering number concludes the proof of Theorem 5.

Theorem 6. *Given a set $X \subset \mathbb{R}^2$ of n points in general position, let $\mathcal{H} \subset \binom{X}{3}$ such that $|\mathcal{H}| = p\binom{n}{3}$ where $p = n^{-\alpha}$ for some $\alpha > 0$. Then there exists a point common to at least $c\,p^s\binom{n}{3}$ triangles of $\mathcal{H}$ where s and c are (large) universal constants.*

Proof. Define $Q(X) := \{\mathrm{aff}(u,v) \cap \mathrm{aff}(x,y) : u,v,x,y \in X \text{ distinct}\}$ to be the *crossings* in X. Then $Q(X)$ contains $\frac{1}{2}\binom{n}{2}\binom{n-2}{2} \approx n^4/8$ crossings.

The main idea of the proof is the following. We are going to show that there exists a crossing q, which is contained in many distinct triangles of $\mathcal{H}$. To this end, we will consider the (q,Δ) pairs, where q is a crossing, and Δ is a triangle of $\mathcal{H}$, so that $q \in \mathrm{int}\, H$.

For any points $a,b,c \in X$, denote the number of crossings in $\mathrm{conv}\{a,b,c\}$ by $N(a,b,c)$. That is, $N(a,b,c) = |Q(X) \cap \mathrm{conv}\{a,b,c\}|$.

As an illustration of the technique that we are going to use, we now show that the average of $N(a,b,c)$ is of order n^4. Because of Tverberg's theorem, any 9 points of X can be partitioned into 3 intersecting triangles. This shows that for of any 9 points, there exists a triangle Δ determined by them, and two further segments determined by them, whose crossing point q lies in the triangle. This configuration (q,Δ) is determined by 7 points (out of the 9). We assign to each 9-tuple such a (q,Δ)-configuration.

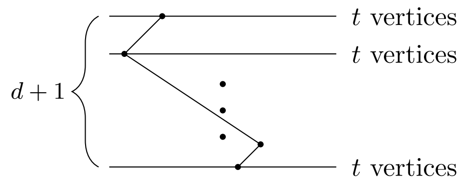
We want to count the number of distinct (q,Δ)-configurations. In the above assignment, each such (q,Δ)-configuration appears at most $\binom{n-7}{2}$ times (since we can add two arbitrary points in order to obtain a 9-tuple). Therefore, the number of distinct (q,Δ)-configurations is at least

$$\binom{n}{9} \Big/ \binom{n-7}{2} \sim n^7.$$

Since there are $\binom{n}{3} \sim n^3$ triangles, on average, they must contain at least $\sim n^7/n^3$ crossings, which is the desired bound.

Next, we transform this argument so that only the triangles of $\mathcal{H}$ are counted.

We will use the following result of Erdős and Simonovits. For preparation, we need to introduce some terminology. A $(d+1)$-*uniform hypergraph* on a base set V is a system of $(d+1)$-tuples of V. Elements of V are called the *vertices* of the hypergraph, while the selected $(d+1)$-tuples are the *edges*. The *complete* $(d+1)$*-partite hypergraph*, denoted by $K(t,\ldots,t)$, is obtained as follows: let $V_1,\ldots,V_{d+1}$ be sets of cardinality t, and let $K(t,\ldots,t)$ contain all the transversals (i.e. all the sets containing one point from each V_i). $K(t,\ldots,t)$ is illustrated in the following figure.



Here comes the result of Erdős and Simonovits.

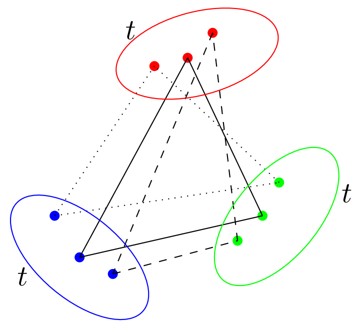
Theorem 7. *For every d and every t, there exists a $b(d,t)>0$ with the following property. Any $(d+1)$-uniform hypergraph $\mathcal{H}$ defined on n vertices with $p\binom{n}{d+1}$ edges, where $n^{-t^d}\le p<1$, contains at least*

$$b\,p^{t^{d+1}}\binom{n}{t,\ldots,t}$$

copies of $K(t,\ldots,t)$.

We provide no proof here, only an intuitive explanation why this bound holds. Look at a random $(d+1)$-uniform hypergraph $\mathcal{H}$ on n vertices with $(d+1)$-tuples chosen randomly with probability x. Then the expected value of $|\mathcal{H}|$ is $\mathbb{E}|\mathcal{H}|=x\binom{n}{d+1}$. Furthermore, the expected value of the number of copies of $K(t,\ldots,t)$ is $x^{t^{d+1}}\binom{n}{t,\ldots,t}$. The proof is based on this observation, and it uses an averaging argument.

Let us return to the proof of Theorem 6. We consider $\mathcal{H}$ as a 3-uniform hypergraph with vertex set X. Applying Theorem 7, we obtain that there are at least $cp^{t^3}\binom{n}{t,t,t}$ copies of $K(t,t,t)$ in $\mathcal{H}$. For each such copy, we consider the three vertex classes of cardinality t as *color classes*. Next, we show that there exists a t so, for every geometric realization of $K_{t,t,t}$, there exist three distinct multi-coloured triangles which intersect (illustrated in the following figure.)



This is exactly the content of the colorful version of Tverberg's theorem, which is cited from the next lecture:

Theorem 8 (Colorful Tverberg Theorem)**.** *Given d and k, there exists a $t(d,k)$ large enough such that for any sets $C_1, C_2, \ldots, C_{d+1} \subset \mathbb{R}^d$, where $|C_i| = t$, there are disjoint sets $S_1, S_2, \ldots, S_k \subset \cup_{i=1}^{d+1} C_i$ such that*

1. $\cap_{i=1}^{k} conv\, S_i \neq \emptyset$
2. $|S_i \cap C_j| = 1$.

In fact, in the above setting ($d = 2, k = 3$), one can show that the statement holds with $t = 4$.

Therefore, as before, we can assign a (q, Δ)-pair for each copy of $K(t,t,t)$ in $\mathcal{H}$ where q is a crossing. Thus, Δ is a triangle of $\mathcal{H}$ with $q \in \operatorname{int} \delta$. How many distinct such crossing pairs are there? Well, the number of distinct copies of $K(t,t,t)$ in $\mathcal{H}$ is at least $cp^{t^3}\binom{n}{t,t,t}$. On the other hand, each crossing pair may be assigned to at most $\binom{n-7}{3t-7}$ copies of $K(t,t,t)$. Therefore, the number of distinct crossing pairs is at least the quotient of these two quantities:

$$\frac{cp^{t^3}\binom{n}{t,t,t}}{\binom{n-7}{3t-7}} = c'p^{t^3}n^7.$$

Yet, there are only $O(n^4)$ crossing points. Therefore, there must be a crossing point which is contained by at least $cp^{t^3}n^3$ triangles of $\mathcal{H}$. □

Analyzing the above argument one obtains that Theorem 6 holds with $s = 64$, and, in turn, Theorem 5 is valid with $\varepsilon = 1/64$. There are stronger bounds; the current best estimate is around $n^{2.5}$.