Combinatorial properties of convex sets

Lecture 5 — 23/01/2015

Lecturer: Imre Bárány *Scribe: Evangelos Anagnostopoulos, Ioannis Psarros*

Theorem 1 (Colored Tverberg)**.** *For every d, $k \geq 1$, there exists $t = t(d,k)$ such that if $C_1, ..., C_{d+1} \subset \mathbb{R}^d$ with $(\forall i)|C_i| = t$ then there exist $S_1, ..., S_k \subset \bigcup_{i=1}^{d+1} C_i$ which are pairwise disjoint,$(\forall j)(\forall i)$ $|S_j \cap C_i| = 1$ and $\bigcap_{j=1}^{k} \text{conv} S_j \neq \emptyset$.*

Theorem 2. *$t(d, 2) = 2$.*

The following proof is due to Lovász [BL92].

Proof. Let $C_i = \{a_i, b_i\}$ for every $i = 1, ..., d+1$. We consider the mapping $f(e_i) = a_i$ and $f(-e_i) = b_i$ for $i = 1, ..., d+1$. Let X be the cross polytope defined by $-e_i, e_i$ in $\mathbb{R}^{d+1}$. Furthermore, we can extend f s.t. it maps points from ∂X to $\mathbb{R}^d$ as follows:

$$f(x) = \sum_{i=1}^{d+1} \alpha_i f(v_i),$$

where $x = \sum_{i=1}^{d+1} \alpha_i v_i \in \partial X$ and for $i = 1, ..., d+1$ $\alpha_i \geq 0$ and $\sum_{i=1}^{d+1} \alpha = 1$. For every $i = 1, ..., d+1$, $v_i \in \{-e_i, e_i\}$.

By the Borsuk-Ulam theorem, there exists $x \in \partial X$ s.t. $f(x) = f(-x)$. Hence, there exist $\alpha_1, ..., \alpha_{d+1}$, $v_1, ..., v_{d+1}$ s.t. $\sum_{i=1}^{d+1} \alpha_i f(v_i) = \sum_{i=1}^{d+1} \alpha_i f(-v_i)$. By definition the two simplices $\text{conv}(f(v_1), ..., f(v_{d+1}))$ and $\text{conv}(f(-v_1), ..., f(-v_{d+1}))$ cover each color exactly once and their intersection contains at least x.

□

We also present an alternative proof of Theorem 1.

Alternative Proof. We consider the vectors $a_1 - b_1, ..., a_{d+1} - b_{d+1}$. Since the number of such vectors is $d+1$ and they are in $\mathbb{R}^d$, they are linearly dependent which implies that there exist $\alpha_1, ..., \alpha_{d+1}$ s.t. $\sum_{i=1}^{d+1} \alpha_i (a_i - b_i) = 0$. Now we define f s.t. for $i = 1, ..., d+1$, if $\alpha_i \geq 0$, $f(i) = a_i$ and if $\alpha_i < 0$ then $f(i) = b_i$ and g s.t. for $i = 1, ..., d+1$ if $f(i) = a_i$ then $g(i) = b_i$ and if $f(i) = b_i$ then $g(i) = a_i$. The two simplices $\text{conv}(f(1), ..., f(d+1))$ and $\text{conv}(g(1), ..., g(d+1))$ cover each color exactly once and their intersection contains the point $x = \sum_{i=1}^{d+1} \alpha_i f(i) = \sum_{i=1}^{d+1} \alpha_i g(i)$.

□

Theorem 3. *Let $\mathcal{H} \subset \binom{X}{d+1}$ where points in $X \subset \mathbb{R}^d$ are in general position, $|X| = n$ and $|\mathcal{H}| = p\binom{n}{d+1}$. Then, there exists point which is common to $\beta p^{t^{d+1}} \binom{n}{d+1}$ simplices in $\mathcal{H}$.*

We prove the above theorem for the case $d = 2$. The proof for the general case can be found in [Mat02, Sec. 9].

Proof. In Fig. 1 we can see three triangles in $\mathbb{R}^2$ with non empty intersection. The vertices are colored and each triangle contains each color exactly once.

We can view $\mathcal{H}$ as a hypergraph H where the vertices correspond to points in X and the edges correspond to simplices in $\mathcal{H}$. Let $K(t, t, t)$ be the complete 3-partite graph where each of the three

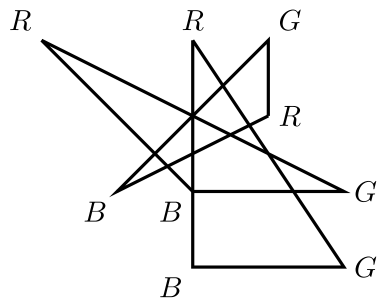

Figure 1: 3 triangles intersecting

classes consists of t vertices. Intuitevely, if H contains many edges then many copies of $K(t,t,t)$ also appear. Each copy of $K(t,t,t)$ contributes with a pair $(p,\Delta) \in (Q(X), \mathcal{H})$ s.t. $q \in \text{int}\Delta$ where $Q(X)$ denotes the crossings in X.

H contains $cp^{t^3}\binom{n}{t,t,t}$ copies of $K(t,t,t)$ for some constant c. The number of copies that we need to take into consideration, that is the number of pairs (p,Δ) equals

$$cp^{t^3}\binom{n}{t,t,t}/(\binom{n}{t-3,t-3,t-3}\binom{n-7}{2}) \sim p^{t^3}n^7.$$

Hence,

$$cp^{t^3}n^7 \le \#\{(p,\Delta),...\} = \sum_{p\in Q(x)} \#\{\Delta \in \mathcal{H} : p \in \Delta\} \implies$$

$$\implies \#\{\Delta : q^* \in \Delta\} \ge cp^{t^3}n^3.$$

□

In the number of halving triangles, if you project them down no point is covered by n^2 triangles.

$$\beta p^{t^{d+1}}\binom{n}{d+1} \Rightarrow p^{3^3}\binom{n}{3} < n^2 \Rightarrow \#\text{ halving triangles } < n^{3-\frac{1}{27}}$$

However this is not the best outcome for $d = 3$. A better bound of $n^{2.5}$ has been proved.

Colorful Tverberg ⇔ Point Selection Theorem: We have seen that the colorful Tverberg implies the Point Selection theorem. We will now also prove the opposite direction.

Proof. Let $\mathcal{H}$ denote the hypergraph depicted in figure 2 and $|C_i| = t, 1 \le i \le d+1$. This means that each edge of $\mathcal{H}$ is a multicolor simplex. For the number of edges in $\mathcal{H}$ it holds that:

$$|\mathcal{H}| = t^{d+1} \gg \binom{n}{d+1}, \text{ where } n = (d+1)t$$

If we apply the point selection theorem we can choose a $\mathcal{H}' \subset \mathcal{H}$, such that $|\mathcal{H}'| \gg \binom{n}{d+1}$. Consider the largest k such that there are $S_1, S_2, \ldots, S_k$ disjoint multicolored simplices in $\mathcal{H}$. Then any other $S \subset \mathcal{H}'$ intersects $\bigcup_i S_i$. The number of such S is at most $k(d+1)t^d + k \ge |\mathcal{H}'| \gg t^{d+1}$, which implies that $k \gg t$. □

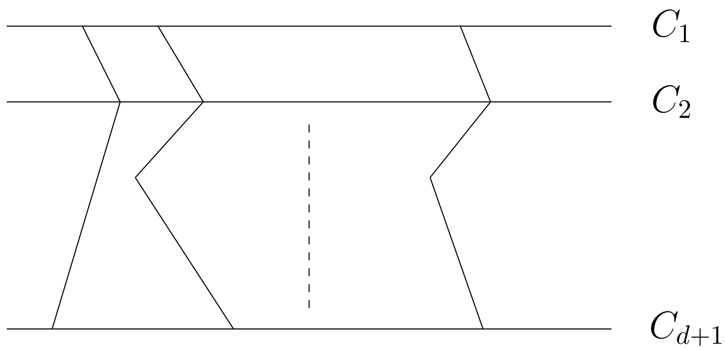

Figure 2: Transversals between the color classes $C_1, C_2, \ldots, C_{d+1}$ in the hypergraph $\mathcal{H}$

(p,q)-property

Definition 4. *A family of sets $\mathcal{C}$ has the* (p,q)-property *if among any p sets in $\mathcal{C}$, there are q intersecting ones.*

Of course this property is only interesting if $p \geq q \geq d+1$.
Let $\mathcal{C}$ be a finite family of n convex sets in $\mathbb{R}^d$ with the (p,q)-property. We want to conclude that under these conditions there exists a small set S of points that covers all sets in the family. So, our goal is to find a set $S \subset \mathbb{R}^d$ of small size such that $S \cap C \neq \emptyset, \forall C \in \mathcal{C}$ and $|S| \leq f(p,q,d)$. Without loss of generality we will consider that $q = d+1$.

Lemma 5. *If a family of sets $\mathcal{C}$ has the (p,q)-property, then there exists a point common to a linear size subset of $\mathcal{C}$.*

Proof. The number of tuples in the set is:

$$\frac{\binom{n}{p}}{\binom{n-(d+1)}{p-(d+1)}} = \binom{n}{d+1}\left[\frac{1}{d^p+1}\right]$$

Let $\alpha = \frac{1}{d^p+1}$. Then we fulfill the requirements of the fractional Helly theorem and so there is a point common to $\frac{\alpha}{d+1}n$ sets. □

Lemma 6. *Consider a finite family of n convex sets $\mathcal{C} = C_1, C_2, \ldots, C_n$ with the (p,q)-property. Then the blown up copy $\mathcal{C}^*$, where $\mathcal{C}^* = k_1$ copies of C_1, k_2 copies of $C_2, \ldots, k_n$ copies of $C_n, k_i \in \mathbb{N}$, has the $(p', d+1)$-property, where $p' = d(p-1)+1$.*

Proof. Pick $B_1, B_2, \ldots, B_{p'} \in \mathcal{C}^*$ and count the number of intersecting sets.
If some set is repeated $d+1$ times we are done.
If no set is repeated $d+1$ times, then every set is repeated at most d times and so we have that $(p-1)d+1$ of them will contain $d+1$ intersecting. □

Lemma 5 implies that $O(\log n)$ points can capture any set with the $(p, d+1)$-property.

Theorem 7 (Alon-Kleitman)**.** *There is a set S intersecting every $C \in \mathcal{C}$ of size $|S| \leq f(p,d)$.*

Proof. We are looking for a finite set $X \subset \mathbb{R}^d$ such that $|X \cap C| \geq \epsilon|X|, \forall C \in \mathcal{C}$, with $\epsilon = f(p,d)$.
If S is a weak ϵ-net for X, then we are done, because if $|X \cap C| \geq \epsilon$, then $S \cap \operatorname{conv}(X \cap C) \neq \emptyset$.
Target: Find such set X (and ϵ)

Choose $I \subset [n]$ s.t. $C(I) := \bigcap_{i \in I} C_i \neq \emptyset$ and pick $z \in C(I)$. Next, let $m(I)$ be a number in $\{0, 1, 2, \ldots\}$ and define $X = \{m(I) \text{ copies of } z(I) \mid \forall I \subset [n] \text{ s.t } C(I) \neq \emptyset\}$. Obviously, $|X| = \sum_{\text{all } I} m(I)$.

Target: choose $\gamma > 0$ and $m(I)$ such that: $\sum_{I \ni i} m(I) \geq \gamma |X|, \forall i \Rightarrow |C_i \cap X| \geq \gamma |X|$.

Target: Find $x(I)$ such that:

$$\frac{1}{\gamma} = \min \sum_{\text{all } I} x(I), \text{ s.t. } \sum_{I \ni i} x(I) \geq 1, \forall i \ \& \ x(I) \geq 0$$

Equivalenly, its dual l.p. is the following:

$$\frac{1}{\gamma} = \max \sum_{\text{all } i} y_i, \text{ s.t. } \sum_{i \in I} y_i \leq 1, \forall I \ \& \ y_i \geq 0$$

Target: Find $y_i \geq 0$ such that:

$$\sum_{\text{all } i} y_i \geq \frac{1}{\gamma} \Rightarrow \sum_{i \in I} y_i > 1 \text{ for some } I \tag{1}$$

We replace $y_i = \frac{k_i}{D}$, in (1) where D is the common denominator:

$$\textbf{Target}: \sum_{\text{all } i} k_i > \frac{D}{\gamma} \implies \sum_{i \in I} k_i > D$$

and now we set $\frac{D}{\gamma} = M$:

$$\sum_{\text{all } i} k_i > M \implies \sum_{i \in I} k_i > \gamma M$$

Create $\mathcal{C}^*$ by taking each set C_i, k_i times. From the last inequality, $\mathcal{C}^*$ has an intersecting subfamily of size γM. Lemma 6 completes the proof. □

Alternative algorithmic proof:

Proof. Start with:
`S = ∅.` $\mathcal{C}_0 = \mathcal{C}$.
`Pick` z_1 `common to` $\beta|\mathcal{C}_0|$ `sets from` $\mathcal{C}_0$.
$S_1 = S_0 \cup \{z_1\}$.
$\mathcal{C}_1 = \mathcal{C}_0 + \{\mathcal{C}_0 \setminus \beta(\mathcal{C}_0)\}$ `--repeat sets not hit by` z_1

In the $(i+1)$-th step:
`Pick` z_{i+1} `common to` $\beta|\mathcal{C}_i|$ `sets from` $\mathcal{C}_i$.
$S_{i+1} = S_i \cup \{z_{i+1}\}$
$\mathcal{C}_{i+1} = \mathcal{C}_i + \{\mathcal{C}_i \setminus \beta(\mathcal{C}_i)\}$ `--repeat sets not hit by` z_{i+1}

In the t-th iteration we will have: $|S_t| = t, |\mathcal{C}_t| = (2-\beta)^t \rho$, because $|C_{i+1}| \leq (2-\beta)|\mathcal{C}_i|$.
Claim: After t steps, with t large enough, $|S_t \cap C| \geq \gamma t, \forall C \in \mathcal{C}$.

Proof of claim: Assume $C \in \mathcal{C}$ is hit k times. Then it was doubled $t-k$ times. That means that 2^{t-k} copies of C are present.

$$\begin{aligned} 2^{t-k} &\leq |\mathcal{C}_t| \leq (2-\beta)^t n \Rightarrow \\ t-k &\leq t\log(2-\beta) + \log n \Rightarrow \\ k &\geq t(1-\log(2-\beta)) - \log n \end{aligned}$$

We solve for

$$t(1-\log(2-\beta)) - \log n = \log n \Rightarrow t = \frac{2\log n}{1-\log(2-\beta)}$$

and so:

$$k \geq \log n \Rightarrow \underbrace{k}_{|S_t \cap C|} \geq \underbrace{\frac{(1-\log(2-\beta))}{2}}_{\gamma} \ t$$

□

Order types and the Same-Type lemma. For completeness we reproduce the following definition which can also be found in [Mat02, Sec. 9].

Definition 8. *Let $p = (p_1, ..., p_n)$ a sequence of points in $\mathbb{R}^d$. The order type of p is defined as the mapping assigning to each $(d+1)$-tuple $(i_1, i_2, ..., i_{d+1})$ of indices, $1 < i_1 < i_2 < ... < i_{d+1} < n$, the orientation of the $(d+1)$-tuple $(p_{i_1}, p_{i_2}, ..., p_{i_{d+1}}))$. Thus, the order type of p can be described by a sequence of $+1$'s, -1's, and 0's with $\binom{n}{d+1}$ terms.*

Theorem 9 (Same-Type lemma)**.** *For any integers d, m there exists $c = c(d,m) > 0$ s.t. the following holds. Let $X_1, ..., X_m \subset \mathbb{R}^d$ finite sets in general position. There exist $Y_1 \subset X_1, ..., Y_m \subset X_m$ s.t. for $i = 1, ..., m$ $|Y_i| \geq c|X_i|$ and for $a_i, b_i \in Y_i$ the order type of $a_1, ..., a_m$ and $b_1, ..., b_m$ is the same.*

Proof. One can observe that it is sufficient to prove the theorem for $m = d+1$. Now list all non trivial partitions ($\emptyset$ excluded) of $[d+1]$:

$$(I_1, J_1), (I_2, J_2), ..., (I_{2^d-1}, J_{2^d-1}).$$

For every i we define a chain of sets $X_i = X_i^0 \supset X_i^1 \supset ... \supset X_i^{2^d-1} = Y_i$. For any $\alpha = 0, ..., 2^d - 1$ it holds $|X_i^\alpha| \geq \frac{1}{2}|X_i^{\alpha-1}|$.

For some α take a hyperplane H_α which halves $X_1^\alpha, ..., X_d^\alpha$ (ham-sandwich theorem) and define halfspace h by choosing the larger from $H_\alpha^+ \cap X_{d+1}^\alpha$, $H_\alpha^- \cap X_{d+1}^\alpha$. Wlog, we assume that $d+1 \in J_{\alpha+1}$. For all $i \in J_{\alpha+1}$ we discard all points of X_i not lying in h and for all $i \in I_{\alpha+1}$ we discard all points of X_i lying in h. This results in subsets of X_i's with at most half their elements. We follow the same procedure $2^d - 1$ times and we obtain $Y_1 = X_1^{2^d-1}, ..., Y_m = X_m^{2^d-1}$. Notice that $|Y_i| \geq 2^{2^d-1}|X_i|$.

We need to show that $a_1, ..., a_{d+1}$ and $b_1, ..., b_{d+1}$ have the same order type. If they do not have the same order type then $z_i = tx_i + (1-t)x_i$ $\forall i \in [d+1]$ all lie in a hyperplane H for some t. Now, $z_0, z_1, ..., z_{d+1}$ have a Radon partition $\{z_i \mid i \in I_\alpha\}$, $\{z_i \mid i \in J_\alpha\}$ $(d+1 \in J_\alpha)$. Obviously, $z_i \in \text{conv}(X_i^\alpha)$ and,

$$\text{conv}(\{z_i \mid i \in I_\alpha\}) \subset \text{conv}(\bigcup_{i \in I_\alpha} X_i^\alpha),$$

$$\text{conv}(\{z_j \mid j \in J_\alpha\}) \subset \text{conv}(\bigcup_{j \in J_\alpha} X_j^\alpha).$$

However, $\bigcup_{i \in I_\alpha} X_i^\alpha$ and $\bigcup_{j \in J_\alpha} X_j^\alpha$ are separated by a hyperplane H_α. □

We also present two related results without proving them.

Theorem 10 (Erdös-Szekeres theorem)**.** *Let $X \subset \mathbb{R}^2$ a finite set of points in general position. If $|X|$ is large enough then it contains the vertices of a convex n-gon.*

Theorem 11. *Let $X_1, X_2, ..., X_{d+1} \subset \mathbb{R}^d$ sets of points in general position and $(\forall i)|X_i| = N$. Then there exists $Y_i \subset X_i$ such that $|Y_i| \geq c(d)N$ and*

$$\bigcap_{\text{all transversals of } Y_i' s} \text{conv}(y_1, ..., y_{d+1}) \neq \emptyset.$$

References

[Mat02] Matousek J., Lectures on Discrete Geometry, 2002.

[BL92] Bárány I. and Larman D. G., A Colored Version of Tverberg's Theorem, Journal of the London Mathematical Society, 1992.