Localization and delocalization of eigenvectors for heavy-tailed random matrices

Charles Bordenave

CNRS & University of Toulouse

Joint work Alice Guionnet (CNRS & ENS Lyon)

Spectrum of random matrices

Let $X = (X_{ij})_{1 \leqslant i,j \leqslant n}$ be a $n \times n$ real symmetric matrix,

$$X_{ji} = X_{ij}$$

Let $\lambda_1, \cdots, \lambda_n$ be its eigenvalues : the empirical spectral distribution (ESD) of X is

$$\mu_X = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}.$$

Random symmetric matrix : the array $(X_{ij})_{i \geqslant j \geqslant 1}$ is iid.

Wigner's semi-circular law

Theorem (Wigner (1958))
If $\mathbb{E}X_{12} = 0, \mathbb{E}X_{12}^2 = 1$ *and*

$$A_n = X/\sqrt{n},$$

then, almost surely,

$$\mu_{A_n} \Longrightarrow \mu_{sc},$$

where

$$\mu_{sc} = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

supported on $[-2, 2]$.

Delocalization of eigenvectors

Let $(v_1(X), \cdots, v_n(X))$ be an orthogonal basis of eigenvectors of X.

For a vector $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$,

$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i| \in \left[n^{-\frac{1}{2}}, 1\right]$$

is a measure of delocalization of x.

$$x = e_k \quad \text{vs} \quad x = (1/\sqrt{n}, \cdots, 1/\sqrt{n}).$$

Delocalization of eigenvectors

Theorem (Erdős-Schlein-Yau (2009-2011))
Assume that $\mathbb{E}X_{12} = 0, \mathbb{E}X_{12}^2 = 1$ *and that for some* $\kappa > 0$, $\mathbb{E}\exp|X_{12}|^\kappa < \infty$. *Then,*

$$\max_{1 \leqslant k \leqslant n} \|v_k(X)\|_\infty \leqslant n^{-\frac{1}{2}}(\log n)^c$$

with overwhelming probability.

Heavy-tailed entries

We now assume that

$$\mathbb{P}(|X_{12}| > t) \sim t^{-\alpha}$$

for some

$$0 < \alpha < 2.$$

In this talk : X_{12} α-stable symmetric

$$\mathbb{E}\exp(itX_{12}) = \exp(-c|t|^{\alpha}).$$

Define

$$A_n = X/n^{1/\alpha}.$$

Some motivation

For any $\varepsilon > 0$,

$$\mathbb{P}\Big(|X_{12}| > n^{1/\alpha}\varepsilon\Big) \sim \frac{\varepsilon^{-\alpha}}{n}.$$

$\Longrightarrow$ On each row of $A_n = X/n^{1/\alpha}$, there are $O(1)$ entries of absolute value at least $\varepsilon > 0$.

$\Longrightarrow$ Entries of order 1 of A_n are the only relevant for the spectral properties.

Similar to the Anderson model in random Schrödinger operators.

Convergence of ESD

Theorem (Ben Arous & Guionnet (2008))
There exists a probability measure μ_{bc} *depending only on* α *such that almost surely,*

$$\mu_{A_n} \Longrightarrow \mu_{bc}.$$

First found non-rigorously by *Bouchaud-Cizeau (1994)*.

Alternative proof and a.s. convergence with *Caputo-Chafaï (2011)*, *Belinschi-Dembo-Guionnet (2009)*.

Properties of the limit measure

The probability measure μ_{bc}

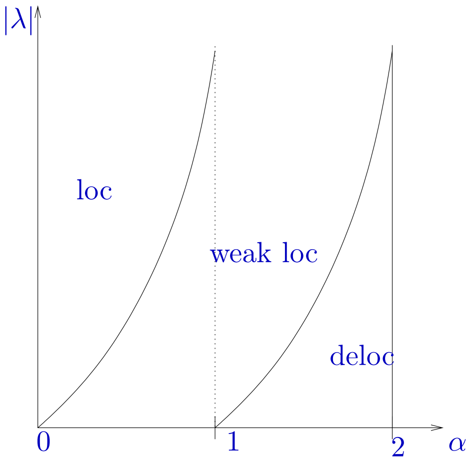
(i) is symmetric and has a bounded positive density f_{bc} on $\mathbb{R}$,

(ii) $f_{bc}(0) = \frac{1}{\pi}\Gamma\left(1+\frac{2}{\alpha}\right)\left(\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\right)^{\frac{1}{\alpha}}$,

(iii) $f_{bc}(t) \sim_{t\to\pm\infty} \frac{\alpha}{2}t^{-\alpha-1}$.

Eigenvectors : what Bouchaud-Cizeau predicted

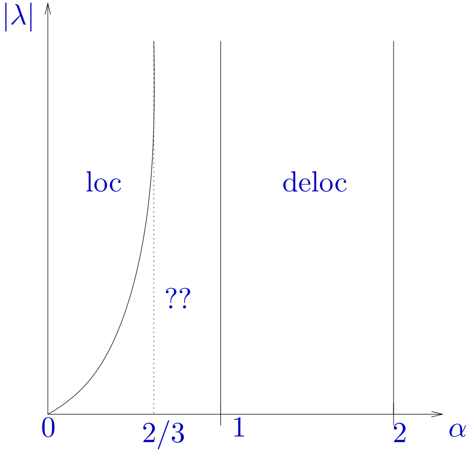
$$A_n v = \lambda v, \qquad \|v\|_2 = 1$$



deloc : $\|v\|_4 = o(1)$, loc : $\|v\|_1 = O(1)$.

Eigenvectors : what we prove

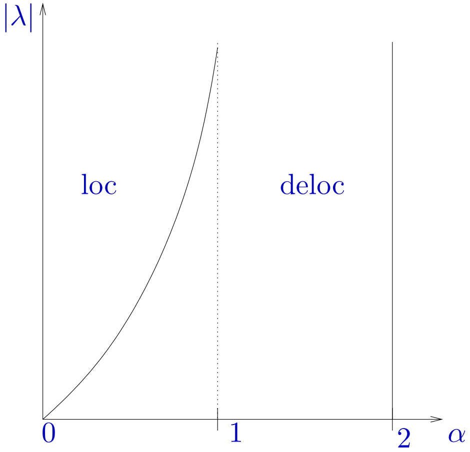
$$A_n v = \lambda v, \qquad \|v\|_2 = 1$$



deloc : $\|v\|_\infty \to 0$, loc : $\cdots$.

Eigenvectors : what should be true

$$A_n v = \lambda v, \qquad \|v\|_2 = 1$$



$$\text{deloc} : \|v\|_\infty \to 0 \quad , \qquad \text{loc} : \|v\|_1 = O(1).$$

First key ingredient : local law for ESD

Define

$$\rho = \begin{cases} \frac{1}{2} & \text{if} \quad \frac{8}{5} \leqslant \alpha < 2 \\ \frac{\alpha}{8-3\alpha} & \text{if} \quad 1 < \alpha < \frac{8}{5} \\ \frac{\alpha}{2+3\alpha} & \text{if} \quad 0 < \alpha \leqslant 1. \end{cases}$$

Theorem

There exists a finite set $\mathcal{E}_\alpha \subset \mathbb{R}$ such that if $I \subset \mathbb{R} \backslash \mathcal{E}_\alpha$ is an interval of length $|I| \geqslant n^{-\rho}(\log n)^c$, then

$$|\mu_{A_n}(I) - \mu_{bc}(I)| = o(|I|),$$

with overwhelming probability.

Take $\eta = n^{-\rho+o(1)}$ and

$$I = \left[E - \frac{\eta}{2}, E + \frac{\eta}{2}\right].$$

Since μ_{bc} has a positive density f_{bc} : it implies a lower bound

$$n\mu_{A_n}(I) = |\{1 \leqslant k \leqslant n : \lambda_k(A_n) \in I\}| \geqslant cn|I|.$$

Delocalization of eigenvectors

If $Av = \lambda v$, $\|v\|_2 = 1$, $\lambda \in I$ and

$$A = \begin{pmatrix} a_{11} & A_1^* \\ A_1 & B \end{pmatrix},$$

then

$$v_1^2 \quad \leqslant \quad \eta^2 \langle A_1, PA_1 \rangle^{-1}$$

where P is the projection on $H = \mathrm{span}\{v_k(B) : \lambda_k(B) \in I\}$,

$$\mathrm{rank}(P) = \dim(H) \geqslant cn|I| - 1.$$

Quadratic form of heavy-tailed vectors

For

$$X = (X_1, \cdots, X_n),$$

iid α-stable symmetric,

$$\langle X, PX \rangle \stackrel{d}{=} S \|P^{1/2} G\|_\alpha^2,$$

with S $\alpha/2$-stable positive, independent of $G \stackrel{d}{\sim} N(0, I_n)$.

(if X Gaussian vector, the law of $\langle X, PX \rangle$ depends only on the eigenvalues of P)

Delocalization of eigenvectors

Theorem
Let $1 < \alpha < 2$ *and a compact* $K \subset \mathbb{R} \backslash \mathcal{E}_\alpha$,

$$\max \{ \|v_k\|_\infty : \lambda_k(A_n) \in K \} \leqslant n^{-\rho(1-\frac{1}{\alpha})+o(1)},$$

with overwhelming probability.

$\Longrightarrow$ delocalization in L^p-norm, $p > 2$,

$$\|v\|_p \leqslant \|v\|_2^{2/p} \|v\|_\infty^{1-2/p}.$$

Localization of eigenvectors

$$\Lambda_I = \{v_k(A) : \lambda_k(A) \in I\}$$

$$|\Lambda_I| = n\mu_A(I).$$

Define

$$W_I(k) = \frac{1}{|\Lambda_I|} \sum_{v \in \Lambda_I} \langle v, e_k \rangle^2.$$

$\Longrightarrow W_I(k) =$ average amplitude of the k-th coordinate of eigenvectors in Λ_I.

$$\sum_{k=1}^{n} W_I(k) = 1.$$

$\Longrightarrow (W_I(k))_{1 \leqslant k \leqslant n} =$ distribution function of the amplitudes of eigenvectors in Λ_I.

Localization of eigenvectors

$$W_I(k) = \frac{1}{|\Lambda_I|} \sum_{v \in \Lambda_I} \langle v, e_k \rangle^2.$$

If eigenvectors in Λ_I are localized then for some k, $W_I(k) \gg 1/n$ while for most k, $W_I(k) \ll 1/n$.

e.g. if for $0 < \kappa < 1$,

$$\frac{1}{n^{1-\kappa}} \sum_{k=1}^{n} W_I(k)^\kappa \leqslant \varepsilon$$

then for some $J \subset \{1, \cdots, n\}$ with $|J| \leqslant n(\delta^{-1}\varepsilon)^{\frac{1}{1-\kappa}}$,

$$\sum_{k \in J} W_I(k) \geqslant 1 - \delta.$$

Localization of eigenvectors

$$\rho = \begin{cases} \frac{1}{2} & \text{if} \quad \frac{8}{5} \leqslant \alpha < 2 \\ \frac{\alpha}{8-3\alpha} & \text{if} \quad 1 < \alpha < \frac{8}{5} \\ \frac{\alpha}{2+3\alpha} & \text{if} \quad 0 < \alpha \leqslant 1. \end{cases}$$

Theorem

Let $0 < \alpha < 2/3$. There exists E_α such that for any interval $I \subset [-E_\alpha, E_\alpha]^c$ of length $|I| \geqslant n^{-\rho}(\log n)^2$,

$$\frac{1}{n^{1-\frac{\alpha}{2}}} \sum_{k=1}^{n} W_I(k)^{\frac{\alpha}{2}} \leqslant c|I|^{\frac{\alpha}{2}-\varepsilon},$$

with overwhelming probability.

($\varepsilon \to 0$ as $\mathrm{dist}(0, I) \to \infty$).

Resolvent matrix

For $z \in \mathbb{C}_+$, define the resolvent matrix

$$R(z) = (A_n - z)^{-1}.$$

$$R(z)_{kk} = \sum_{j=1}^{n} \frac{\langle v_j, e_k \rangle^2}{\lambda_j - z} \ \in \mathbb{C}_+.$$

$$\frac{1}{n}\mathrm{Tr} R(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_j - z} = g_{\mu_{A_n}}(z),$$

is the Cauchy-Stieltjes transform of the ESD.

From the resolvent matrix

For $z = E + i\eta$ and $I = [E - \eta, E + \eta]$.

In the local law we start from

$$\frac{\mu_{A_n}(I)}{|I|} \simeq \frac{1}{\pi}\frac{1}{n}\sum_{k=1}^{n} \mathfrak{Im} R(z)_{kk},$$

and for the localization from

$$\frac{1}{n^{1-\frac{\alpha}{2}}}\sum_{k=1}^{n} W_I(k)^{\frac{\alpha}{2}} \leqslant \frac{|I|}{\mu_{A_n}(I)}\frac{1}{n}\sum_{k=1}^{n} (\mathfrak{Im} R(z)_{kk})^{\frac{\alpha}{2}}.$$

From the resolvent

$z = E + i\eta$ and $I = [E - \eta, E + \eta]$.

For $\eta = n^{-\rho}(\log n)^2$, we prove that (local law)

$$\frac{1}{n}\sum_{k=1}^{n} \Im\mathfrak{m} R(z)_{kk} \simeq \pi f_{bc}(E) > 0,$$

but for $0 < \alpha < 2/3$ and $|E|$ large enough, (localization)

$$\frac{1}{n}\sum_{k=1}^{n} \left(\Im\mathfrak{m} R(z)_{kk}\right)^{\frac{\alpha}{2}} \leqslant c\eta^{\frac{\alpha}{2}+o(1)}.$$

From the resolvent

The resolvent matrix is bounded

$$\|R(z)\| \leqslant 1/(\Im\mathfrak{m} z)^{-1}.$$

From concentration inequalities, with overwhelming probability,

$$\frac{1}{n}\sum_{k=1}^{n} R(z)_{kk} \simeq \mathbb{E}R(z)_{11},$$

and

$$\frac{1}{n}\sum_{k=1}^{n} (\Im\mathfrak{m} R(z)_{kk})^{\frac{\alpha}{2}} \simeq \mathbb{E}(\Im\mathfrak{m} R(z)_{11})^{\frac{\alpha}{2}}.$$

Local operator convergence

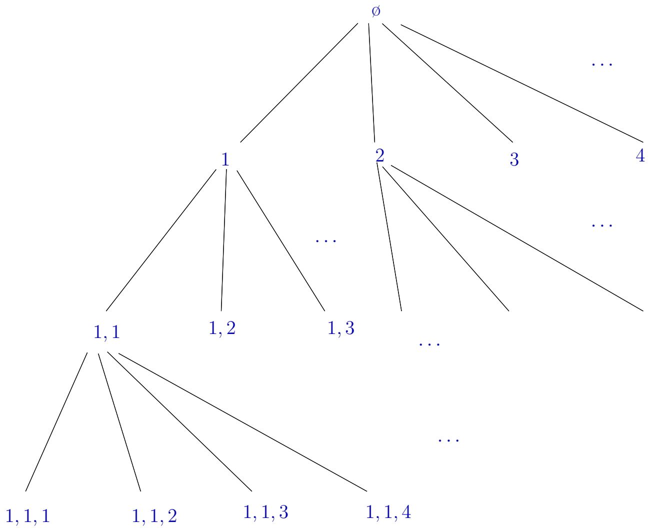
We need to understand the limit of $\mathbb{E}R(z)_{11}$ and $\mathbb{E}(\Im\mathfrak{m} R(z)_{11})^{\frac{\alpha}{2}}$.

The sequence of matrices (A_n) has a random limit operator *(with Caputo and Chafaï)*.

It is built on the Poisson Weighted Infinite Tree (Aldous 92).

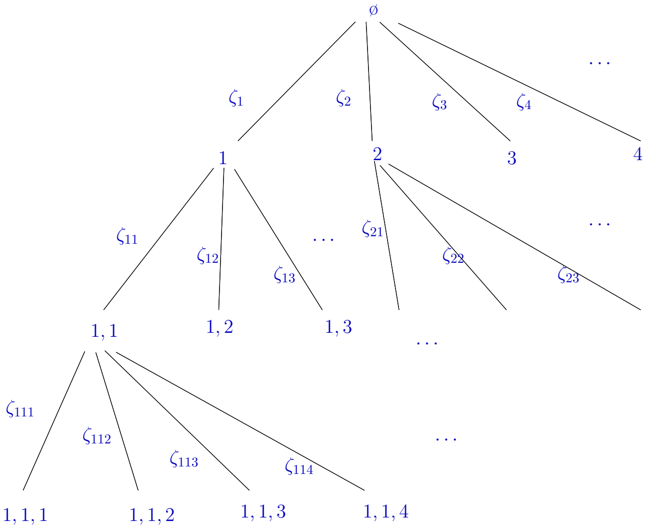
Aldous' PWIT

Set $V = \cup_{k \geqslant 0} \mathbb{N}^k$ with $\mathbb{N}^0 = \{\emptyset\}$, $\mathbb{N} = \{1, 2, \cdots\}$. Consider the infinite tree on V



Aldous' PWIT

Let Z_v be iid Poisson processes of intensity measure 1 on $\mathbb{R}_+$
$Z_v = \{0 \leqslant \zeta_{v1} \leqslant \zeta_{v1} \leqslant \cdots\}$.



Operator on the PWIT

Let $(\varepsilon_v)_{v \in V}$ be iid ± 1 Ber$(1/2)$ variables.

Define the operator on compactly supported function of $L^2(V)$,

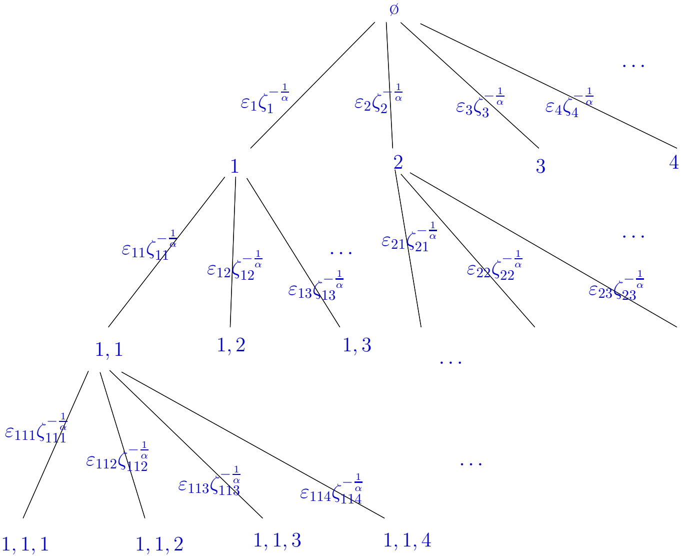
$$\mathbf{A} e_v = \sum_{k \geqslant 1} \varepsilon_{vk} \zeta_{vk}^{-1/\alpha} e_{vk} + \varepsilon_v \zeta_v^{-1/\alpha} e_{a(v)},$$

where $a(v)$ is the ancestor of $v \neq \varnothing$.

$$\sum_{k \geqslant 1} \zeta_{vk}^{-2/\alpha} < \infty \quad \text{implies that} \quad \mathbf{A} e_v \in L^2(V).$$

Moreover, with probability one, the operator $\mathbf{A}$ is essentially self-adjoint.

Operator on the PWIT



Local operator convergence

There exists a sequence of bijections $\sigma_n : V \to \mathbb{N}$, $\sigma_n(\emptyset) = 1$ and, for all $\phi \in L^2(V)$ with compact support, weakly,

$$\sigma_n^{-1} A_n \sigma_n \phi \to \mathbf{A}\phi.$$

For all $z \in \mathbb{C}_+$

$$R(z)_{11} \Rightarrow g(z) = \langle e_\emptyset, (\mathbf{A} - z)^{-1} e_\emptyset \rangle = \int \frac{\mu(dx)}{x - z}.$$

$\Longrightarrow$ μ is the spectral measure with vector $e_\emptyset$ of the operator $\mathbf{A}$.

Convergence of resolvent

For any $z \in \mathbb{C}_+$,

$$R(z)_{11} \Rightarrow g(z) = \langle e_\varnothing, (\mathbf{A} - z)^{-1} e_\varnothing \rangle = \int \frac{\mu(dx)}{x - z}.$$

We find

$$\mathbb{E}\mu = \mu_{bc} \quad \text{and} \quad \mathbb{E}g(z) = \int \frac{\mu_{bc}(dx)}{x - z}.$$

(In the Wigner case, i.e. $\mathbb{E}X_{12}^2 < \infty$, $\mu = \mu_{sc}$ *is not random!)*

Recursive Distributional Equation

The variable $g(z)$ satisfies the recursive distributional equation (RDE)

$$g \stackrel{d}{=} -\left(z + \sum_{k\geqslant 1} \xi_k g_k \right)^{-1},$$

where $g, (g_k)_{k\geqslant 1}$ iid independent of $\{\xi_k\}_{k\geqslant 1}$, a Poisson point processes of $\mathbb{R}_+$ with intensity

$$\frac{\alpha}{2} x^{-\frac{\alpha}{2}-1} dx.$$

$(\xi_k = \zeta_k^{-2/\alpha})$

Recursive Distributional Equation

$S = \sum_k \xi_k$ is a positive $\alpha/2$-stable random variable, and if $y_k \geqslant 0$ iid,

$$\sum_{k \in \geqslant 1} \xi_k y_k \stackrel{d}{=} \mathbb{E}[y^{\frac{\alpha}{2}}]^{\frac{2}{\alpha}} S.$$

$\Longrightarrow$ For any $\kappa > 0$,

$$\mathbb{E}[g(z)^\kappa] = \text{ function of } \mathbb{E}[g(z)^{\frac{\alpha}{2}}].$$

$$\mathbb{E}[g(z)^{\frac{\alpha}{2}}] = F\Big(\mathbb{E}[g(z)^{\frac{\alpha}{2}}]\Big) \quad \text{and} \quad \mathbb{E}[g(z)] = G\Big(\mathbb{E}[g(z)^{\frac{\alpha}{2}}]\Big).$$

Approximate fixed point equation

To prove the local law, we prove that

$$\mathbb{E}[R(z)_{11}^{\alpha/2}] \simeq F\Big(\mathbb{E}[R(z)_{11}^{\alpha/2}]\Big),$$

and

$$\mathbb{E}[R(z)_{11}] \simeq G\Big(\mathbb{E}[R(z)_{11}^{\alpha/2}]\Big),$$

+ use an implicit function theorem.

Localization on the RDE

Theorem

Let $0 < \alpha < 2/3$. *There exists* E_α *such that for all* $|z| \geqslant E_\alpha$, *the RDE*

$$g(z) \overset{d}{=} -\left(z + \sum_{k\geqslant 1} \xi_k g_k(z)\right)^{-1}$$

has a unique solution in $L^{\frac{\alpha}{2}}$. *This solution satisfies*

$$\mathbb{E}(\mathfrak{Im} g(E + i\eta))^{\frac{\alpha}{2}} \leqslant c\eta^{\frac{\alpha}{2} - \varepsilon}.$$

$\Longrightarrow$ Moments $\mathbb{E}[g(z)^\kappa]$ are not enough. The characteristic function of g is function of

$$\gamma : (u, v) \mapsto \mathbb{E}(ug + v\bar{g})^{\frac{\alpha}{2}}, \qquad u^2 + v^2 = 1, u, v \geqslant 0.$$

Localization on the RDE

Recall

$$g(z) = \int \frac{d\mu}{x - z}.$$

and

$$\mathbb{E}\mu = \mu_{bc}.$$

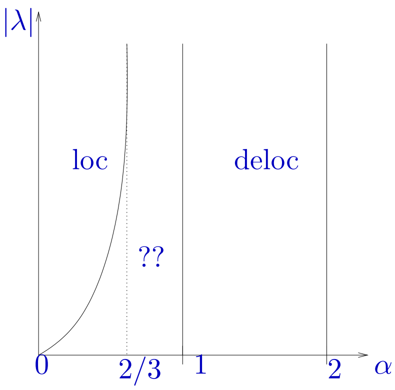
With probability 1, the absolutely continuous part of μ is contained in $[-E_\alpha, E_\alpha]$

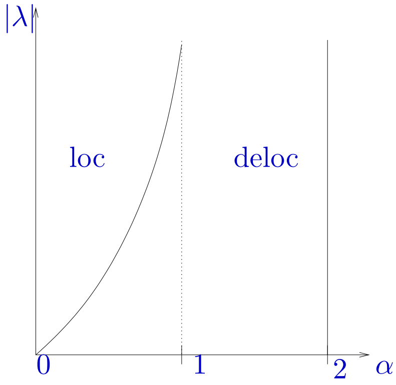
but

$\mathbb{E}\mu = \mu_{bc}$ is absolutely continuous and support $\mathbb{R}$.

Final remarks

$$A_n v = \lambda v, \qquad \|v\|_2 = 1$$





Final remarks

For the largest eigenvalues of X (of order $n^{2/\alpha}$) : *Soshnikov (2004), Auffinger-Ben Arous-Péché (2009).*

Final remarks

Localization/delocalization in other models : adjacency matrix of random graphs.

Erdős-Rényi graphs : show that for all $c > c_*$, κn *eigenvectors of the adjacency matrix of* $G(n, c/n)$ *are strongly delocalized.*

Uniform regular graphs : show that for any $d \geqslant 3$, $n - o(n)$ *eigenvectors of the adjacency matrix of* $G(n, d)$ *are strongly delocalized.*

For $c \gg 1$ and $d \gg 1$, *Pal-Dumitriu (2010), Tran-Vu-Wang (2012), Erdős-Knowles-Yau-Yin (2012).*