Hardness of Grothendieck Problems

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The Grothendieck Inequality

Given a matrix $M \in \mathbb{R}^{d \times d}$ (bilinear form on ℓ_∞) let

$$\mathrm{OPT}(M) = \max_{\alpha_i, \beta_j \in \{\pm 1\}} \sum M_{ij} \alpha_i \beta_j$$

$$\mathrm{SDP}(M) = \sup_{\|a_i\|_{\ell_2} = \|b_j\|_{\ell_2} = 1} \sum M_{ij} \langle a_i, b_j \rangle$$

Thm. (Grothendieck '53)

There exists an absolute constant K_G such that for any M,

$$\mathrm{OPT}(M) \leq \mathrm{SDP}(M) \leq K_G \, \mathrm{OPT}(M)$$

$K_G \leq 1.78\ldots$ (Krivine '79) $-\ \varepsilon$ (Braverman et al. '12)

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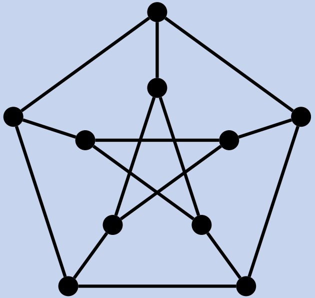
Maximum Cut Problem

Given a graph $G = (V, E)$, compute the maximum number of edges crossing a bi-partition of V:



$L = \text{Deg}_G - \text{Adj}_G$ (Laplacian)

$\text{MAX CUT(G)} = \frac{1}{4}\,\text{OPT}(L)$

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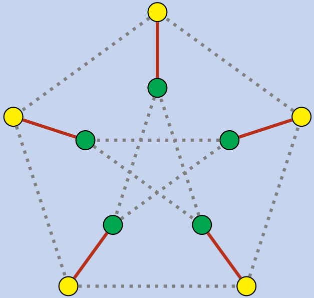

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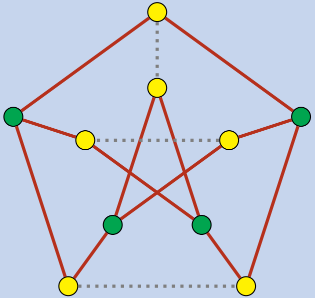

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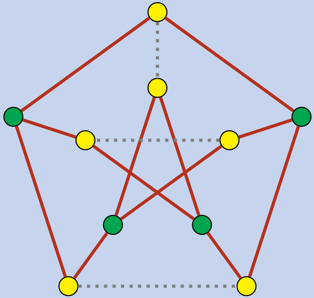

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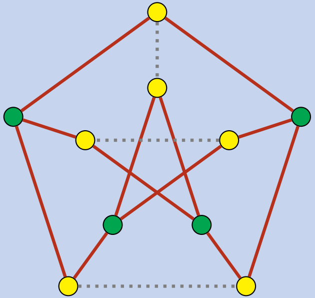

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- Computing OPT(M) is interesting!
  - ⋆ MAX CUT.
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  - ⋆ Statistical physics; ground state energy in spin glasses.

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- Better approximation algorithms for OPT(M)?
- Probably not if Khot's *Unique Games Conjecture* holds.

Theorem (Khot–O'Donnell '09), (Raghavendra–Steurer '09).

If the UGC is true and P$\neq$NP, then for any $\varepsilon > 0$, there is **no** efficient algo. that approximates OPT(M) to within a factor $K_G - \varepsilon$.

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Given a tensor $T \in \mathbb{R}^{d \times d \times d \times d}$ (bilinear form on matrices) let

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Hardness of the *Little* NCG

- We prove our main result by proving something stronger: optimal hardness for a special case.

Theorem 2 (B–Regev–Saket).

If P≠NP, then for any $\varepsilon > 0$, there is no efficient $(\sqrt{2} - \varepsilon)$-approx. algo. for the *Little NCG*.

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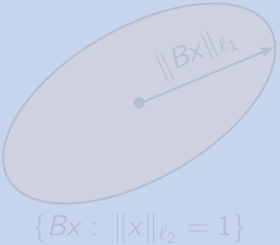

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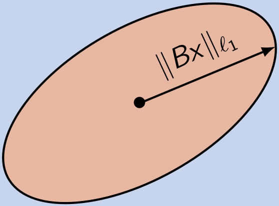

$\{Bx : \|x\|_{\ell_2} = 1\}$

Why is this a special case?

The Little Groth. Problem.

$$\begin{aligned}\|B\|_{\ell_2\to\ell_1} &:= \max_{\|x\|_{\ell_2}=1} \|Bx\|_{\ell_1} \\ &= \max_{\|x\|_{\ell_2}=\|y\|_{\ell_\infty}=1} y^\mathsf{T} Bx \\ &= \max_{\|y\|_{\ell_\infty}=1} \|y^\mathsf{T} B\|_{\ell_2} \qquad \|B\|_{\ell_2\to\ell_1} = \|B^*\|_{\ell_\infty\to\ell_2} \\ &= \max_{\|y\|_{\ell_\infty}=1} \sqrt{y^\mathsf{T} BB^\mathsf{T} y} \\ &= \max_{\|y\|_{\ell_\infty}=\|z\|_{\ell_\infty}=1} \sqrt{y^\mathsf{T} BB^\mathsf{T} z} \quad \text{Cauchy–Schwarz}\end{aligned}$$

Hence, for $M = BB^\mathsf{T}$, we have $\|B\|^2_{\ell_2\to\ell_1} = \mathrm{OPT}(M)$.

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Hence, for $M = BB^\top$, we have $\|B\|^2_{\ell_2\to\ell_1} = \mathrm{OPT}(M)$.

Why is this a special case?

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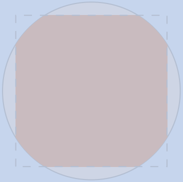

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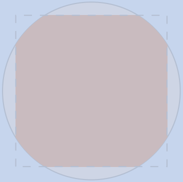

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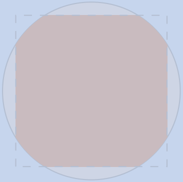

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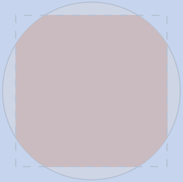

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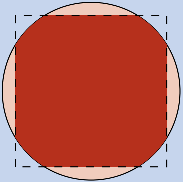

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Define $f : \mathbb{R}^n \to L_1(\{-1,1\}^n)$ by

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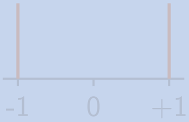

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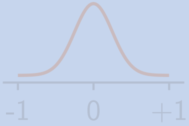

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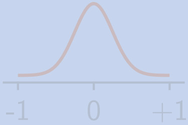

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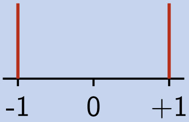

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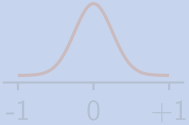

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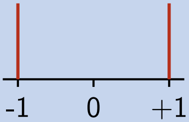

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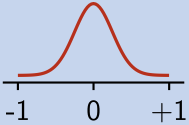

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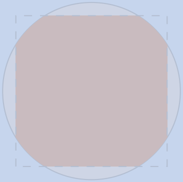

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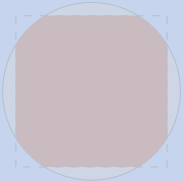

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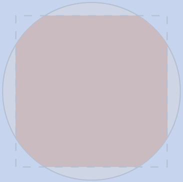

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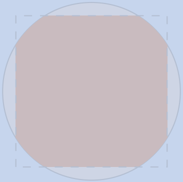

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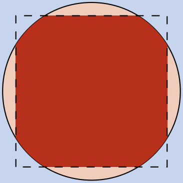

Attempt 1: Random unitaries

- The matrix analogue of Bernoullis are random unitaries.
- So a natural guess for the function f would be

$$f(a)(U_1, \ldots, U_n) = a_1 U_1 + \cdots + a_n U_n.$$

- Then, $\|f(e_i)\|_{S_1} = 1$. ✓
- But if $a \in \mathbb{C}^n$ has no large coordinates then $f(a)$ is roughly a random Gaussian matrix G.
- By Wigner's semi-circle law, $\mathbb{E}[\|G\|_{S_1}] \approx 0.848 > 1/\sqrt{2}$. ✗

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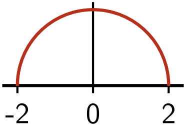

The Haagerup–Itoh example

- Integrality-gap examples often provide useful hints for hardness constructions.
- Haagerup–Itoh use an *isometric embedding* $f : \mathbb{C}^n \to S_1$ based on the generators of the C.A.R. algebra.
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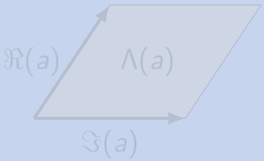
The Magic Parallelogram Lemma

For $a \in \mathbb{C}^n$, the map $f(a) = a_1 C_1 + \cdots + a_n C_n$ satisfies

$$\|f(a)\|_{S_1} = \frac{1}{2}\sqrt{\|a\|_{\ell_2}^2 + 2\Lambda(a)} + \frac{1}{2}\sqrt{\|a\|_{\ell_2}^2 - 2\Lambda(a)}$$



$\Lambda(e_i) = 0 \quad \Rightarrow \quad \|f(e_i)\|_{S_1} = 1$ ✓

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For $a \in \mathbb{C}^n$, the map $f(a) = a_1 C_1 + \cdots + a_n C_n$ satisfies

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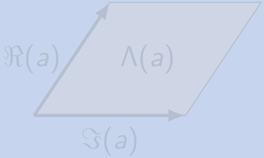

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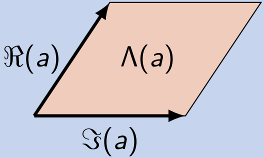

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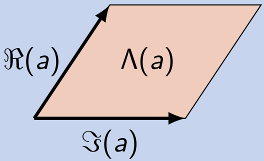

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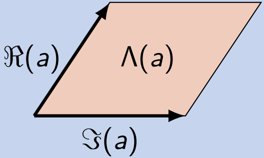

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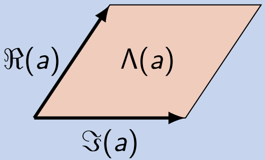

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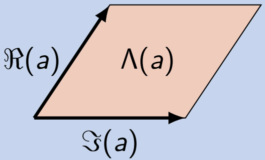

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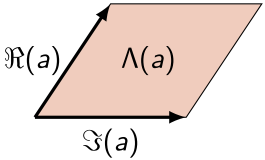

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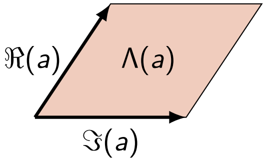

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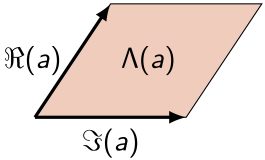

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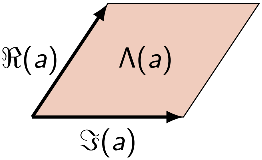

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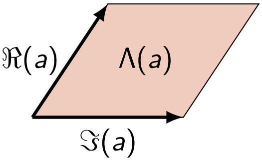

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For any $\varepsilon > 0$, given a tensor $T \in \mathbb{R}^{d\times d\times d\times d}$, it is NP-hard to approximate the value

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to within a factor $2-\varepsilon$. Moreover, this is optimal.

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