On the connection between the Thin-Shell conjecture and the KLS Conjecture.

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Phenomena in high dimensions, Roscoff

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Isotropic log-concave measures

- A probability density $\rho : \mathbb{R}^n \to [0, \infty)$ is called *log-concave* if it takes the form $\rho = \exp(-H)$ for a convex function $H : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. A probability measure is log concave if its density is log concave.
- Typical examples: the uniform measure on a convex body, a Gaussian measure.
- For a probability measure μ on $\mathbb{R}^n$, define

$$b(\mu) = \int_{\mathbb{R}^n} x d\mu(x), \qquad Cov(\mu) = \int_{\mathbb{R}^n} (x - b(\mu)) \otimes (x - b(\mu)) d\mu(x),$$

a probability measure μ is called *isotropic* if its barycenter lies at the origin and its covariance matrix is the identity matrix.

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Isoperimetry for log-concave measures

- Define for a measurable $A \subset \mathbb{R}^n$,

$$\mu^+(A) = \liminf_{\varepsilon \to 0^+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}$$

  where $A_\varepsilon := \{x \in \mathbb{R}^n; \ \ \exists y, \ |x - y| \leq \varepsilon\}$.

- If μ is isotropic and log-concave and $H \subset \mathbb{R}^n$ is an $(n-1)$-dimensional subspace, then

$$C^{-1} \leq \mu^+(H) \leq C$$

  for some universal constant $C > 0$.

- Define the constant G_n by the equation,

$$G_n^{-1} = \inf_{\mu, T} \mu^+(T)$$

  where the infimum is taken over all isotropic log-concave probability measures μ and measurable sets satisfying $\mu(T) = \frac{1}{2}$.

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The KLS conjecture

- As we have seen above (above = 2 days ago), by works of Buser, Cheeger, Gromov-V.Milman, Ledoux and E.Milman the constant G_n has several equivalent definitions related to the optimal Cheeger constant, spectral gap, exponential concentration and first moment concentration a log concave measure.

Conjecture(Kannan-Lovász-Simonovits)

There exists a universal constant $C > 0$ such that $G_n < C$.

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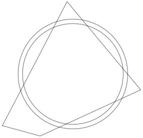
Thin-shell concentration

- Observe that when X is an isotropic random vector, $\mathbb{E}\left[|X|^2\right] = \sum_{i=1}^n \mathbb{E}\left[X_i^2\right] = n$.
- Define,

$$\sigma_n = \sup_X \sqrt{Var\left[|X|\right]}$$

where X runs over all isotropic log-concave random vectors.
- It turns out that,



$$\sqrt{Var[|X|]} \leq \sigma_n \ll \mathbb{E}[|X|]$$

These are usually referred to as *thin-shell* estimates. The first result of this kind was proved by B.Klartag. Several alternative proofs and improvements were introduced by Fleury, Guédon, E.Milman and Paouris. The currently best known bound is due to Guédon-Milman who show that $\sigma_n \leq Cn^{1/3}$.

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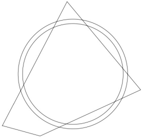

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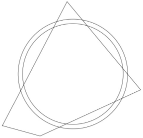

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There exists a universal constant $C > 0$ such that $\sigma_n < C$.

- By a theorem of Cheeger, when μ is isotropic, for any smooth function φ,

$$\int_{\mathbb{R}^n} \left(\varphi - \int \varphi d\mu \right)^2 d\mu \leq C G_n^2 \int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu$$

- By taking $\varphi(x) = |x|^2$, we see that $\sigma_n \leq C G_n$. In particular, the KLS conjecture implies the thin-shell conjecture.
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The connection between the two conjectures

- Right, so KLS implies Thin-Shell. What about the other way around?

Theorem (Bobkov)

For any log-concave random vector X and any smooth function φ, one has

$$\frac{Var[\varphi(X)]}{\mathbb{E}\left[|\nabla\varphi(X)|^2\right]} \leq C\mathbb{E}[|X|]\sqrt{Var[|X|]}.$$

- The theorem implies $G_n \leq Cn^{1/4}\sqrt{\sigma_n}$.
- Under the *thin-shell* hypothesis, we would attain $G_n \leq Cn^{1/4}$.

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There exists a universal constant $C > 0$ such that,

$$G_n \leq C\sigma_n \log n$$

- In fact, we prove something slightly stronger: Up to a factor of $\sqrt{\log n}$, the worst isoperimetric sets are ellipsoids.
- Along with the thin-shell bound by Guédon-Milman, we get $G_n \leq Cn^{1/3} \log n$.
- Our estimate does not work body-wise, hence, in order to obtain an isoperimetric inequality for a specific convex body, one needs a thin-shell bound for an entire family of convex bodies.
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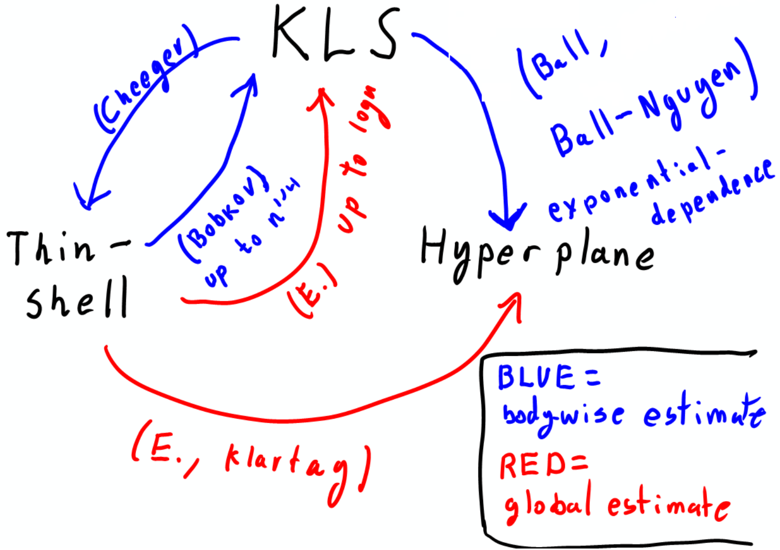
So what implies what, exactly?



Playing Fruit-Ninja with convex sets

- Let $K \subset \mathbb{R}^n$ be an isotropic convex body and $T \subset K$ with $\frac{Vol(T)}{Vol(K)} = \frac{1}{2}$.
  Let us consider the following iterative process:
- We take a random direction θ_1, uniformly distributed in S^{n-1} and define,

$$K_1 = K \cap \{\langle x, \theta \rangle \geq 0\}$$

  Denote, $b_1 = b(K_1), A_1 = Cov(K_1)$.
- Generate another random direction θ_1, uniformly distributed in S^{n-1} and define,

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  And so on.

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- Define,

$$f(\theta_1) = \frac{Vol(T \cap K_1)}{Vol(K_1)}$$

  it is easy to check that $f(\theta_1)$ is C-Lipschitz for some universal constant $C > 0$.
- By well-known concentration properties of the sphere, $f(\theta_1)$ is concentrated around the value $\frac{1}{2}$. Its standard deviation is of the order $\frac{1}{\sqrt{n}}$.
- Even if we iterate the process n times, the relative volume of T will remain between 0.4 and 0.6 with a non-negligible probability.
- What does K_n look like?

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Localization via multiplying by linear functions

Instead of truncating an entire halfspace in every step, consider the following idea:

- Start with f_0, defined as the density of the uniform measure on K. Fix some small $\epsilon > 0$.
- Again, we take a random direction θ_1, uniformly distributed in S^{n-1} and define,

$$f_1(x) = f_0(x)(1 + \epsilon\langle x, \theta\rangle)$$

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How is this better?

- Suppose, for a minute, that the two first random directions that were generated were $+x_1, -x_1$. Then by the second step, we have multiplied the original density by:
$$(1+\epsilon x_1)(c_0 - \epsilon c_1 x_1) = c_0 + \epsilon(1-c_1)x_1 - \epsilon^2 c_1 x_1^2$$
for some constants c_0, c_1.
- After roughly $N = 1/\epsilon^2$ such iterations, the function will have been multiplied by,
$$C \prod_{j=1}^{N} (1 + \epsilon(1-c_j)x_1 - \epsilon^2 c_j x_1^2) \simeq C e^{d_1 x_1 - d_2 x_1^2}$$
for some constants C, c_j, d_1, d_2.
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Localization via multiplying by linear functions - continued

How is this better?

- Suppose, for a minute, that the two first random directions that were generated were $+x_1, -x_1$. Then by the second step, we have multiplied the original density by:

$$(1+\epsilon x_1)(c_0 - \epsilon c_1 x_1) = c_0 + \epsilon(1-c_1)x_1 - \epsilon^2 c_1 x_1^2$$

  for some constants c_0, c_1.
- After roughly $N = 1/\epsilon^2$ such iterations, the function will have been multiplied by,

$$C\prod_{j=1}^{N}(1+\epsilon(1-c_j)x_1 - \epsilon^2 c_j x_1^2) \simeq Ce^{d_1 x_1 - d_2 x_1^2}$$

  for some constants C, c_j, d_1, d_2.
- The function approximately takes the form $f_0 e^{\langle v,x\rangle - \langle Ax,x\rangle^2}$ for some positive definite martix A.

The continuous version of the localization

- We begin with an isotropic log-concave density $f(x)$.
- We construct a 1-parameter family of functions f_t, in the following way:
  Let W_t be a standard Wiener process in $\mathbb{R}^n$. Define the process $F_t(x)$ by the equations:
  $$F_0(x) = 1, \quad dF_t(x) = \langle dW_t, A_t^{-1/2}(x - a_t)\rangle F_t(x) \qquad (1)$$
  where,
  $$a_t = \int_{\mathbb{R}^n} x f(x) F_t(x) dx$$
  is the barycenter of fF_t, and,
  $$A_t = \int_{\mathbb{R}^n} (x - a_t) \otimes (x - a_t) f(x) F_t(x) dx$$
  is its covariance matrix.
- Finally, we write $f_t(x) = f(x)F_t(x)$.

Some basic properties of the localization

- The function f_t is well defined, finite and non-negative for all $t > 0$.
- For all $x \in \mathbb{R}^n$, $f_t(x)$ is a martingale. Consequently, for all $E \subset \mathbb{R}^n$ and $t > 0$,

$$\mathbb{E}[f_t^+(E)] = f^+(E)$$

- With probability 1,

$$\int_{\mathbb{R}^n} f_t(x)dx = 1, \quad \forall t > 0$$

- It has a semigroup property: running the process for t seconds and then another s seconds is like running it for $t + s$ seconds.

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Log-concavity is preserved

- By Itö's formula,

$$d\log F_t(x) = \frac{dF_t(x)}{F_t(x)} - \frac{\left|A_t^{-1/2}(x-a_t)\right|^2 F_t(x)^2}{2F_t(x)^2}dt = \qquad (2)$$

$$= \left\langle dW_t, A_t^{-1/2}(x-a_t)\right\rangle - \frac{1}{2}\left|A_t^{-1/2}(x-a_t)\right|^2 dt$$

- This shows that F_t is actually of the form:

$$F_t(x) = C_t \exp\left(\langle x, c_t\rangle - \frac{1}{2}|B_t x|^2\right)$$

for some $C_t > 0$, $c_t \in \mathbb{R}^n$, where $B_t^2 := \int_0^t A_s^{-1} ds$.

- By results of Buser, Brascamp-Lieb, Gross, Bakry-Emery, Ledoux etc., whenever B_t is larger than some multiple of the identity, the function f_t enjoys even nicer concentration properties.

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How does the hypercontractivity help us?

- Let μ_t be the measure whose density is f_t.
- The last equation along with, for instance, the result of Bakry-Emery, shows that if $||A_t||_{OP} \leq \alpha^2$ for $t \in [0, T]$, then the function f_t will satisfy,

$$\mu_t^+(E) \geq \frac{CT}{\alpha}$$

whenever $0.1 \leq \mu_t(E) \leq 0.9$.

- Our goal is, therefore, to ensure that $||A_t||_{OP}$ remains small for long enough.

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- Our goal is, therefore, to ensure that $||A_t||_{OP}$ remains small for long enough.

Analyzing the eigenvalues of A_t

- Fix a time $t > 0$ and choose a basis such that A_t is diagonal.
- Denote the entries of A_t by $\alpha_{i,j}$.

We have (omitting some not-so-important terms),

$$d\alpha_{i,j} = d\int_{\mathbb{R}^n} x_i x_j f_t(x + a_t)dx \simeq$$

$$\int_{\mathbb{R}^n} x_i x_j df_t(x + a_t)dx = \int_{\mathbb{R}^n} x_i x_j \langle x, A_t^{-1/2} dW_t\rangle f_t(x + a_t)dx =$$

$$\sqrt{\lambda_i \lambda_j} \int_{\mathbb{R}^n} x_i x_j \langle x, dW_t\rangle \sqrt{\det A_t} f_t(A_t^{1/2}(x + a_t))dx$$

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- We conclude,

$$d\alpha_{i,j} = \sqrt{\lambda_i \lambda_j} \langle \xi_{i,j}, dW_t \rangle$$

where,

$$\xi_{i,j} = \int_{\mathbb{R}^n} x_i x_j x \tilde{f}_t(x) dx$$

with $\tilde{f}_t$ isotropic.

- By differentiating the eigenvalues with respect to the entries of the matrix, we get,

$$d\lambda_i = \langle \lambda_i \xi_{i,i}, dW_t \rangle + \sum_{j=1, j \neq i}^{n} \lambda_i \lambda_j \frac{|\xi_{i,j}|^2}{\lambda_i - \lambda_j} dt.$$

- The diagonal terms contribute to the quadratic variation of the respective eigenvalues. While the off-diagonal terms contribute to the repulsion between eigenvalues.

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Controlling the norm of the covariance matrix

- The maximal eigenvalue is then controlled by analyzing the Itö process,

$$S_t = \sum_{i=1}^{n} \lambda_i^{\log n}.$$

- It turns out that the repulsion between eigenvalues is typically more significant than the martingale components.
- It comes down to bounding the expression,

$$\sum_{j=1}^{n} |\xi_{i,j}|^2 = \left\| \int_{\mathbb{R}^n} x \otimes x x_i \tilde{f}(x) dx \right\|_{HS}^2$$

- The above expression can be regarded as the Hilbert-Schmidt norm of the difference between the covariance matrices corresponding to the two "halves" of a convex body, separated by the hyperplane $\{x_i = 0\}$.

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Merci à tous,
Felicitations, Alain,
Bon courage, Espagne,
Et vive la France.