Reductions of the slicing problem and related questions

Roscoff 2012

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Log-concave measures

1. A measure μ on $\mathbb{R}^n$ is called log-concave if

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$$

   for any Borel subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$.

2. The barycenter of μ is

$$\mathrm{bar}(\mu) := \frac{\int_{\mathbb{R}^n} x d\mu(x)}{\int_{\mathbb{R}^n} d\mu(x)}.$$

   We say that μ is centered if $\mathrm{bar}(\mu) = 0$.

3. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if $\log f$ is concave on its support $\{f > 0\}$.
   If μ is a log-concave probability measure and $\mu(H) < 1$ for every hyperplane H of $\mathbb{R}^n$, then μ has a log-concave density f_μ.

Isotropic Constant

Covariance matrix

$\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx}.$$

We set

$$\|\mu\|_\infty := \sup_{x \in \mathbb{R}^n} f_\mu(x).$$

Isotropic constant

The isotropic constant of μ is

$$L_\mu := \left(\frac{\|\mu\|_\infty}{\int_{\mathbb{R}^n} f_\mu(x) dx} \right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

Isotropic Measures

Isotropic measure

We say that μ is isotropic (and we write $\mu \in \mathcal{IL}_{[n]}$) if it is centered, $\mu(\mathbb{R}^n) = 1$ and $\mathrm{Cov}(\mu)$ is the identity matrix. Then,

$$L_\mu = \|\mu\|_\infty^{1/n}.$$

- If K is a convex body in $\mathbb{R}^n$, then the Brunn-Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure. We denote this by μ_K.
- Usually, we say that K is isotropic if it is centered, it has volume 1 and $\mathrm{Cov}(\mathbf{1}_K) = L_K^2 \mathrm{Id_n}$ for some constant $L_K > 0$. Thus, K is isotropic if the measure with density $L_K^n \mathbf{1}_{\frac{1}{L_K}K}$ is isotropic.
- The definition of the isotropic constant L_K of K is consistent with the definition of L_{μ_K}.

Marginals

Marginal

The marginal of μ with respect to $F \in G_{n,k}$ is defined by

$$\pi_F(\mu)(A) := \mu(P_F^{-1}(A)) = \mu(A + F^{\perp})$$

for all Borel subsets of F. The density of $\pi_F \mu$ is the function

$$f_{\pi_F \mu}(x) = \int_{x+F^{\perp}} f_\mu(y)dy, \qquad x \in F.$$

If μ is centered, log-concave or isotropic, then $\pi_F \mu$ is respectively also centered, log-concave or isotropic. In particular, if μ is isotropic then

$$\det \mathrm{Cov}(\pi_F \mu) = \det \mathrm{Cov}(\mu) = 1$$

for every $1 \leq k \leq n$ and every $F \in G_{n,k}$.

Hyperplane conjecture

We set

$$L_n = \max\{L_\mu : \mu \in \mathcal{IL}_{[n]}\}.$$

Hyperplane conjecture

There exists an absolute constant $C > 0$ such that $L_n \leq C$ for all $n \geq 1$.

Classical reference: Milman and Pajor.

Estimates

1. Bourgain, 1990: $L_n \leq C\sqrt[4]{n}\log n$.
2. Klartag, 2005: $L_n \leq C\sqrt[4]{n}$.

L_q-centroid bodies

L_q-centroid bodies

If μ is a probability measure on $\mathbb{R}^n$, the L_q-centroid body $Z_q(\mu)$, $q \geq 1$, is the symmetric convex body with support function

$$h_{Z_q(\mu)}(y) := \left(\int |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}, \qquad y \in \mathbb{R}^n.$$

- μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$.
- From Hölder's inequality it follows that $Z_1(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leq p \leq q < \infty$.
- From Borell's lemma, $Z_q(\mu) \subseteq c\frac{q}{p} Z_p(\mu)$ for all $1 \leq p < q$.
- If μ is isotropic, then $R(Z_q(\mu)) := \max\{h_{Z_q(\mu)}(\theta) : \theta \in S^{n-1}\} \leq cq$.

A formula for L_μ

Theorem (Paouris, $\sim$ 2007)

If μ is a centered and log-concave probability measure in $\mathbb{R}^n$, then

$$[f_\mu(0)]^{1/n} \cdot |Z_n(\mu)|^{1/n} \simeq 1.$$

In particular, if μ is isotropic then

$$L_\mu \cdot |Z_n(\mu)|^{1/n} \simeq 1.$$

Consequence

For every $q \leq n$,

$$L_\mu \leq \frac{C}{|Z_q(\mu)|^{1/n}}.$$

Basic parameters

$I_q(\mu)$

The quantity $I_q(\mu)$ is defined for every $q \in (-n, \infty)$, $q \neq 0$, by

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q f(x) dx \right)^{1/q}.$$

Note that if μ is isotropic then $I_2(\mu) = \sqrt{n}$.

q-width

If C is a convex body in $\mathbb{R}^n$ then, for each $-\infty < q < \infty$, $q \neq 0$, we define the q-width of C by

$$w_q(C) := \left(\int_{S^{n-1}} h_C^q(\theta) \, d\sigma(\theta) \right)^{1/q}.$$

Note that $w_1(C) = w(C)$ is the mean width of C.

$I_{\pm k}(\mu)$ and $w_{\pm k}(Z_k(\mu))$

Formula (Alesker, 1995)

For every positive integer $k \leq n-1$,

$$I_k(\mu) \simeq \sqrt{\frac{n}{k}} w_k\big(Z_k(\mu)\big).$$

Formula (Paouris, 2008)

For every positive integer $k \leq n-1$,

$$I_{-k}(\mu) \simeq \sqrt{\frac{n}{k}} w_{-k}\big(Z_k(\mu)\big).$$

Isotropic case

$q_*(\mu)$

The parameter $q_*(\mu) := \sup\{1 \leq q \leq n : k_*(Z_q(\mu)) \geq q\}$ was introduced by Paouris, $\sim$ 2002.

Theorem (Paouris, 2002-12)

- One has $I_q(\mu) \simeq I_2(\mu) = \sqrt{n}$ for all $|q| \leq q_*(\mu)$.
- If μ is isotropic, then $q_*(\mu) \geq c\sqrt{n}$.

Paouris, 2005

The "positive part" of the previous theorem leads to the inequality

$$\mu(\{x : \|x\|_2 \geq ct\sqrt{n}\}) \leq \exp(-Ct\sqrt{n})$$

for all $t \geq 1$.

First Reduction: DP

Dafnis and Paouris, 2009

For every n-dimensional isotropic log-concave measure μ and every $\delta \geq 1$, set

$$q_{-c}(\mu, \delta) := \max\{1 \leq p \leq n-1 : I_{-p}(\mu) \geq \delta^{-1} I_2(\mu) = \delta^{-1}\sqrt{n}\}.$$

Note that $q_{-c}(\mu, \delta) \geq q_*(\mu) \geq c\sqrt{n}$ in the isotropic case.

Main result

For every $\delta \geq 1$,

$$L_n \leq C\delta \sup_{\mu \in \mathcal{IL}_{[n]}} \sqrt{\frac{n}{q_{-c}(\mu, \delta)}} \log\left(\frac{en}{q_{-c}(\mu, \delta)}\right),$$

where $C > 0$ is an absolute constant.

Main tools for DP

Regular bodies with maximal isotropic constant

There exist absolute constants κ, τ and $\zeta > 0$ such that, for every $n \in \mathbb{N}$ there exists an isotropic convex body K_0 in $\mathbb{R}^n$ such that $L_{K_0} \geq \zeta L_n$, and

$$\log N(K_0, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2} \text{ for all } t \geq \tau\sqrt{n \log n}.$$

One can also guarantee that $R(K_0) \leq \gamma\sqrt{n} L_{K_0}$, where $\gamma > 0$ is an absolute constant.

Covering numbers and $I_{-q}(K)$

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. If $r_s := \log N(K, sB_2^n) < n$ for some $s > 0$ then

$$I_{-r_s}(K) \leq 3es. \tag{1}$$

The proof of DP (more or less)

- It is enough to consider the class of convex bodies.
- Let K_0 be a body with $L_{K_0} \simeq L_n$ and

$$\log N(K_0, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2} \text{ for all } t \geq \tau\sqrt{n \log n}.$$

- Choose $t = \sqrt{\frac{n}{q_{-c}(K_0,\delta)}}\sqrt{\kappa}\sqrt{n}\log n$.
- Then,

$$\log N(K_0, tB_2^n) \leq q_{-c}(K_0, \delta),$$

and hence

$$\begin{aligned}\sqrt{n}L_{K_0} &\leq \delta I_{-q_{-c}(K_0,\delta)}(K_0) \leq 3e\delta t \\ &= 3e\delta\sqrt{\kappa}\sqrt{\frac{n}{q_{-c}(K_0,\delta)}}\sqrt{n}\log n.\end{aligned}$$

- Since $L_{K_0} \simeq L_n$, the result follows.

The converse is also true

Theorem

There exist absolute constants $c, C > 0$ such that, for every n and for every $\mu \in \mathcal{IL}_{[n]}$,

$$q_{-c}(\mu, CL_\mu) \geq cn.$$

Question

Is it true that

$$q_{-c}(K, C_1) \geq c_2 n$$

for every isotropic convex body K in $\mathbb{R}^n$ (and some absolute constants C_1, c_2)?

- This is equivalent to the fact that $L_n \leq C$.

$I_q(K,C)$

$I_1(K, Z_q^\circ(K))$

Let K be an isotropic convex body in $\mathbb{R}^n$. Consider the parameter

$$I_1(K, Z_q^\circ(K)) = \int_K \|\langle \cdot, x\rangle\|_{L_q(K)} dx.$$

Generally, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then for every symmetric convex body C in $\mathbb{R}^n$ and for every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(K, C) := \left(\int_K \|x\|_C^q dx \right)^{1/q}.$$

The notation $I_1(K, Z_q^\circ(K))$ is then justified by the fact that $\|\langle \cdot, x\rangle\|_{L_q(K)}$ is the norm induced on $\mathbb{R}^n$ by the polar body $Z_q^\circ(K)$ of the L_q-centroid body of K.

Expected order

For an isotropic convex body K in $\mathbb{R}^n$:

1. In the range $2 \leq q \leq \sqrt{n}$ we know that
$$w(Z_q(K)) = \int_{S^{n-1}} \|\langle \cdot, x\rangle\|_{L_q(K)} d\sigma(x) \simeq \sqrt{q} L_K.$$
2. One would hope that
$$\|\langle \cdot, x\rangle\|_{L_q(K)} \simeq \sqrt{q} L_K \|x\|_2$$
for most $x \in K$.
3. Then,
$$I_1(K, Z_q^{\circ}(K)) \simeq \sqrt{q} L_K \int_K \|x\|_2 dx \simeq \sqrt{qn} L_K^2.$$

Second Reduction: GPV

Regular bodies (again)

Let $\kappa, \tau > 0$. We say that an isotropic convex body K in $\mathbb{R}^n$ is (κ, τ)-regular if $\log N(K, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2}$ for all $t \geq \tau\sqrt{n \log n}$.

Theorem (G, Paouris and Vritsiou, 2011)

There exists an absolute constant $\rho \in (0,1)$ with the following property: given $\kappa, \tau \geq 1$, for every $n \geq n_0(\tau)$ and every (κ, τ)-regular isotropic convex body K in $\mathbb{R}^n$ we have that, if $q \geq 2$ satisfies

$$I_1(K, Z_q^\circ(K)) \leq \rho n L_K^2,$$

then

$$L_K^2 \leq C\kappa \sqrt{\frac{n}{q}} \log^2 n \max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn} L_K^2}\right\}.$$

Second Reduction: GPV

1. We know that, for some absolute constants κ, τ and $\delta > 0$, there exists a (κ, τ)-regular isotropic convex body K in $\mathbb{R}^n$ with $L_K \geq \delta L_n$.
2. For every isotropic convex body K in $\mathbb{R}^n$, we have that $I_1(K, Z_q^\circ(K)) \leq cq\sqrt{n}L_K^2 \leq \rho n L_K^2$ is satisfied when $q \ll \sqrt{n}$.
3. Applying the theorem with $q = 2$ we recover Bourgain's bound $L_n \leq C\sqrt[4]{n}\log n$.
4. Any improvement of the exponent of q in the upper bound $I_1(K, Z_q^\circ(K)) \leq cq\sqrt{n}L_K^2$ would lead to an estimate $L_n \leq Cn^\alpha$ with $\alpha < \frac{1}{4}$.

Second Reduction: GPV

It seems plausible that one could even have $I_1(K, Z_q^\circ(K)) \le c\sqrt{qn}L_K^2$, at least when q is small, say $2 \le q \ll \sqrt{n}$. Some evidence is given by the following facts:

1. If K is an unconditional isotropic convex body in $\mathbb{R}^n$, then

$$c_1\sqrt{qn} \le I_1(K, Z_q^\circ(K)) \le c_2\sqrt{qn}\log n$$

   for all $2 \le q \le n$.

2. If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $2 \le q \le \sqrt{n}$, there exists a set $A_q \subseteq O(n)$ with $\nu(A_q) \ge 1 - e^{-q}$ such that $I_1(K, Z_q^\circ(U(K))) \le c_3\sqrt{qn}\, L_K^2$ for all $U \in A_q$.

Klartag and Milman, 2011

Klartag and Milman define a "hereditary" variant of $q_*(\mu)$:

$$q_*^H(\mu) := n \inf_k \inf_{E \in G_{n,k}} \frac{q_*(\pi_E \mu)}{k},$$

where $\pi_E \mu$ is the marginal of μ with respect to the subspace E.
Note that $q_*^H(\mu) \geq c\sqrt{n}$ in the isotropic case.

Main Result: KM

For every isotropic measure μ on $\mathbb{R}^n$ and for every $p \leq q_*^H(\mu)$,

$$|Z_p(\mu)|^{1/n} \geq c\sqrt{\frac{p}{n}} \left[\det \mathrm{Cov}(\mu)\right]^{\frac{1}{2n}} = c\sqrt{\frac{p}{n}}.$$

Consequence

One has

$$L_\mu \simeq \frac{1}{|Z_n(\mu)|^{1/n}} \leq \frac{1}{|Z_{q_*^H(\mu)}(\mu)|^{1/n}} \leq C\sqrt{\frac{n}{q_*^H(\mu)}}$$

Vritsiou, 2012

1. A disadvantage of the result of Klartag and Milman is that the parameter $q_*^H(\mu)$ may be of order $\sqrt{n}$. An example is given by the ℓ_1^n-ball.
2. In a recent work, Vritsiou defines two new hereditary parameters: the first one is a hereditary version of $q_{-c}(\mu, \delta)$, while the second one is related to the highest dimension of marginals with bounded isotropic constant.
3. Both of them could be of order n in the isotropic case.
4. Vritsiou shows that they are more or less equivalent and that the results from DP and KM can be extended to hold for every q up to these parameters.
5. This gives a third reduction of the slicing problem and unifies most of the previous results: after all, DP and KM are equivalent.

New hereditary parameters: V

Recall that

$$q_{-c}(\mu,\delta) := \max\{1 \leq p \leq n-1 : I_{-p}(\mu) \geq \delta^{-1} I_2(\mu) = \delta^{-1}\sqrt{n}\}.$$

$q^H_{-c}(\mu,\delta)$

An hereditary variant of $q_{-c}(\mu,\delta)$ can be defined as follows:

$$q^H_{-c}(\mu,\delta) := n \inf_k \inf_{E \in G_{n,k}} \frac{q_{-c}(\pi_E \mu,\delta)}{k}$$

New hereditary parameters: V

$r_\sharp^H(\mu, A)$

First define

$$r_\sharp(\mu, A) := \max\{1 \le k \le n-1 : \exists\, E \in G_{n,k} \text{ such that } L_{\pi_E \mu} \le A\}$$

for every $A \ge 1$. Then, set

$$r_\sharp^H(\mu, A) := n \inf_k \inf_{E \in G_{n,k}} \frac{r_\sharp(\pi_E \mu, A)}{k}$$

Third Reduction: V

Theorem (Vritsiou, 2012)

There exist absolute constants $C_1, C_2 > 0$ such that for every $\mu \in \mathcal{IL}_{[n]}$ and every $A \geq 1$,

$$r_\sharp^H(\mu, A) \leq q_{-c}^H(\mu, C_1 A) \leq r_\sharp^H(\mu, C_2 A).$$

Moreover, for every $p \leq r_\sharp^H(\mu, A)$ we have that

$$|Z_p(\mu)|^{1/n} \geq \frac{c}{A}\sqrt{\frac{p}{n}}.$$

Corollary

$$L_\mu \leq CA\sqrt{\frac{n}{r_\sharp^H(\mu, A)}} \leq CA\sqrt{\frac{n}{q_{-c}^H\left(\mu, \frac{C_1}{C_2}A\right)}}$$

Comparing the two parameters

Theorem (Vritsiou, 2012)

There exist absolute constants $C_1, C_2 > 0$ such that for every $\mu \in \mathcal{IL}_{[n]}$ and every $A \geq 1$,

$$r_\sharp^H(\mu, A) \leq q_{-c}^H(\mu, C_1 A) \leq r_\sharp^H(\mu, C_2 A).$$

Comment 1

Note that there exists an absolute constant $\delta_0 > 0$ such that

$$q_{-c}^H(\mu, \delta_0) \gg q_*^H(\mu) \geq c_1 \sqrt{n}.$$

The comparison theorem tells us that $r_\sharp^H(\mu, A_1)$ as well is at least of the order of $\sqrt{n}$ for some $A_1 \simeq 1$ and every isotropic measure μ on $\mathbb{R}^n$. Thus, one can remove the logarithmic term in the main result of DP and slightly improve the bounds for L_n that this approach can give.

Comparing the two parameters

Comment 2

The example of the uniform measure on the unit ball of ℓ_1^n, shows that there exist isotropic log-concave measures μ on $\mathbb{R}^n$ for which $q_*(\mu) \simeq \sqrt{n}$, and hence $q_*^H(\mu) \simeq \sqrt{n}$. The choice of the parameters $r_\sharp^H(\mu, \cdot)$ and $q_{-c}^H(\mu, \cdot)$ permits us to extend the range of p with which the method of Klartag and Milman can be applied.

Comment 3

It is interesting to study the parameter $r_\sharp(\mu, A)$, the highest dimension $k \leq n-1$ in which we can find marginals of μ with isotropic constant bounded above by A.

Bad measures

Proposition (Vritsiou, 2012)

If μ is an isotropic measure on $\mathbb{R}^n$ with $L_\mu \simeq L_n$ then for every $\lambda \in (0,1)$ and every positive integer $k = \lambda n$, we have that

$$L_{\pi_E \mu} \geq C^{-\frac{1}{\lambda}} L_\mu$$

for every subspace $E \in G_{n,k}$, where $C \geq 1$ is an absolute constant.

Main Ideas: KM

$\Lambda_\mu(\xi)$

The logarithmic Laplace transform of μ is defined by

$$\Lambda_\mu(\xi) := \log\left(\int_{\mathbb{R}^n} e^{\langle x,\xi\rangle}\frac{d\mu(x)}{\mu(\mathbb{R}^n)}\right), \qquad \xi \in \mathbb{R}^n.$$

$\Lambda_p(\mu)$

The level-sets of the logarithmic Laplace transform of μ are the bodies

$$\Lambda_p(\mu) := \{x \in \mathbb{R}^n : \Lambda_\mu(x) \leq p \text{ and } \Lambda_\mu(-x) \leq p\}, \quad p \geq 0.$$

One has

$$\Lambda_p(\mu) \simeq p(Z_p(\mu))^\circ$$

for every $p \geq 1$.

The measures μ'_x

1. For every $x \in \{\Lambda_\mu < \infty\}$, we write μ'_x for the probability measure whose density is

$$f_{\mu'_x}(z) := \frac{e^{\langle z,x \rangle} f_\mu(z)}{\int_{\mathbb{R}^n} e^{\langle z,x \rangle} d\mu(z)}.$$

2. The barycenter and the covariance matrix of μ'_x are exactly the first and second derivatives of Λ_μ at x:

$$\mathrm{bar}(\mu'_x) = \nabla \Lambda_\mu(x) \text{ and } \mathrm{Cov}\,\mu'_x = \mathrm{Hess}\,\Lambda_\mu(x).$$

3. We now write μ_x for the centered probability measure with density $f_{\mu_x}(z) := f_{\mu'_x}(z + \mathrm{bar}(\mu'_x))$.

Basic Facts: KM

Fact I

If $x \in \frac{1}{2}\Lambda_p(\mu)$ then $\Lambda_q(\mu) \simeq \Lambda_q(\mu_x)$ for every $q \geq p$.
Equivalently, $Z_q(\mu) \simeq Z_q(\mu_x)$ for every $q \geq p$.

Fact II

If μ is a centered, log-concave probability measure on $\mathbb{R}^n$, then for every $p \in [1, n]$ we have that

$$|Z_p(\mu)|^{1/n} \simeq \sqrt{\frac{p}{n}} \left(\frac{1}{\left|\frac{1}{2}\Lambda_p(\mu)\right|} \int_{\frac{1}{2}\Lambda_p(\mu)} \det \mathrm{Cov}(\mu_x)\, dx \right)^{\frac{1}{2n}}$$

$$\simeq \sqrt{\frac{p}{n}} \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} [\det \mathrm{Cov}(\mu_x)]^{\frac{1}{2n}}.$$

The proof: V

1. We want to show is that if $p \leq r_\sharp^H(\mu, A)$, then

$$|Z_p(\mu)|^{1/n} \geq \frac{c}{A}\sqrt{\frac{p}{n}}\,[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}} = \frac{c}{A}\sqrt{\frac{p}{n}}.$$

2. Recall that

$$|Z_p(\mu)|^{1/n} \simeq \sqrt{\frac{p}{n}} \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} [\det \mathrm{Cov}(\mu_x)]^{\frac{1}{2n}}.$$

3. Therefore, we have to show that

$$[\det \mathrm{Cov}(\mu_x)]^{\frac{1}{2n}} \geq \frac{c'}{A}$$

for every $x \in \frac{1}{2}\Lambda_p(\mu)$.

The proof: V

Denote the eigenvalues of $\mathrm{Cov}(\mu_x)$ by $\lambda_1^x \leq \lambda_2^x \leq \cdots \leq \lambda_n^x$, and write E_k for the k-dimensional subspace which is spanned by eigenvectors corresponding to the first k eigenvalues of $\mathrm{Cov}(\mu_x)$.

Lemma 1 (KM)

For all $1 \leq s \leq k \leq n$,

$$\sqrt{\lambda_k^x} \geq c_1 \sup_{F \in G_{E_k,s}} |Z_s(\pi_F \mu_x)|^{1/s}.$$

Lemma 2

Set $s_k^x := r_\sharp(\pi_{E_k}\mu, A)$. Then

$$\sup_{F \in G_{E_k,s_k^x}} |Z_{s_k^x}(\pi_F \mu)|^{1/s_k^x} \geq \frac{c_2}{A} \left[\det \mathrm{Cov}(\mu)\right]^{\frac{1}{2n}} = \frac{c_2}{A}. \tag{2}$$

The proof: V

Now, use the fact that:

1. If $p \le s_k^x = r_\sharp(\pi_{E_k}\mu, A)$ then $Z_{s_k^x}(\mu_x) \simeq Z_{s_k^x}(\mu)$, and hence
$$Z_{s_k^x}(\pi_F \mu_x) \simeq Z_{s_k^x}(\pi_F \mu).$$
2. If $s_k^x < p$, then we can write
$$Z_{s_k^x}(\pi_F \mu_x) \supseteq c_0 \frac{s_k^x}{p} Z_p(\pi_F \mu_x) \supseteq c_0' \frac{s_k^x}{p} Z_p(\pi_F \mu) \supseteq c_0' \frac{s_k^x}{p} Z_{s_k^x}(\pi_F \mu).$$

This gives:

Lemma 3

For every $F \in G_{E_k, s_k^x}$,

$$Z_{s_k^x}(\pi_F \mu_x) \supseteq c_0'' \min\left\{1, \frac{s_k^x}{p}\right\} Z_{s_k^x}(\pi_F \mu) \supseteq c_0'' \frac{k}{n} Z_{s_k^x}(\pi_F \mu).$$

Third Reduction: V

Theorem (Vritsiou, 2012)

Let $A \geq 1$. For every $p \in [1, r_\sharp^H(\mu, A)]$,

$$|Z_p(\mu)|^{1/n} \geq \frac{c}{A}\sqrt{\frac{p}{n}}.$$

Proof. For every $p \in [1, r_\sharp^H(\mu, A)]$ and for every $x \in \frac{1}{2}\Lambda_p(\mu)$,

$$[\det \mathrm{Cov}(\mu_x)]^{1/2} = \prod_{k=1}^{n} \sqrt{\lambda_k^x} \geq \prod_{k=1}^{n} \frac{c}{A}\frac{k}{n} = \frac{c^n}{A^n}\frac{n!}{n^n}.$$

If we take n-th roots, the theorem then follows.