Brascamp-Lieb inequality and quantitative versions of Helly's theorem

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If $\mathcal{P} = \{P_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if any $n+1$ members of $\mathcal{P}$ have non-empty intersection then $\bigcap_{i \in I} P_i \neq \emptyset$.

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We shall discuss a quantitive (volume) version of this result.

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We shall discuss a quantitive (volume) version of this result.

Bárány, Katchalski and Pach, 1982

Let $\mathcal{P} = \{P_i : i \in I\}$ be a finite family of convex sets in $\mathbb{R}^n$. If the intersection of any $2n$ or fewer members of $\mathcal{P}$ has volume greater than or equal to 1, then $\left|\bigcap_{i \in I} P_i\right| \geqslant c_n$, where $c_n > 0$ is a constant depending only on n.

Quantitative Helly's theorem

Using the fact that every (closed) convex set is the intersection of a family of closed half-spaces and a simple compactness argument one can remove the restriction that $\mathcal{P}$ is finite and also assume that each P_i is a closed half-space:

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Let $\mathcal{P} = \{P_i : i \in I\}$ be a family of closed half-spaces in $\mathbb{R}^n$ such that $\left|\bigcap_{i\in I} P_i\right| > 0$. There exist $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant n^{2n^2} \left|\bigcap_{i\in I} P_i\right| .$$

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Note that the cube $[-1, 1]^n$ in $\mathbb{R}^n$ can be written as the intersection of $2n$ closed half-spaces and that the intersection of any $2n - 1$ of these half-spaces has infinite volume.

Quantitative Helly's theorem

In the same work, Bárány, Katchalski and Pach conjectured that instead of the bound $C_n \leqslant n^{2n^2}$ for the constant C_n it may be possible to have $C_n \leqslant n^{cn}$ for an absolute constant $c > 0$.

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In the same work, Bárány, Katchalski and Pach conjectured that instead of the bound $C_n \leqslant n^{2n^2}$ for the constant C_n it may be possible to have $C_n \leqslant n^{cn}$ for an absolute constant $c > 0$.

Theorem (Naszódi, 2015)

Let $\mathcal{P} = \{P_i : i \in I\}$ be a family of closed half-spaces such that $\left|\bigcap_{i\in I} P_i\right| > 0$. We may find $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^{2n} \left|\bigcap_{i\in I} P_i\right|,$$

where $C > 0$ is an absolute constant.

Naszódi's argument

- We start with a family $\mathcal{P} = \{P_i : i \in I\}$ of closed half-spaces

$$P_i = \{x : \langle x, u_i \rangle \leqslant 1\}$$

such that $0 < \left|\bigcap_{i \in I} P_i\right| < \infty$.

- We may assume that $\mathcal{P}$ is a finite family, therefore $P = \bigcap_{i \in I} P_i$ is a polytope.

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- We may assume that $\mathcal{P}$ is a finite family, therefore $P = \bigcap_{i \in I} P_i$ is a polytope.
- By affine invariance, we may also assume that P is in John's position.
- From John's theorem there exists $J \subseteq I$ such that u_j, $j \in J$ are contact points of P and B_2^n, and $a_j > 0$, $j \in J$ such that

$$I_n = \sum_{j \in J} a_j u_j \otimes u_j \quad \text{and} \quad \sum_{j \in J} a_j u_j = 0.$$

Naszódi's argument

- By the Dvoretzky-Rogers lemma, we may choose n of these contact points, which we denote by $v_1, \ldots, v_n$, so that

$$\mathrm{dist}(v_k, \mathrm{span}(v_1, v_2, \ldots, v_{k-1})) \geqslant \sqrt{\frac{n-k+1}{n}}$$

for all $k = 2, \ldots, n$.

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- It follows that the simplex $S = \mathrm{conv}\{v_0 = 0, v_1, \ldots, v_n\} \subseteq P$ has volume

$$|S| = \frac{1}{n!} \prod_{k=1}^{n} \mathrm{dist}(v_k, \mathrm{span}(v_1, v_2, \ldots, v_{k-1})) \geqslant \frac{1}{n^{\frac{n}{2}} \sqrt{n!}}.$$

Naszódi's argument

- Now we use the fact that if w is the center of mass of S then S contains a symmetric (with respect to w) convex body T with

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- Consider the ray ℓ from the origin in the direction of $-w$. Then, ℓ intersects the boundary of $\mathrm{conv}\{u_j, j \in J\}$ at a point

$$z \in \mathrm{conv}\{v_{n+1}, \dots, v_{n+k}\}$$

for some $v_{n+i} \in \{u_j, j \in J\}$ and $k \leqslant n$.

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- Also, note that

$$\mathrm{conv}\{u_j, j \in J\} \supseteq \frac{1}{n} B_2^n,$$

and hence

$$\|z\|_2 \geqslant \frac{1}{n}.$$

Naszódi's argument

- Applying a contraction with center z and ratio

$$\lambda = \frac{\|z\|_2}{\|z-w\|_2} = \frac{\|z\|_2}{\|z\|_2 + \|w\|_2} \geqslant \frac{\|z\|_2}{1+\|z\|_2} \geqslant \frac{1}{n+1}$$

to T, we obtain an origin symmetric convex body

$$Q \subseteq \mathrm{conv}\{z, v_1, \dots, v_n\} \subseteq \mathrm{conv}\{v_1, \dots, v_n, v_{n+1}, \dots, v_{n+k}\}$$

with volume

$$|Q| \geqslant \frac{1}{(n+1)^n}|T| \geqslant \frac{1}{2^n(n+1)^n}|S| \geqslant \frac{1}{2^n(n+1)^n n^{\frac{n}{2}}\sqrt{n!}}.$$

Naszódi's argument

- Consider the intersection of $n + k \leqslant 2n$ half-spaces

$$R = \bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leqslant 1\}.$$

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- Using the Blaschke-Santaló inequality for Q and the fact that $B_2^n \subseteq P$ and $R \subseteq Q^\circ$ we get

$$\frac{|R|}{|P|} \leqslant \frac{|Q^\circ|}{|B_2^n|} \leqslant \frac{|B_2^n|}{|Q|}.$$

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$$\frac{|R|}{|P|} \leqslant \frac{|Q^\circ|}{|B_2^n|} \leqslant \frac{|B_2^n|}{|Q|}.$$

- Since $|Q| \geqslant \frac{1}{2^n(n+1)^n n^{\frac{n}{2}} \sqrt{n!}}$, we see that

$$|R| \leqslant \frac{\pi^{\frac{n}{2}} 2^n (n+1)^n n^{\frac{n}{2}} \sqrt{n!}}{\Gamma\left(\frac{n}{2}+1\right)} |P|$$

and the result follows (in fact, one has $C_n \leqslant (Cn)^{\frac{3n}{2}}$).

Linear number of half-spaces

If we relax the condition on the number s of half-spaces that we use (but still require that it is proportional to the dimension) we are able to improve the exponent in the constant C_n from $\frac{3n}{2}$ to n:

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Theorem (Brazitikos, 2015)

There exists an absolute constant $\beta > 1$ with the following property: for every family $\{P_i : i \in I\}$ of closed half-spaces

$$P_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leqslant 1\}$$

in $\mathbb{R}^n$, such that $P = \bigcap_{i \in I} P_i$ has positive volume, there exist $s \leqslant \beta n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^n \, |P|,$$

where $C > 0$ is an absolute constant.

Linear number of half-spaces

- We may assume that $P = \bigcap_{i\in I}\{x \in \mathbb{R}^n : \langle x, v_i\rangle \leqslant 1\}$ has finite volume and, since the statements are affinely invariant, that P is in John's position, i.e. the ellipsoid of maximal volume inscribed in P is the Euclidean unit ball B_2^n.
- Then, we have John's decomposition of the identity: there exists $J \subseteq I$ such that v_j, $j \in J$ are contact points of P and B_2^n and there are positive scalars a_j, $j \in J$ such that

$$I_n = \sum_{j\in J} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j\in J} a_j v_j = 0.$$

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- Assuming that we have this decomposition for a set J of cardinality $|J| \leqslant \beta n$ we may use the Brascamp-Lieb inequality in order to estimate the volume of

$$\bigcap_{i \in J} \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leqslant 1\}.$$

Linear number of half-spaces

- For any $\beta > 1$ we can extract a subset σ of J, of cardinality βn, which still forms an approximate John's decomposition of the identity with suitable weigths.

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- For any $\beta > 1$ we can extract a subset σ of J, of cardinality βn, which still forms an approximate John's decomposition of the identity with suitable weigths.
- This is a result of Srivastava; using this, he showed that for any convex body K in $\mathbb{R}^n$ and any $\epsilon > 0$ there exists a convex body T such that $T \subseteq K \subseteq (\sqrt{5} + \epsilon)T$ and T has at most $O_\epsilon(n)$ contact points with its John ellipsoid.

Approximate John's decomposition

Theorem (Srivastava, 2012)

Let $v_1, \dots, v_m \in S^{n-1}$ *and* $a_1, \dots, a_m > 0$ *such that*

$$I_n = \sum_{j=1}^{m} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j=1}^{m} a_j v_j = 0.$$

Given $\varepsilon > 0$ *we can find a subset* σ *of* $\{1, \dots, m\}$ *with* $|\sigma| = O_\varepsilon(n)$*, positive scalars* c_i*,* $i \in \sigma$*, and a vector* v *with*

$$\|v\|_2^2 \leqslant \frac{\varepsilon}{\sum_{i\in\sigma} c_i},$$

such that

$$I_n \preceq \sum_{i\in\sigma} c_i (v_i + v) \otimes (v_i + v) \preceq (4+\varepsilon) I_n \quad \text{and} \quad \sum_{i\in\sigma} c_i (v_i + v) = 0.$$

Brascamp-Lieb inequality

Theorem (Brascamp-Lieb)

Let $m \geqslant n$, and let $u_1, \ldots, u_m \in \mathbb{R}^n$ and $c_1, \ldots, c_m > 0$ with $c_1 + \cdots + c_m = n$. Then,

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{c_j}(\langle x, u_j \rangle) dx \leqslant D \prod_{j=1}^{m} \left(\int_{\mathbb{R}} f_j \right)^{c_j}$$

for all integrable functions $f_j : \mathbb{R} \to [0, \infty)$, where $D = 1/\sqrt{F}$ and

$$F = \inf \left\{ \frac{\det \left(\sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right)}{\prod_{j=1}^{m} \lambda_j^{c_j}} : \lambda_j > 0 \right\}.$$

Brascamp-Lieb inequality

BL-constant (K. Ball)

If $v_1, \dots, v_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy

$$I_n = \sum_{j=1}^{m} c_j v_j \otimes v_j$$

then

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Geometric BL-inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{c_j}(\langle x, u_j \rangle) dx \leqslant \prod_{j=1}^{m} \left(\int_{\mathbb{R}} f_j \right)^{c_j}.$$

Approximate BL-inequality

Approximate BL-constant

Let $\gamma > 1$. If $u_1, \ldots, u_s \in S^{n-1}$ and $c_1, \ldots, c_s > 0$ satisfy

$$I_n \preceq A := \sum_{j=1}^{s} c_j u_j \otimes u_j \preceq \gamma I_n$$

then

$$\gamma^n \det\left(\sum_{j=1}^{s} \kappa_j \lambda_j u_j \otimes u_j\right) \geqslant \det\left(\sum_{j=1}^{s} c_j \lambda_j u_j \otimes u_j\right) \geqslant \prod_{j=1}^{s} \lambda_j^{\kappa_j}$$

for all $\lambda_1, \ldots, \lambda_s > 0$, where $\kappa_j = c_j \langle A^{-1} u_j, u_j \rangle > 0$, $1 \leqslant j \leqslant s$.

Approximate BL-inequality

Theorem

Let $\gamma > 1$*. Let* $u_1, \ldots, u_s \in S^{n-1}$ *and* $c_1, \ldots, c_s > 0$ *satisfy*

$$I_n \preceq A := \sum_{j=1}^{s} c_j u_j \otimes u_j \preceq \gamma I_n$$

and set $\kappa_j = c_j \langle A^{-1} u_j, u_j \rangle > 0$, $1 \leqslant j \leqslant s$*. If* $f_1, \ldots, f_s : \mathbb{R} \to \mathbb{R}^+$ *are integrable functions then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^{s} f_j^{\kappa_j}(\langle x, u_j \rangle) dx \leqslant \gamma^{\frac{n}{2}} \prod_{j=1}^{s} \left(\int_{\mathbb{R}} f_j(t) dt \right)^{\kappa_j}.$$

Helly's theorem

- We assume that P is in John's position, and we find $J \subseteq I$ so that the vectors u_j, $j \in J$ are contact points of P and S^{n-1} and there exist $a_j > 0$, $j \in J$, such that

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$$I_n = \sum_{j\in J} a_j u_j \otimes u_j \quad \text{and} \quad \sum_{j\in J} a_j u_j = 0.$$

- Using Srivastava's theorem we find a subset $\sigma \subseteq J$ with $|\sigma| \leqslant \alpha_1(\varepsilon)n$, positive scalars c_j, $j \in \sigma$, and a vector u such that

$$I_n \preceq \sum_{j\in\sigma} c_j(u_j + u) \otimes (u_j + u) \preceq (4+\varepsilon) I_n$$

and

$$\sum_{j\in\sigma} c_j(u_j + u) = 0 \;\text{ and }\; \|u\|_2^2 \leqslant \frac{\varepsilon}{\sum_{j\in\sigma} c_j}.$$

Helly's theorem

- Taking traces we check that

$$n \leqslant \sum_{j\in\sigma} c_j \leqslant (4+2\varepsilon)n.$$

- In particular,

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- In particular,

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- Recall that $\mathrm{conv}\{u_j, j\in J\} \supseteq \frac{1}{n}B_2^n$. Then, for the vector $w = \frac{u}{\sqrt{\varepsilon n}}$ we have $\|w\|_2 \leqslant \frac{1}{n}$ and hence $w \in \mathrm{conv}\{u_j, j\in J\}$. Carathéodory's theorem shows that there exist $\tau \subseteq J$ with $|\tau| \leqslant n+1$ and $\rho_i > 0$, $i\in\tau$ such that

$$w = \sum_{i\in\tau} \rho_i u_i \quad \text{and} \quad \sum_{i\in\tau} \rho_i = 1.$$

Helly's theorem

- Note that
$$\left(\sum_{j\in\sigma} c_j\right)(-u) = \sum_{j\in\sigma} c_j u_j,$$
and this shows that $-u \in \mathrm{conv}\{u_j : j \in \sigma\}$. It follows that the segment
$$\left[-u, \frac{u}{\sqrt{\varepsilon n}}\right] \subset \mathrm{conv}\{u_j : j \in \sigma \cup \tau\}.$$

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- For $j \in \sigma$ we set

$$v_j = \sqrt{\frac{n}{n+1}}\left(-u_j, \frac{1}{\sqrt{n}}\right) \quad \text{and} \quad b_j = \frac{n+1}{n}c_j.$$

We also set $-v = \sqrt{\frac{n}{n+1}}(u,0)$. Then,

$$I_{n+1} \preceq \sum_{j\in\sigma} b_j(v_j+v)\otimes(v_j+v) \preceq (4+2\varepsilon)I_{n+1}.$$

Helly's theorem

- Choosing $\varepsilon = 10^{-3}$ we get

$$\left\| \sum_{j\in\sigma} b_j v_j \otimes v + \sum_{j\in\sigma} v \otimes b_j v_j + \left(\sum_{j\in\sigma} b_j \right) v \otimes v \right\| \leqslant \frac{1}{2},$$

therefore

$$\frac{1}{2} I_{n+1} \preceq \sum_{j\in\sigma} b_j v_j \otimes v_j \preceq 5 I_{n+1}.$$

Helly's theorem

- Now, we set $\kappa_j > 0 = b_j\langle A^{-1}u_j, u_j\rangle$, $j \in \sigma$. Then, if $f_j : \mathbb{R} \to \mathbb{R}^+$ are measurable functions, we have
$$\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle)dy \leqslant 10^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j}.$$
- For $j \in \sigma$ we define $f_j(t) = e^{-\frac{b_j}{k_j}t}\mathbf{1}_{[0,\infty)}(t)$. Then,
$$\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle)dy \leqslant 10^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j}$$
$$= 10^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\frac{\kappa_j}{c_j}\right)^{\kappa_j} \leqslant 40^{\frac{n+1}{2}},$$
where $\kappa_j/b_j = \langle A^{-1}u_j, u_j\rangle \leqslant 2$ because
$$\frac{1}{2}I_{n+1} \preceq A = \sum_{j\in\sigma} b_j v_j \otimes v_j.$$

Helly's theorem

- Let $Q = \{x \in \mathbb{R}^n : \langle x, u_j \rangle < 1, \;\; j \in \sigma \cup \tau\}$.
- If $y = (x, r) \in \mathbb{R}^{n+1}$ then the conditions $r > 0$ and $x \in \frac{r}{\sqrt{n}} Q$ imply that
$$\prod_{j \in \sigma} f_j^{\kappa_j}(\langle y, v_j \rangle) > 0.$$

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$$\prod_{j \in \sigma} f_j^{\kappa_j}(\langle y, v_j \rangle) > 0.$$

- We also have

$$\frac{1}{\left(\sum_{j\in\sigma} c_j\right)} \left\langle \sum_{j\in\sigma} c_j u_j, x \right\rangle = \langle -u, x \rangle = \sqrt{\varepsilon n} \langle -w, x \rangle$$

$$= \sqrt{\varepsilon n} \left\langle -\sum_{i\in\tau} \rho_i u_i, x \right\rangle \geqslant -\sqrt{\varepsilon} r,$$

where the last inequality holds since $x \in \frac{r}{\sqrt{n}} Q$. It follows that

$$\left\langle \sum_{j\in\sigma} c_j u_j, x \right\rangle \geqslant -5\sqrt{\varepsilon} rn.$$

Helly's theorem

- Using the above (and recalling our choice of $\varepsilon = 10^{-3} < 1$) we see that if $y = (x, r) \in \frac{r}{\sqrt{n}} Q \times (0, \infty)$ then

$$
\begin{aligned}
&\prod_{j \in \sigma} f_j^{\kappa_j}(\langle y, v_j \rangle) \\
&= \exp\left(-\sum_{j \in \sigma} b_j \left(\frac{r}{\sqrt{n}} - \sqrt{\frac{n}{n+1}} \langle x, u_j \rangle \right) \right) \\
&= \exp\left(-\frac{r}{\sqrt{n}} \sum_{j \in \sigma} b_j \right) \exp\left(\left\langle x, \sum_{j \in \sigma} b_j u_j \right\rangle \right) \\
&\geqslant \exp\left(-5r \frac{n+1}{\sqrt{n}} - 5\sqrt{\varepsilon} r (n+1) \right) \geqslant \exp\left(-10 r (n+1) \right).
\end{aligned}
$$

- Now, the approximate BL-inequality gives us

$$\frac{|Q|}{n^{\frac{n}{2}}} \int_0^\infty r^n e^{-10r(n+1)}\, dr = \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q} e^{-10r(n+1)} dx\, dr$$
$$\leqslant \int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) dy \leqslant 40^{\frac{n+1}{2}}.$$

- Direct computation and then Stirling's approximation show that

$$|Q| \leqslant C_1^n \frac{n^{\frac{3n}{2}}}{n!} \leqslant C_2^n n^{\frac{n}{2}}$$

  and Q is the intersection of $\leqslant |\sigma| + |\tau| \leqslant \alpha_1(10^{-3})n + n + 1$ half-spaces.
- Since $B_2^n \subseteq P \subseteq Q$, the result follows.

Continuous version: Ball-Barthe lemma

A Borel measure ν on S^{n-1} is called isotropic if

$$I_n = \int_{S^{n-1}} u \otimes u \, d\nu(u).$$

Lutwak, Yang and Zhang, 2004

if ν is an isotropic measure on S^{n-1} and $t : \mathrm{supp}(\nu) \longrightarrow (0, \infty)$ is continuous, then

$$\det\left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u)\right) \geqslant \exp\left[\int_{S^{n-1}} \log t(u) \, d\nu(u)\right].$$

Continuous version: BL-inequalities

Theorem (Barthe)

Let ν be an isotropic Borel measure in $\mathbb{R}^n$ and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ that satisfy (H). Then,

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle) d\nu(u)\right) dx \leqslant \exp\left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) \, d\nu(u)\right).$$

Also, if h is a measurable function such that

$$h\left(\int_{S^{n-1}} \theta(u) u \, d\nu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u)) \, d\nu(u)\right)$$

for every integrable function θ, then

$$\int_{\mathbb{R}^n} h(x) \, dx \geqslant \exp\left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) \, d\nu(u)\right).$$

Continuous version: approximate Ball-Barthe lemma

We say that a Borel measure ν on S^{n-1} is a γ-*approximation of an isotropic measure* (for some $\gamma > 1$) if

$$I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u) \preceq \gamma I_n.$$

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Theorem

Let ν be a γ-approximation of an isotropic measure on S^{n-1}. For every continuous function $t : \mathrm{supp}(\nu) \longrightarrow (0, \infty)$ one has

$$\gamma^n \det \left(\int_{S^{n-1}} t(u) u \otimes u \left\langle T_\nu^{-1} u, u \right\rangle d\nu(u) \right) \geqslant \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right)$$

$$\geqslant \exp \left[\int_{S^{n-1}} \log t(u) \left\langle T_\nu^{-1} u, u \right\rangle d\nu(u) \right].$$

Continuous version: approximate Ball-Barthe lemma

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$$\geqslant \exp \left[\int_{S^{n-1}} \log t(u) \, \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right].$$

The proof involves mixed discriminants as in Lutwak, Yang and Zhang.

Continuous version: approximate BL-inequalities

Lemma

Let $\gamma > 1$ and let ν be a γ-approximation of an isotropic measure on S^{n-1}. Let μ be the measure on S^{n-1} with $d\mu(u) = \langle T_\nu^{-1} u, u \rangle d\nu(u)$. If $(f_u), (g_u)$, $u \in S^{n-1}$ are two families of functions that satisfy (H) then

$$\exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} g_u\right) d\mu(u)\right) \int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle) d\mu(u)\right) dx$$

$$\leqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) d\mu(u)\right)$$

$$\times \int_{\mathbb{R}^n}^{*} \sup_{y = \int \theta(u) u d\nu(u)} \exp\left(\int_{S^{n-1}} \log g_u(\theta(u)) d\mu(u)\right) dy.$$

Continuous version: approximate BL-inequalities

Theorem

Let ν be a γ-approximation of an isotropic Borel measure in $\mathbb{R}^n$ and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ that satisfy (H). Then,

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle)\langle T_\nu^{-1}u, u\rangle d\nu(u)\right) dx$$
$$\leqslant \gamma^n \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right)\langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right).$$

Continuous version: approximate BL-inequalities

Theorem

Let ν *be a* γ*-approximation of an isotropic Borel measure in* $\mathbb{R}^n$ *and let* (f_u), $u \in S^{n-1}$ *be a family of functions* $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ *that satisfy* (H)*. If* h *is a measurable function such that*

$$h\left(\int_{S^{n-1}} \theta(u)u\, d\nu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u))\langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right)$$

for every integrable function θ*, then*

$$\gamma^{\frac{n}{2}} \int_{\mathbb{R}^n} h(y)\, dy \geqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right)\langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right).$$

Condition (H)

We say that a family of functions (f_u), $u \in S^{n-1}$, satisfies condition (H) if

- There exist a continuous function $F : S^{n-1} \times \mathbb{R} \longrightarrow (0, +\infty)$ and two functions a, b on S^{n-1} with $a < b$ (a, b are either real-valued continuous or constant with value $\pm\infty$) such that for all $(u, t) \in S^{n-1} \times \mathbb{R}$

$$f_u(t) = \mathbf{1}_{a(u) \leqslant t \leqslant b(u)} F(u, t).$$

- There exists a function $U \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ such that $0 \leqslant f_u \leqslant U$ for all $u \in S^{n-1}$.