Spaces with Ricci curvature bounded from below

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Lessons

1) On the definition of spaces with Ricci curvature bounded from below

2) Analytic properties of $RCD(K, N)$ spaces

3) Geometric properties of $RCD(K, N)$ spaces

Quoting the first sentence of Cheng-Yau '75

'Most of the problems in differential geometry can be reduced to problems in differential equations on Riemannian manifolds'

‘Rules’ we will follow to make analysis on mm spaces

Forget about:	Focus on:
Lipschitz functions	Sobolev functions
Charts	Intrinsic calculus
Trying to define who Df and ∇f really are (for the moment)	Understanding the duality relation $Df(\nabla g)$

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on mm spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

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Differentials on $\mathbb{R}^d$

Given $f : \mathbb{R}^d \to \mathbb{R}$ smooth, its differential $Df : \mathbb{R}^d \to T^*\mathbb{R}^d$ is intrinsically defined by

$$Df(x)(v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \qquad \forall x \in \mathbb{R}^d,\ v \in T_x\mathbb{R}^d$$

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A way to get it is starting from the observation that for any tangent vector w it holds

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Then we can say that $v = \nabla f(x)$ provided $=$ holds, or equivalently

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Rmk.
Uniqueness holds iff the norm is strictly convex

Linearity holds iff the norm comes from a scalar product.

An important identity

$$\max_{v \in \nabla g(x)} Df(v) = \inf_{\varepsilon > 0} \frac{\|D(g + \varepsilon f)\|_*^2(x) - \|Dg\|_*^2(x)}{2\varepsilon}$$

$$\min_{v \in \nabla g(x)} Df(v) = \sup_{\varepsilon < 0} \frac{\|D(g + \varepsilon f)\|_*^2(x) - \|Dg\|_*^2(x)}{2\varepsilon}.$$

The object $D^{\pm} f(\nabla g)$ in mm spaces

For $f, g \in S^2$, the functions $D^{\pm} f(\nabla g) : X \to \mathbb{R}$ are defined by

$$D^{+} f(\nabla g) := \inf_{\varepsilon>0} \frac{|D(g+\varepsilon f)|^2 - |Dg|^2}{2\varepsilon}$$

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Notice that

$$D^- f(\nabla g) \leq D^+ f(\nabla g), \qquad \mathfrak{m} - a.e.$$
$$|D^{\pm} f(\nabla g)| \leq |Df||Dg| \in L^1(X, \mathfrak{m}),$$
$$D^+(-f)(\nabla g) = -D^- f(\nabla g) = D^+ f(\nabla(-g)), \qquad \mathfrak{m} - a.e.$$

Calculus rules

Locality

$$D^{\pm}f(\nabla g) = D^{\pm}\tilde{f}(\nabla\tilde{g}), \qquad \mathfrak{m}\text{-}a.e.\ on\ \{f = \tilde{f}\} \cap \{g = \tilde{g}\}$$

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Chain rule

$$D^{\pm}(\varphi \circ f)(\nabla g) = \varphi' \circ f\, D^{\pm \mathrm{sign}(\varphi' \circ f)} f(\nabla g),$$
$$D^{\pm}f(\nabla(\varphi \circ g)) = \varphi' \circ g\, D^{\pm \mathrm{sign}(\varphi' \circ g)} f(\nabla g)$$

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Leibniz rule

$$D^{+}(f_1 f_2)(\nabla g) \leq f_1 \, D^{\ \mathrm{sign}(f_1)} f_2(\nabla g) + f_2 \, D^{\ \mathrm{sign}(f_2)} f_1(\nabla g),$$
$$D^{-}(f_1 f_2)(\nabla g) \geq f_1 \, D^{-\mathrm{sign}(f_1)} f_2(\nabla g) + f_2 \, D^{-\mathrm{sign}(f_2)} f_1(\nabla g)$$

For $f_1, f_2 \in S^2 \cap L^\infty$, and $g \in S^2$.

Infinitesimally Hilbertian spaces

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In this case

$$D^+ f(\nabla g) = D^- f(\nabla g) = D^+ g(\nabla f) = D^- g(\nabla f), \qquad \mathfrak{m} - a.e.$$

and we denote these quantities by $\nabla f \cdot \nabla g$.

Plan representing gradients: definition

For $g \in S^2$ and $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ test plan it holds

$$\varlimsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi} \le \frac{1}{2}\int |Dg|^2(\gamma_0)\,\mathrm{d}\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t}\iint_0^t |\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}$$

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We say that $\boldsymbol{\pi}$ *represents* ∇g, provided it holds

$$\varliminf_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} \geq \frac{1}{2} \int |Dg|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} + \varlimsup_{t \downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}$$

Plan representing gradients: existence

Thm (G. '12. Ambrosio, G., Savaré '11. G., Kuwada, Ohta '10).
For $g \in S^2(X)$ and $\mu \in \mathscr{P}(X)$ such that $\mu \leq C\mathfrak{m}$, a plan $\boldsymbol{\pi}$ representing ∇g and such that $\mathrm{e}_{0\,\sharp}\boldsymbol{\pi} = \mu$ exists.

Horizontal and vertical derivatives, a.k.a.: First order differentiation formula

Let $f, g \in S^2$, and $\boldsymbol{\pi}$ which represents ∇g.
Then

$$\varlimsup_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi}$$
$$\geq \varliminf_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi}$$

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$$\int D^+ f(\nabla g)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi} \geq \varlimsup_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}$$

$$\geq \varliminf_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi} \geq \int D^- f(\nabla g)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}$$

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A property of GF of K-convex functions on $\mathbb{R}^d$

Let $E : \mathbb{R}^d \to \mathbb{R}$ be K-convex and $t \mapsto x_t$ be such that

$$x_t' = -\nabla E(x_t).$$

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Pick $y \in \mathbb{R}^d$ and notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 = x_t' \cdot (x_t - y) = \nabla E(x_t) \cdot (y - x_t)$$

and for $y_{t,s} := (1-s)x_t + sy$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} E(y_{t,s}) = \nabla E(x_t) \cdot (y - x_t).$$

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Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 \leq E(y) - E(x_t) - \frac{K}{2}|x_t - y|^2$$

EVI_K gradient flows

Def. On a metric space (Y, d_Y), we say that $(x_t) \subset Y$ is an EVI_K-GF of $E : Y \to [0, \infty]$ if it is loc. abs. cont. and for every $y \in Y$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}^2(x_t, y) \leq E(y) - E(x_t) - \frac{K}{2}\mathsf{d}^2(x_t, y), \qquad a.e.\ t > 0$$

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(Savaré) If (x_t) is an EVI_K gradient flows it satisfies

$$E(x_0) = E(x_t) + \frac{1}{2}\int_0^t |x_s'|^2 + |\partial^- E|^2(x_s)\,\mathrm{d}s, \qquad \forall t > 0$$

Note: The viceversa is not true

The heat flow as EVI_K gradient flow of the entropy

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We want to compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} W_2^2(\mu_t, \nu) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} \mathrm{Ent}_{\mathfrak{m}}(\nu_{t,s})$$

where $s \mapsto \nu_{t,s}$ is a geodesic joining μ_t to ν.

Derivative of $\frac{1}{2}W_2^2(\mu_t, \nu)$

Fix t_0 a point of differentiability of $t \mapsto \frac{1}{2}W_2^2(\mu_t, \nu)$ and let φ be a Kantorovich potential from μ_{t_0} to ν.
Then

$$\frac{1}{2}W_2^2(\mu_{t_0}, \nu) = \int \varphi \, \mathrm{d}\mu_{t_0} + \int \varphi^c \, \mathrm{d}\nu$$

$$\frac{1}{2}W_2^2(\mu_{t_0+h}, \nu) \geq \int \varphi \, \mathrm{d}\mu_{t_0+h} + \int \varphi^c \, \mathrm{d}\nu$$

Recalling that $\mu_t = \rho_t \mathfrak{m}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \frac{1}{2}W_2^2(\mu_t, \nu) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \int \varphi \, \mathrm{d}\mu_t = \int \varphi \Delta \rho_{t_0} \, \mathrm{d}\mathfrak{m}$$

Some properties of W_2-geodesics

Thm. (Regularity of interpolated densities Rajala '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $CD(K, \infty)$ space and $\mu, \nu \in \mathscr{P}(X)$ s.t. $\mu, \nu \leq C\mathfrak{m}$.

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Then there exists a geodesic (μ_t) such that $\mu_t \leq C'\mathfrak{m}$ for every $t \in [0, 1]$ and $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is K-convex.

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Thm. (Metric Brenier's theorem Ambrosio, G., Savaré '11) Let (μ_t) be a geodesic such that $\mu_t \le C\mathfrak{m}$ for every $t \in [0,1]$, $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ a lifting of it and φ a Kantorovich potential inducing it.

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Then $\boldsymbol{\pi}$ represents the gradient of $-\varphi$.

Derivative of $\mathrm{Ent}_{\mathfrak{m}}(\nu_s)$

Let $s \mapsto \nu_s$ be a geodesic s.t. $\nu_s \leq C\mathfrak{m}$ for every s and such that $\nu_0 = \eta\mathfrak{m}$ with $\eta \geq c > 0$, $\eta \in W^{1,2}(X)$.
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Let φ be a Kantorovich potential inducing it.
Then

$$\begin{aligned}
\varliminf_{s\downarrow 0} \frac{\mathrm{Ent}_{\mathfrak{m}}(\nu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_0)}{s} &\geq \varliminf_{s\downarrow 0} \frac{1}{s} \int \log \eta \, \mathrm{d}(\nu_s - \nu_0) \\
&= \lim_{s\downarrow 0} \int \frac{\log \eta(\gamma_s) - \log \eta(\gamma_0)}{s} \, \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \nabla(\log \eta) \cdot \nabla\varphi(\gamma_0) \, \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \nabla(\log \eta) \cdot \nabla\varphi \; \eta \, \mathrm{d}\mathfrak{m} \\
&= -\int \nabla\eta \cdot \nabla\varphi \, \mathrm{d}\mathfrak{m}
\end{aligned}$$

The heat flow is an EVI_K gradient flow of the entropy

We (Ambrosio, G., Savaré '11) conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) \le \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathrm{Ent}_{\mathfrak{m}}(\nu_{t,s})$$

$$\le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t) - \frac{K}{2}W_2^2(\mu_t,\nu)$$

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$$\le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t) - \frac{K}{2}W_2^2(\mu_t,\nu)$$

We deduce that for $(\mu_t),(\nu_t) \subset \mathscr{P}(X)$ heat flows we have

$$W_2^2(\mu_t,\nu_t) \le e^{-2Kt}W_2^2(\mu_0,\nu_0)$$

Heat Kernel and Bronian motion

We deduce that there exists the heat flow $t \mapsto \mu_t[x]$ starting from δ_x for any $x \in X$.

General constructions related to the theory of Dirichlet forms then grant existence and uniqueness of a Markov process $\mathbf{X}_t$ with transition probabilities $\mu_t[x]$, i.e.:

$$\mathbb{P}(\mathbf{X}_{t+s} \in A | \mathbf{X}_t = x) = \mu_t[x](A)$$

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A duality result

Thm. (Kuwada '09)
Let $\mathsf{H}_t : \mathscr{P}(X) \to \mathscr{P}(X)$ be the heat flow at level of measures and $\mathsf{h}_t : L^1 \to L^1$ the one for densities.

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Then TFAE:

$$W_2^2(\mathsf{H}_t(\mu), \mathsf{H}_t(\nu)) \le e^{-2Kt} W_2^2(\mu, \nu), \qquad \forall t \ge 0,\ \mu, \nu \in \mathscr{P}(X)$$

$$\mathrm{lip}^2(\mathsf{h}_t(f)) \le e^{-2Kt}\, \mathsf{h}_t(\mathrm{lip}^2(f)), \qquad \forall t \ge 0,\ f : X \to \mathbb{R} \text{ Lipschitz}$$

where

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}$$

Density in energy in $W^{1,2}$ of Lipschitz functions

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Density in energy in $W^{1,2}$ of Lipschitz functions

Thm. (Ambrosio, G., Savaré '11) Let $(X, \mathsf{d}, \mathfrak{m})$ be a mms. Then:

- for every $(f_n) \subset \mathrm{LIP}(X)$ converging in L^2 to some f, we have

$$|Df| \leq G, \qquad \text{where } G \text{ is any } L^2\text{-weak limit of } (\mathrm{lip}(f_n))$$

- for every $f \in W^{1,2}(X)$ there exists $(f_n) \subset \mathrm{LIP}(X)$ L^2-converging to f such that

$$|Df| = \lim_n \mathrm{lip}(f_n) \qquad \text{the limit being intended strong in } L^2$$

Bochner inequality ($N = \infty$)

(G., Kuwada, Ohta '10. Ambrosio, G., Savaré '11)
Starting from

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and by relaxation we deduce

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which gives

$$\int \Delta g \frac{|Df|^2}{2}\,\mathrm{d}\mathfrak{m} \geq \int (\nabla f \cdot \nabla \Delta f + K|Df|^2) g\,\mathrm{d}\mathfrak{m}$$

for every $f \in W^{1,2}(X) \cap D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in L^\infty(X) \cap D(\Delta)$ with $g \geq 0$ and $\Delta g \in L^\infty(X)$.

Bochner inequality ($N = \infty$)

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for every $f \in W^{1,2}(X) \cap D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in L^\infty(X) \cap D(\Delta)$ with $g \ge 0$ and $\Delta g \in L^\infty(X)$.

Also the converse implication from Bochner to $RCD(K, \infty)$ holds (Ambrosio, G., Savaré '12)

Bochner inequality ($N < \infty$)

(Erbar, Kuwada, Sturm '13) On an $RCD(K, N)$ space we have

$$\int \Delta g \frac{|Df|^2}{2} \, d\mathfrak{m} \geq \int \Big(\frac{(\Delta f)^2}{N} + \nabla f \cdot \nabla \Delta f + K|Df|^2 \Big) g \, d\mathfrak{m}$$

(see also (Ambrosio, Mondino, Savaré - in progress))

Related results

(Mondino, Garofalo '13) Li-Yau inequality: for $f \geq 0$ on $RCD(0, N)$ spaces we have

$$\Delta(\log(\mathrm{h}_t f)) \geq \frac{N}{2t}$$

(Kell '13, Jiang '11, Koskela, Rajala, Shanmugalingam '03) Local Lipschitz regularity of harmonic functions on $RCD(K, N)$ spaces

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Optimal maps

Thm. (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be $RCD(K, N)$, $\mu, \nu \in \mathscr{P}(X)$ with $\mu \ll \mathfrak{m}$.

Optimal maps

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Then:

- There is only one optimal plan
- Such plan is induced by a map T
- For μ-a.e. x there is only one geodesic γ^x from x to $T(x)$
- For μ-a.e. $x \neq y$ we have $\gamma^x_t \neq \gamma^y_t$ for every $t \in [0, 1)$

Analytic properties of $RCD(K, N)$ spaces

- Differential calculus on mm spaces
- The heat flow on $RCD(K, \infty)$ spaces again
- Bochner inequality
- Optimal maps
- Distributional Laplacian

Distributional Laplacian

Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and locally compact, $\Omega \subset X$ open, $g \in S^2(\Omega)$

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Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and locally compact, $\Omega \subset X$ open, $g \in S^2(\Omega)$

We say that $g \in D(\Delta, \Omega)$ if there exists a Radon measure μ on Ω such that

$$-\int_\Omega \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m} = \int_\Omega f \, \mathrm{d}\mu,$$

holds for every f Lipschitz with $\mathrm{supp}(f) \subset\subset \Omega$.

In this case we put $\Delta g_{|_\Omega} := \mu$

Calculus rules

Linearity

$$\Delta(\alpha_1 g_1 + \alpha_2 g_2) = \Delta g_1 + \Delta g_2$$

Chain rule

$$\Delta(\varphi \circ g) = \varphi' \circ g \, \Delta g + \varphi'' \circ g |Dg|^2 \mathfrak{m}$$

Leibniz rule

$$\Delta(g_1 g_2) = g_1 \Delta g_2 + g_2 \Delta g_1 + 2\nabla g_1 \cdot \nabla g_2 \mathfrak{m}$$

Relations with nonlinear potential theory

Theorem (G. '12. G. Mondino '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be inf. Hilb., with doubling measure and supporting a 2-Poincaré inequality.
Let $\Omega \subset X$ and $g \in S^2(\Omega)$.

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Then TFAE:

- $g \in D(\Delta, \Omega)$ and $\Delta g \leq 0$
- For every Lipschitz $f \geq 0$ with $\mathrm{supp}(f) \subset\subset \Omega$ we have

$$\int_\Omega |Dg|^2 \, \mathrm{d}\mathfrak{m} \leq \int_\Omega |D(g+f)|^2 \, \mathrm{d}\mathfrak{m}$$

Laplacian comparison

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The same holds on $RCD(0, N)$ spaces:
Thm (G. '12) For $(X, \mathsf{d}, \mathfrak{m})$ $RCD(0, N)$ and $\overline{x} \in X$ we have

$$\Delta \frac{\mathsf{d}^2(\cdot, \overline{x})}{2} \leq N\mathfrak{m}$$

Idea of the proof (1/2)

Pick $f \geq 0$ Lipschitz with compact support and let $\rho := cf^{\frac{N}{N-1}}$

$$\mu_0 := \rho\mathfrak{m}, \qquad \mu_1 := \delta_{\overline{x}}, \qquad t \mapsto \mu_t \text{ the geodesic connecting them}$$

The geodesic convexity of $\mathcal{U}_N$ gives

$$\varlimsup_{t\downarrow 0} \frac{\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0)}{t} \leq \mathcal{U}_N(\mu_1) - \mathcal{U}_N(\mu_0) = c^{1-\frac{1}{N}} \int f \, \mathrm{d}\mathfrak{m}$$

Idea of the proof (2/2)

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ be the lifting of (μ_t) and notice that

$$\begin{aligned}\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0) &\geq \int u_N'(\rho)\,\mathrm{d}(\mu_t - \mu_0) \\ &= \int u_N'(\rho)(\gamma_t) - u_N'(\rho)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\end{aligned}$$

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Notice that $\boldsymbol{\pi}$ represents the gradient of $\varphi := -\frac{\mathsf{d}^2(\cdot,\bar{x})}{2}$ to get

$$\begin{aligned}\lim_{t\downarrow 0} \frac{\mathcal{U}_N(\mu_t) - \mathcal{U}_N(\mu_0)}{t} &\geq \int \nabla(u_N'(\rho)) \cdot \nabla\varphi\,(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= \frac{c^{1-\frac{1}{N}}}{N} \int \nabla f \cdot \nabla\varphi\,\mathrm{d}\mathfrak{m}\end{aligned}$$

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Hence

$$-\frac{1}{N}\int \nabla f\cdot\nabla\frac{\mathrm{d}^2(\cdot,\overline{x})}{2}\,\mathrm{d}\mathfrak{m} \leq \int f\,\mathrm{d}\mathfrak{m}, \qquad \forall f \geq 0,\ \text{Lip with cpt supp}$$

Thank you