Vanishing of ℓ^p-cohomology via optimal transport

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Why ℓ^2-cohomology?

Theorem (Gaboriau, 2001)

Let G be a discrete group of infinite cardinality which admits a presentation with k generators and r relations. If $1 + r - k < 0$ then any normal subgroup of G is either

- finite,
- of finite index,
- not-finitely generated (as a group).

Corollary of a vanishing result for the ℓ^2-cohomology in degree 1.

Hopf's conjecture ($(-1)^n \chi(M^{2n}) > 0$ if the curvature of M is negative) is true for Kähler manifolds, thanks to result on the vanishing of ℓ^2-cohomology in all degrees (but the last). [Gromov 1991]

Why ℓ^p-cohomology?

It is an invariant of quasi-isometry...

Quasi-isometry

Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is a quasi-isometry if there exists two constants $L \geq 1$ and $C \geq 0$ such that

$$\begin{array}{ll} \forall x, x' \in X & \frac{1}{L} d(x, x') - C \leq d\big(f(x), f(x')\big) \leq L d(x, x') + C \\ \forall y \in Y, \exists x \in X & d\big(y, f(x)\big) \leq C. \end{array}$$

Definition

Let Γ be a finitely generated group. Let S be a finite generating set (*i.e.* $s \in S \implies s^{-1} \in S$). The Cayley graph of Γ (for S) $G = \mathcal{C}\mathrm{ay}(\Gamma, S)$ is a graph with vertex set $V = \Gamma$. Two vertices x and y are linked by an edge if $\exists s \in S$ such that $sx = y$.

Example of quasi-isometries

Example: If S and S' are two [finite symmetric] generating sets, the identity $\mathcal{C}\mathrm{ay}(G,S) \to \mathcal{C}\mathrm{ay}(G,S')$ is a quasi-isometry. More precisely (if d_S et $d_{S'}$ are, respectively, the distance on $\mathcal{C}\mathrm{ay}(G,S)$ and on $\mathcal{C}\mathrm{ay}(G,S')$)

$$d_{S'}(x,y) \leq K d_S(x,y)$$

where

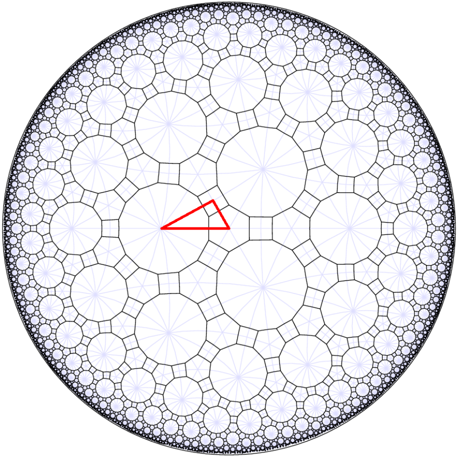
$$K = \max_{s \in S} d_{S'}(e,s)$$

Theorem (Kanai, 1985)

Any Riemanniann manifold of bounded geometry is quasi-isometric to a graph of bounded valency.

Example of quasi-isometries

Example: Let M be a compact manifold, $\pi_1(M)$ its fundamental group and $\widetilde{M}$ its universal covering. Then any Cayley graph of $\pi_1(M)$ is quasi-isometric to $\widetilde{M}$.



Why ℓ^p-cohomology?

1- Quasi-isometries. Pansu has computed and used it to show that some δ-pinched homogeneous manifolds cannot be quasi-isometric to any δ'-pinched manifold with $\delta' < \delta$. Gives many examples of rigidity (mostly for semi-direct products $N \rtimes \mathbb{R}$, where N is nilpotent).

2- Hyperbolic boundary. In hyperbolic groups, there is a $p_c \geq 1$ so that the ℓ^p-cohomology is non-trivial $\forall p > p_c$. This p_c gives a critical exponent for groups acting on $CAT(-1)$-spaces: $e(\Gamma) \geq p_c$ [Pansu + Bourdon, Martin & Valette]

$$e(\Gamma) = \inf\{s \in \mathbb{R}_{>0} \mid \sum_{g \in \Gamma} e^{-s|go-o|} < +\infty\}$$

Also, p_c is less than the conformal dimension of the ideal boundary (least Hausdorff dimension of a metric in the natural conformal structure). [Bourdon & Pajot]

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3- Poisson boundary. Under some condition on the isoperimetric profile, the reduced ℓ^p-cohomology in degree 1 lives inside the Poisson boundary; giving a quasi-isometric invariant part of a quasi-isometry sensitive concept. (G.)

What is ℓ^p-cohomology?

It is the cohomology of the complex of smooth forms with differential in L^p (interesting for contractible non-compact manifold M with bounded geometry)

$$L^p H^k(M) = \frac{\text{closed } k-\text{forms in } L^p}{\mathrm{d}((k-1)-\text{forms in } L^p)}$$

It turns out the definition is [even] simpler for graphs $G = (V, E)$ in degree 1. Take $E \subset V \times V$ symmetric, and let

$$\begin{array}{rccc} \nabla : & \{V \to \mathbb{R}\} & \to & \{E \to \mathbb{R}\} \\ & f & \mapsto & \nabla f(x,y) = f(y) - f(x) \end{array}$$

Definition

The ℓ^p-cohomology (resp. reduced) in degree 1 of a graph is

$$\ell^p H^1(G) = \frac{\mathrm{Im}\,\nabla \cap \ell^p(E)}{\nabla \ell^p(V)} \qquad \text{resp.} \quad \underline{\ell^p H}^1(G) = \frac{\mathrm{Im}\,\nabla \cap \ell^p(E)}{\overline{\nabla \ell^p(V)}}.$$

ℓ^p-cohomology

The space of p-Dirichlet functions is $\mathrm{D}^p(G) = \{f : V \to \mathbb{R} \mid \nabla f \in \ell^p(E)\}$.

It is endowed with a semi-norm $\|f\|_{\mathrm{D}^p} = \|\nabla f\|_{\ell^p}$. ("semi-" $\to$ constant functions).

In "integrated expression", the reduced ℓ^p-cohomology in degree 1 is also

$$\underline{\ell^p H}^1(G) = \frac{\mathrm{Im}\,\nabla \cap \ell^p(E)}{\overline{\nabla \ell^p(V)}^{\ell^p}} = \frac{\mathrm{D}^p(G)}{\overline{\ell^p(V) + \mathrm{cst}}^{\mathrm{D}^p}}$$

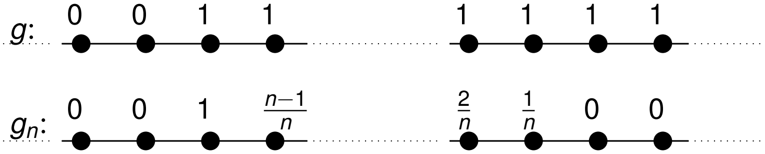
Theorem (Élek 1998, Pansu ∅)

Fix a bound on the geometry (valency, curvature and injectivity radius). Then the [reduced] ℓ^p-cohomology [in degree 1] is an invariant of quasi-isometry.

A simple (yet important) example.

$$\underline{\ell^p H}^p(G) = \mathrm{D}^p(G) / \overline{\ell^p(V) + \mathbb{R}}^{\mathrm{D}^p(G)}$$

Example: Let $G = \mathcal{C}\mathrm{ay}(\mathbb{Z}, \{\pm 1\})$ be the line.



Then $g_n \overset{\mathrm{D}^p}{\to} g$ if $p > 1$. Thus $\forall 1 < p < \infty, [g] = 0 \in \underline{\ell^p H}^1(G)$.

But, $[g] \neq 0 \in \underline{\ell^1 H}^1(G)$.

Ends and $\underline{\ell^1 H}^1$

$\underline{\ell^1 H}^1(G)$ is intimately related to the ends of a graph.

Definition

Let $n \in \mathbb{Z}_{\geq 0}$. A graph $\Gamma = (V, E)$ has $\leq n$ ends if for all finite $F \subset V$, $V \setminus F$ has $\leq n$ infinite connected components.

Examples:

- A finite graph has 0 ends.
- $\mathcal{C}\mathrm{ay}(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\})$ (infinite grid) has 1 end.
- $\mathcal{C}\mathrm{ay}(\mathbb{Z}, \{\pm 1\})$ (infinite line) has 2 ends.
- $\mathcal{C}\mathrm{ay}(F_k, \{a_1^{\pm}, \ldots, a_k^{\pm 1})$ has ∞ many ends.

Ends and $\ell^1 H^1$

Lemma

The number of ends is a quasi-isometry invariant.

If $f : X \to Y$ sends one end into two ends, there is a sequence of neighbours in X, say x_n and x_n', which are sent into two distinct ends of Y. Thus $d_Y\big(f(x_n), f(x_n')\big) \to \infty$ even if $d_X(x_n, x_n') = 1$. So f is not a quasi-isometry.

Ends and $\ell^1 H^1$

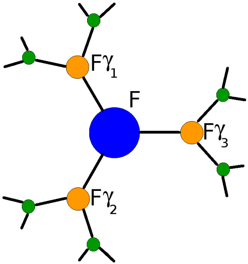
Lemma

The number of ends is a quasi-isometry invariant.

Theorem (Hopf, 1944)

The number of ends of a Cayley graph is 0, 1, 2 or ∞.

Idea: 3 ends $\Longrightarrow$ ∞ ends



Theorem (Stallings, 1971)

[The Cayley graph of] a group has 2 ends iff it contains $\mathbb{Z}$ as a finite index subgroup. It has ∞ many ends iff it is a "non-trivial" amalgamated product or HNN extension.

Ends and $\underline{\ell^1 H}^1$

$$\underline{\ell^1 H}^1(G) = \mathrm{D}^1(G)/\overline{\ell^1(V)+\mathbb{R}}^{\mathrm{D}^1(G)}$$

Lemma ("well-known")

If G has finitely many ends, $\underline{\ell^1 H}^1(G) \cong \mathbb{R}^{\mathrm{ends}(G)-1}$.

For P a path from x to y, $f(y) = f(x) + \sum_{e \in P} \nabla f(e)$, thus $\mathrm{D}^1(G) \subset \ell^\infty(V)$. Furthermore $\|g\|_{\mathrm{D}^1}$ is small outside a finite set F. This means g is almost constant outside$F$. Thence $g \in \mathrm{D}^1(G)$ has a value on each end.

The map "value on the ends" β is linear and continuous on $\mathrm{D}^1(G)$. Consequently, $\ell^1(V) \subset \mathrm{Ker}\,\beta \implies \overline{\ell^1(V)}^{\mathrm{D}^1(G)} \subset \mathrm{Ker}\,\beta$.

If $g \in \mathrm{D}^1(G)$ takes 0 value on the ends: let g_t be the g truncated to values $> t$, then $g - g_t$ has finite support and tends to g if $t \to 0$. Thus, $g \in \overline{\ell^1(V)}^{\mathrm{D}^1(G)}$

Lastly, constant functions has the same value on all ends.

Link with p-harmonic functions

Given $f \in \underline{\ell^p H^p}(G) = \mathrm{D}^p(G) / \overline{\ell^p(V) + \mathbb{R}}^{\mathrm{D}^p(G)}$, what does the element of minimal D^p-norm looks like in $[f]$?

In other words, what is a minimiser of the p-energy $\|\nabla g\|^p_{\ell^p} = \int |\nabla g|^p$?

A quick computation, if f is of minimal norm then, $\forall g \in \ell^p(V)$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \|f + tg\|^p_{\mathrm{D}^p} \bigg|_{t=0} = p \sum_{e \in E} \nabla g(e) |\nabla f(e)|^{p-2} \nabla f(e) = p \langle \nabla g \mid \mu_{p,p'} \nabla f \rangle$$

where p' is the Hölder conjugate of p and $\mu_{p,q} : \ell^p(\mathbb{N}) \to \ell^q(\mathbb{N})$ is the [Mazur] map defined by $\mu_{p,q} F(n) = |F(n)|^{\frac{p}{q} - 1} F(n)$.

The p-harmonic equations (for $p \in \mathbb{Z}_{\geq 2}$) originally came up when looking at conformal maps $\mathbb{R}^p \to \mathbb{R}^p$.

Link with p-harmonic functions

f minimal D^p-norm in $[f] \iff \forall g \in \ell^p(V)$, $\langle \nabla g \mid \mu_{p,p'} \nabla f \rangle = 0$

For a countable X, there is a “canonical” pairing between finitely supported functions $f, g : X \to \mathbb{R}$, defined by

$$\langle f \mid g \rangle = \sum_{x \in X} f(x) g(x)$$

The dual of the gradient is the divergence:

$$\begin{array}{rccl} \nabla^* : & \{E \to \mathbb{R}\} & \to & \{G \to \mathbb{R}\} \\ & f & \mapsto & \nabla^* f(y) = \sum_{x \sim y} f(x,y) - \sum_{x \sim y} f(y,x) \end{array}$$

(Recall that, for a k-valent graph, $\nabla^* \nabla = \Delta = k(\mathrm{Id} - P)$.)

Theorem (Puls 2006)

· f minimises the p-energy in its reduced class iff it satisfies $\nabla^* \mu_{p,p'} \nabla f \equiv 0$.
· $\underline{\ell^p H}^1(\Gamma)$ is also the reduced cohomology of the left-regular representation on $\ell^p \Gamma$.

A small trick

$$f \in \underline{\ell^p H^p}(G) = \mathrm{D}^p(G)/\overline{\ell^p(V)+\mathbb{R}}^{\mathrm{D}^p(G)}.$$

Let $G = \mathcal{C}\mathrm{ay}(\Gamma, S)$. The left-regular representation is the action of Γ on functions $f : \Gamma \to \mathbb{R}$ defined by $\lambda_\gamma f(x) = \delta_\gamma * f(x) = f(\gamma^{-1}x)$.

The gradient of f is made up of $(\delta_s - \delta_1) * f$ as s runs over S.

For $s \in S$, $[\lambda_s f] = [f]$. Indeed,

$$f(g) - \lambda_s f(g) = f(g) - f(s^{-1}g) = \nabla f(s^{-1}g, g).$$

By writing γ as a word in S, $[\lambda_\gamma f] = [f]$. Then going to convex combinations, $[\mu * f] = [f]$ for any finitely supported probability measure on Γ.

A small trick

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Lemma (G.)

If there is a sequence μ_n of finitely supported probability measures, and constants $c_n \in \mathbb{R}$ with

- $\mu_n * f - c_n \to 0$ point-wise;
- $\forall n, \|\mu_n * f\|_{\mathrm{D}^p} \leq K$;

then $[f] = 0 \in \underline{\ell^p H}^1(\Gamma)$.

A small trick

Lemma (G.)

Let $1 < p < \infty$. If there is a sequence μ_n of finitely supported probability measures, and constants $c_n \in \mathbb{R}$ with

- $\mu_n * f - c_n \to 0$ point-wise
- $\forall n, \|\mu_n * f\|_{\mathrm{D}^p} \leq K$

then $[f] = 0 \in \underline{\ell^p H}^1(\Gamma)$.

Proof: in $\ell^p(\mathbb{N})$, boundedness + pointwise convergence implies weak* convergence.

In the reflexive regime, this implies weak convergence.

By Hahn-Banach, weakly closed convex sets are norm closed $\implies$ there is a sequence $\{h_n\}$ so that each h_n is a convex combination of the $\{\mu_i * f_i - c_i\}_{i=1}^n$ and $h_n \to 0$ in norm.

But $[h_n] = [f]$, so $[f] = 0$.

A simple result

Lemma (G.)

Let $1 < p < \infty$. If there is a sequence μ_n of finitely supported probability measures, and constants $c_n \in \mathbb{R}$ with

- $\mu_n * f - c_n \to 0$ point-wise
- $\forall n, \|\mu_n * f\|_{\mathrm{D}^p} \leq K$

then $[f] = 0 \in \underline{\ell^p H}^1(\Gamma)$.

This lemma applies easily to groups with infinitely many finite conjugation classes: take μ_n to be the uniform measure on C_n.

The gradient is made of $\{\lambda_s f - f\}_{s \in S}$. Since $\mu_n * \lambda_s * f = \lambda_s * \mu_n * f$, $\|\mu_n * f\|_{\mathrm{D}^p(\Gamma)} \leq \|\mu_n\|_{\ell^1(\Gamma)} \|f\|_{\mathrm{D}^p(\Gamma)}$.

Pick $c_n = \mu_n * f(1)$, then $\mu_n * f - c_n$ tends pointwise to 0 near 1.

Corollary

Theorem (Gromov, Puls, Martin-Valette, Kappos, ...)

Let $1 < p < \infty$. If Γ has infinitely many finite conjugacy class (*e.g.* Γ is nilpotent) then $\underline{\ell^p H}^1(\Gamma) = 0$.

Passes to virtually such groups by quasi-isometry.

To go further, need:

Lemma (Holopainen & Soardi, 1994)

$$\frac{\mathrm{D}^p(G) \cap \ell^\infty(V)}{\overline{\ell^p(V) + \mathbb{R}}^{\mathrm{D}^p(G)}} = \{0\} \iff \frac{\mathrm{D}^p(G)}{\overline{\ell^p(V) + \mathbb{R}}^{\mathrm{D}^p(G)}} = \{0\}$$

In other words, if all bounded functions in $\mathrm{D}^p(G)$ have a trivial class, then the reduced ℓ^p-cohomology is trivial.

Amenable groups

Lemma (Holopainen & Soardi, 1994)

$$\frac{\mathrm{D}^p(G)\cap\ell^\infty(V)}{\overline{\ell^p(V)+\mathbb{R}}^{\mathrm{D}^p(G)}}=\{0\}\iff\frac{\mathrm{D}^p(G)}{\overline{\ell^p(V)+\mathbb{R}}^{\mathrm{D}^p(G)}}=\{0\}$$

This is interesting since

Definition (Følner 1955/Reiter)

A group is amenable if there is a sequence ξ_n of finitely supported probability measures with $\|\xi_n-\lambda_s\xi_n\|_{\ell^1(G)}\to 0$. A Følner sequence is the specific case when ξ_n are uniform measures on some finite sets F_n.

Historical note on amenability

Amenability was introduced by von Neumann to counter the (Hausdorff-)Banach-Tarski paradox. This paradox stems from the fact that $\mathrm{SO}_3(\mathbb{R})$ contains a free subgroup (consider two irrational rotation with independant axis).

Definition (Von Neumann 1929)

A group is amenable if there is a linear functional $m : \ell^\infty(\Gamma) \to \mathbb{R}$ such that $m(\lambda_\gamma f) = m(f)$, $m(f) \geq 0$ if $f \geq 0$.

If Γ is not finite, the existence of such an element requires the axiom of choice, and is consequently completely non-explicit. However, the existence of sequence of almost-invariant probability measures can be very explicit: if the group does not have exponential growth, then a subsequence of $\mathbb{1}_{B_n}/|B_n|$ (where B_n are balls of radius n) will do.

Amenable groups

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A group is amenable if there is a sequence ξ_n of finitely supported probability measures with $\|\xi_n-\lambda_s\xi_n\|_{\ell^1(G)}\to 0$. A Følner sequence is the specific case when ξ_n are uniform measures on some finite sets F_n.

Indeed, if $f\in\mathrm{D}^p(\Gamma)\cap\ell^\infty(\Gamma)$, then

$$\|\xi_n*f-\lambda_s\xi_n*f\|_{\ell^\infty(\Gamma)}\leq\|\xi_n-\lambda_s\xi_n\|_{\ell^1(\Gamma)}\|f\|_{\ell^\infty(\Gamma)}$$

So that ξ_n*f tends pointwise [even uniformly] to a constant.

Amenable groups

Indeed, if $f \in \mathrm{D}^p(\Gamma) \cap \ell^\infty(\Gamma)$, then

$$\|\xi_n * f - \lambda_s \xi_n * f\|_{\ell^\infty(\Gamma)} \leq \|\xi_n - \lambda_s \xi_n\|_{\ell^1(\Gamma)} \|f\|_{\ell^\infty(\Gamma)}$$

So that $\xi_n * f$ tends pointwise [uniformly] to a constant.

It remains to check which condition comes up when one tries to verify the second condition in the "trick": $\|\xi_n * f\|_{\mathrm{D}^p} \leq K$ for all n.

$$\begin{aligned}(\delta_s - \delta_e) * \xi_n * f(\eta) &= \sum_{\gamma \in sX_n} \xi_n(s^{-1}\gamma)\lambda_\gamma f(\eta) - \sum_{\gamma \in X_n} \xi_n(\gamma)\lambda_\gamma f(\eta) \\ &= \sum_{\gamma \in X_n} \xi_n(\gamma) f(\gamma^{-1}s^{-1}\eta) - \sum_{\gamma \in X_n} \xi_n(\gamma) f(\gamma^{-1}\eta),\end{aligned}$$

where X_n is the support of ξ_n.

Amenable groups

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where X_n is the support of ξ_n.

Putting $\phi^\vee(\gamma) = \phi(\gamma^{-1})$, the expression

$$(\delta_s - \delta_e) * \xi_n * f(\eta) = \int f d(\lambda_s \xi_n)^\vee - \int f d\xi_n^\vee$$

indicates that estimates on transport cost would be welcome.

Transport cost

The pairing $\langle f \mid g \rangle = \sum_{x \in X} f(x)g(x)$ extends to more general spaces of functions (*e.g.* $f \in \ell^p(X)$ and $g \in \ell^{p'}(X)$, *e.g.* f arbitrary and g finitely supported).

The adjoint of ∇ (for this pairing) is the divergence:

$$\begin{array}{rccl} \nabla^* : & \{E \to \mathbb{R}\} & \to & \{G \to \mathbb{R}\} \\ & f & \mapsto & \nabla^* f(y) = \sum_{x \sim y} f(x,y) - \sum_{x \sim y} f(y,x) \end{array}$$

A "transport pattern" from ξ to ϕ (two finitely supported probability measures) is a finitely supported function $m : E \to \mathbb{R}_{\geq 0}$ such that $\nabla^* m = \phi - \xi$.

It always exists since the graph is assumed connected. In pedantic terms, ∇^* is the boundary operator ∂_1 from 1-chains to 0-chains. In finite connected complexes the image of this operator is the kernel of ∂_0, *i.e.* functions which sum to 0.

Transport cost

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A "transport pattern" from ξ to ϕ (two finitely supported probability measures) is a finitely supported function $m : E \to \mathbb{R}_{\geq 0}$ such that $\nabla^* m = \phi - \xi$.

The assumption that m is positive is also always true: if m is not positive, consider $m'(x,y) = \max(m(x,y) - m(y,x), 0)$. Then, m' is positive and $\nabla^* m' = \nabla^* m$.

Given a "price" on the edges (a function $f' : E \to \mathbb{R}$) the transport cost for m is $\mathrm{TC}_{f';m}(\xi,\phi) = \langle f' \mid m \rangle$, and the optimal transport cost $W_1(\xi,\mu) = \inf_m \mathrm{TC}_{\mathbb{1},m}(\xi,\phi) = \inf_m \langle \mathbb{1} \mid m \rangle$.

Transport cost

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By design, $\mathrm{TC}_{\nabla f;m}(\xi,\phi) = \langle \nabla f \mid m \rangle = \langle f \mid \nabla^* m \rangle = \langle f \mid \phi - \xi \rangle$.

Let’s rewrite

$$(\delta_s - \delta_e) * \xi_n * f(\eta) = \mathrm{TC}_{\rho_\eta \nabla f,m}\big((\lambda_s \xi_n)^\vee, \xi_n^\vee\big)$$

where ρ_η is the right-regular representation and $\xi^\vee(\gamma) = \xi(\gamma^{-1})$

Transport cost

$$(\delta_s - \delta_e) * \xi_n * f(\eta) = \mathrm{TC}_{\rho_\eta \nabla f, m}\big((\lambda_s \xi_n)^\vee, \xi_n^\vee\big)$$

Define a map T which sends a function $f' \in \ell^p(E)$ to $(Tf')(\eta) = \mathrm{TC}_{\rho_\eta f', m_n}\big((\lambda_s \xi_n)^\vee, \xi_n^\vee\big)$ (a function on Γ). If one can show that T is bounded from $\ell^1(E)$ to $\ell^1(\Gamma)$ and from $\ell^\infty(E)$ to $\ell^\infty(\Gamma)$ then, by Riesz-Thorin interpolation, this map is bounded for all $1 \leq p \leq \infty$.

A finitely generated group G will be said **transport amenable** if there exists S a finite generating set, a positive constant $K \in \mathbb{R}$ and a sequence of finitely supported probability measure ξ_n converging to a left-invariant mean such that, $\forall s \in S$ and $\forall n$, the transport cost

$$\mathrm{TC}(\rho_s \xi_n^\vee, \xi_n^\vee) \leq K,$$

ℓ^∞ bound

Take m_n a transport pattern from $(\lambda_s \xi_n)^\vee$ to $\xi_n^\vee$ so that $\mathrm{TC}_{\mathbb{1},m_n}\big((\lambda_s\xi_n)^\vee,\xi_n^\vee\big) \leq 2K$.

$$\|Tf'\|_{\ell^\infty(\Gamma)} = \sup_{\eta\in\Gamma} |Tf'(\eta)| = |\mathrm{TC}_{\rho_\eta f',m_n}\big((\lambda_s\xi_n)^\vee,\xi_n^\vee\big)|$$

By replacing the price of each edge by the maximal price of an edge $\|f'\|_{\ell^\infty}$, this last quantity is bounded by $\|f'\|_{\ell^\infty(E)}|\mathrm{TC}_{\mathbb{1},m_n}\big((\lambda_s\xi_n)^\vee,\xi_n^\vee\big)| \leq 2K\|f'\|_{\ell^\infty(E)}$.

ℓ^1 bound

Given: m_n so that $\mathrm{TC}_{\mathbb{1},m_n}\big((\lambda_s\xi_n)^\vee,\xi_n^\vee\big) \leq 2K$.

$$\begin{aligned} \|Tf'\|_{\ell^1(\Gamma)} &= \sum_{\eta\in\Gamma} |Tf'(\eta)| &&= \sum_{\eta\in\Gamma} |\langle \rho_\eta f' \mid m_n\rangle| \\ &\leq \sum_{\eta\in\Gamma} \langle \rho_\eta |f'| \mid m_n\rangle &&= \langle \sum_{\eta\in\Gamma} \rho_\eta |f'| \mid m_n\rangle \\ &= \|f'\|_{\ell^1(E)} \langle \mathbb{1} \mid m_n\rangle &&= 2K\|f'\|_{\ell^1(E)} \end{aligned}$$

Thus $T : \ell^p(E) \to \ell^p(\Gamma)$ is bounded.

Corollary and example

Theorem (G.)

If Γ is transport amenable then $\underline{\ell^p H}^1(\Gamma) = \{0\}$.

If $F \subset V$ is a subset of vertices, ∂F is the subset of edges incident with F and F^c.

A group is CF (for controlled Følner) if there is a sequence of finite sets such that $\frac{|\partial F_n|}{|F_n|} \leq \frac{K}{\mathrm{Diam}\, F_n}$.

Tessera showed virtually polycyclic groups, solvable Baumslag-Solitar, lampighter $\mathbb{Z}_2 \wr \mathbb{Z}$, ... have this property. In fact, Tessera showed it implies that $\mathbb{E}(|W_n|) \sim n^{1/2}$ and (if it is also of exponential growth) $P_1^{(n)}(1) \sim e^{-n^{1/3}}$. In particular, lamplighters $\mathbb{Z} \wr \mathbb{Z}$ or $\mathbb{Z}_2 \wr \mathbb{Z}^2$ do not have this property.

Examples

Recall: multiplication on the left defines edges, multiplication on the right is a Cayley graph automorphism.

Example 0: Abelian groups are transport amenable.

Take F_n a sequence of balls of radius n and $\xi_n = \mathbb{1}_{F_n}/|F_n|$.

Must transport F_n^{-1} to $F_n^{-1}s$.

Since the group law is commutative $F_n^{-1}s = sF_n^{-1}$. It suffice to move all elements by s.

This gives a transport cost of 1.

This strategy will always fail in a non-Abelian group: the action on the right (automorphism) may *a priori* distort the distance (defined by multiplication on the left) in an uncontrolled fashion.

Examples

CF groups: $\dfrac{|\partial F_n|}{|F_n|} \leq \dfrac{K}{\mathrm{Diam}\, F_n}$

Example 1: CF groups are transport amenable.
WLOG $F_n \subset B_{2d_n}(1)$ with $d_n = \mathrm{Diam}\, F_n$.
Pick $\xi_n = |F_n|^{-1} \mathbb{1}_{F_n}$.
Let σ be a bijection from $F_n s \setminus F_n$ to $F_n \setminus F_n s$.
Then m is defined by taking the weight at $x \in F_n s \setminus F_n$ and sending it to $\sigma(x)$.
There are $|\partial F_n|$ such x to be transported, the weight of each one is $|F_n|^{-1}$ and the distance they are carreid along is at most $\mathrm{Diam}\, F_n$.
Thus the cost of the transport is $\leq \frac{|\partial F_n|}{|F_n|} \mathrm{Diam}\, F_n \leq K, \forall n$.

Examples

Example 2: Lamplighter groups $F \wr H$ where H is of polynomial growth and F finite.

$L \wr H = H \ltimes_\rho (\oplus_H L)$ where H acts on the infinite sum $\oplus_H L$ by shifting indices: $\rho_h \phi(k) = \phi(kh)$. For $h, g \in H$ and $\phi, \psi : H \to L$ finitely supported, the product is

$$(h, \{\phi_k\}_{k \in H}) \cdot (g, \{\psi_k\}_{k \in H}) = (hg, \{\phi_{kg}\psi_k\}_{k \in H})$$

· Følner sequence: let A_n be a CF sequence of diameter d_n in H and let B_k be the functions $H \to L$ which are supported on A_k. Then $F_n := A_n \times B_n$ is a Følner sequence. Take $\xi_n = \mathbb{1}_{F_n}/|F_n|$.

· $r \in \{e_H \to S_L\} \times \{e_H\}$ does not displace the Følner set at all.

· For $z \in \{0\} \times S_H$, displace elements from $F_n^{-1} z$ to F_n^{-1} with a cost which is bounded by a constant multiple of $|F_n|$.

· The F_n^{-1} are ungainly, invert everything: namely displace $z^{-1} F_n$ to F_n, where the cost of displacing γ to γ' is the length of the element η such that $\gamma\eta = \gamma'$.

Examples

Example 2: Lamplighter groups $F \wr H$ where H is of polynomial growth and F finite.

Goal: the set F_n has been displaced to zF_n, and one must transport it back to F_n by multiplying on the right.

Difficulty: For example, when H is Abelian, when one looks at zF_nz^{-1} then the lamplighter positions are OK, but there are going to be lamps states which might be "on" on the set A_nz^{-1}.

One must change the lamp states before going to the final position.

· Fix a bijection $\sigma : z^{-1}A_n \setminus A_n \to A_n \setminus z^{-1}A_n$.

· Given i non-trivial lamp states, take the lamplighter (with the lamps) along a path so that the i lamps (and their image by σ) pass at the identity. When these lamps are at the identity, interrupt this path so as to change their states. Then return the lamplighter to a neighbouring position. Make at most $2i(d_n + k) + d_n$ steps where the diameter of the "lamp states group" F is k;

Examples

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· For i fixed, the number of such elements is $c^{|A_n|-|\Delta_z A_n|}(c-1)^i\binom{|\Delta_z A_n|}{i}$ where $c=|F|$;

· So the total of used edges is at most (recall $\sum_{i=1}^{N}(c-1)^i i\binom{N}{i} = Nc^{N-1}$)

$$c^{|A_n|-|\Delta_z A_n|}\sum_{i=0}^{|\Delta_z A_n|}(c-1)^i\binom{|\Delta_z A_n|}{i}\left(2i(d_n+k)+d_n\right)$$
$$\leq c^{|A_n|-1}\left(2(d_n+k)|\Delta_z A_n|+d_n c\right);$$

· As $|F_n| = c^{|A_n|}|A_n|$, one gets a bound of $(2|\Delta_z A_n|(d_n+k)+d_n c)/|A_n|c$.

Examples

The essential point in the previous example was that the lamplighter position group has a sequence which is left-Følner and right-controlled-Følner (sequence of balls).

Example 3: Lamplighter groups $G \wr H$ where H is of polynomial growth and G amenable.

Question: Are there amenable groups which are not transport amenable?

Examples 2 & 3 are important as these groups have non-trivial bounded harmonic functions.

Under some assumptions on the isoperimetric profile, one has that the non-trivial bounded elements of $\underline{\ell^p H}^1(G)$ can be mapped to bounded harmonic functions.

Isoperimetric profiles

Note: Though our upcoming examples and applications are mostly going to be about Cayley graphs, G is no longer necessarily such a graph.

Definition

Let $d \in \mathbb{Z}_{\geq 1}$. A graph G satisfies a d-dimensional isoperimetric profiles (noted IS_d) if $\exists K > 0$ such that, $\forall F \subset V$ finite,

$$|F|^{1-\frac{1}{d}} \leq K|\partial F|$$

It has a strong isoperimetric profile (noted IS_ω) if $\exists K > 0$ such that, $\forall F \subset V$ finite, $|F| \leq K|\partial F|$

Examples: $\mathcal{C}\mathrm{ay}(\mathbb{Z}^d, S)$ satisfait IS_d.

A group is amenable iff its Cayley graph does not satisfy IS_ω. (Restatement of Følner)

Isoperimetric profiles

Satisfying IS_α (for $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\omega\}$) is invariant under quasi-isometries.

Hyperbolic $\implies \mathrm{IS}_\omega \implies \mathrm{IS}_d, \forall d$.

But $\mathrm{IS}_d, \forall d \not\Longrightarrow \mathrm{IS}_\omega$. For example, Cayley graphs of $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ where $\alpha(1) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Theorem (Varopoulos 1985 + Gromov 1981 + Wolf 1968)

Γ has polynomial growth of degree $\leq d \iff \mathcal{C}\mathrm{ay}(\Gamma, S)$ does not have IS_{d+1}.

$\implies$ groups which are not virtually nilpotent have IS_d for all d.

Harmonic functions

Let $P_x^{(n)}$ be the measure defined by $P_x^{(n)}(y) =$ the probability that a simple random walker starting at x ends up in y after n steps.

This gives a kernel: $P^{(n)}g(x) := \int g(y) \mathrm{d}P_x^{(n)}(y)$.

A function $g : V \to \mathbb{R}$ is harmonic if $P^{(1)}g = g$ (mean value property).

Definition

Here the "Poisson boundary" is the space of bounded harmonic functions. $\mathcal{H}_b(G)$ will denote this space modulo constant functions.

Harmonic functions

Definition

Here the "Poisson boundary" is the space of bounded harmonic functions. $\mathcal{H}_b(G)$ will denote this space modulo constant functions.

$\underline{\ell^1 H}^1$ is related to the ends ["well-known"]. In the hypoerbolic case, there a strong link between ℓ^p-cohomology in degree 1 and some space of functions on the hypoerbolic boundary [Bourdon & Pajot 2003]. A natural idea, to attack amenable groups, is to try to look at the "values" of this function on the "Poisson boundary".

To define a value of g "on" the Poisson boundary, the natural idea is to look at $\lim_{n\to\infty} P^{(n)} g$.

Transport again: to infinity and beyond!

$P^{(n)}g(x)$ Cauchy? $\to P^{(n)}g(x) - P^{(m)}g(x) = \int g \mathrm{d}P_x^{(n)} - \int g \mathrm{d}P_x^{(m)}$.

With a transport pattern, one gets a bound:

$$\begin{aligned} \int g\mathrm{d}\xi - \int g\mathrm{d}\phi &= \langle g \mid \xi - \phi \rangle &= \langle g \mid \nabla^* \tau_{\phi,\xi} \rangle \\ &= \langle \nabla g \mid \tau_{\phi,\xi} \rangle &\leq \|\nabla g\|_{\ell^p} \|\tau_{\phi,\xi}\|_{\ell^{p'}} \end{aligned}$$

Pointwise convergence of $P^{(n)}g$ (in x) can be deduced from a $\ell^{p'}$ bound on the transport from $P_x^{(n)}$ to $P_x^{(m)}$. There is a very natural transport patter:

take $m - n$ random steps!

Transport again: to infinity and beyond!

So it essentially suffices to check that $\sum_{n\geq 0} P_x^{(n)}$ is in $\ell^{p'}(V)$.

Theorem (Varopoulos 1985)

If G has IS_d, then for some $K > 0$, $\|P_x^{(n)}\|_{\ell^\infty(V)} \leq Kn^{-d/2}$.

Thus

$$\|P_x^{(n)}\|_{\ell^q(V)}^q \leq \|P_x^{(n)}\|_{\ell^\infty(V)}^{q-1} \|P_x^{(n)}\|_{\ell^1(V)} \leq K' n^{-d(q-1)/2}$$

and $\sum P^{(n)}$ converges in ℓ^q if $q' < d/2$.

Boundary values

Theorem (G.)

If G satisfies IS_d and $1 \leq p < d/2$. There is a map π from $\mathrm{D}^p(G)$ into harmonic functions modulo constants such that

- $[g] = [h] \in \underline{\ell^p H}^1(G) \iff \pi(g) = \pi(h)$.
- if $g \in \mathrm{D}^p(G) \cap \ell^\infty(V)$, then $\pi(g) \in \ell^\infty(V)$.
- if $g \in \mathrm{D}^p(G)$, then $\pi(g) \in \mathrm{D}^q(G)$ for all $q > \dfrac{dp}{d-2p}$.

Using vanishing of reduced ℓ^2-cohomology for all amenable groups (Cheeger-Gromov 1986):

Corollary (G.)

If G is the Cayley graph of an amenable group Γ and $1 < p < q < \infty$. Then the identity map $\underline{\ell^p H}^1(G) \to \underline{\ell^q H}^1(G)$ is injective. In particular, $\underline{\ell^p H}^1(G) = \{0\}$ for all $1 < p \leq 2$.

Corollaries

Corollary (G.)

If G satisfies IS_d and $\mathcal{H}_b(G) = \{0\}$, then $\underline{\ell^p H}^1(G) = \{0\}$ for all $p \in [1, \frac{d}{2}[$.

Indeed, $\pi(g) = 0$, $\forall g \in \mathrm{D}^p(G) \cap \ell^\infty(V)$, so the theorem gives $[g] = 0$.

Corollary (G.)

If Γ is of intermediate growth then $\underline{\ell^p H}^1(\Gamma) = \{0\}$ for all $p \in [1, \infty[$.

Vanishing via this second theorem also contains strictly CF (except for polynomial growth).

However, transport amenability and this criterion are not contained in each other.

Corollaries

Using further techniques:

Corollary (G.)

An amenable wreath product $L \wr H$ may have non-constant bounded harmonic functions, but if $\underline{\ell^p H}^1(H) = \{0\}$ their gradient is never in $\ell^p(E)$, $\forall p \in [1,\infty[$.

As mentioned at the beginning, isoperimetric profiles as well as reduced ℓ^p-cohomology are quasi-isometry invariants. But Lyons (1987) showed the Poisson boundary (*i.e.* $\mathcal{H}_b(G)$) is not invariant under quasi-isometries.

Corollary (G.)

If G satisfies IS_d then $\cup_{p<d/2}\pi\big(D^p(G)\cap\ell^\infty(V)\big)$ is a part of the Poisson boundary which is invariant under quasi-isometries.

Some questions

Question 1: Is there an amenable group which is not transport amenable?

Question 2: Is there an amenable group with a non-constant (bounded or not) harmonic functions whose gradient is in c_0?

It is easy to make harmonic functions with gradient in ℓ^∞ (*i.e.* Lipschitz harmonic functions) on ANY group. For example, in $\mathbb{Z}^d$, any linear map to $\mathbb{R}$ will be harmonic and Lipschitz.

More generally, any nilpotent groups has an infinite center, hence such a map. But, Kleiner showed that the space of Lipschitz harmonic functions is finite dimensional on [virtually] nilpotent groups. In particular, they may not have gradient in c_0.

However, even for other amenable groups, there are no known example of a harmonic function with gradient in c_0.

Some questions

Question 1: Is there an amenable group which is not transport amenable?

Question 2: Is there an amenable group with a non-constant (bounded or not) harmonic functions whose gradient is in c_0?

In the group case, most of the arguments above carry if one has a sequence of measures ξ_n which tends to a left-invariant mean and a bound $\|\tau_{\delta_e,\xi_n}\|_{\ell^p} \leq K, \forall n$.

Γ has no bounded harmonic functions $\iff$ the $P^{(n)}$ tend to a left-invariant mean.

Question 3: Take an amenable group Γ which has bounded harmonic functions and ξ_n tending to a left-invariant mean. Can one find a transport pattern τ_{δ_e,ξ_n} with uniform bound (in n) for its $\ell^q(E)$-norm, $q < 2$?

Indeed, the previous pattern is extremely bad. For example, one can easily improve the above in $\mathbb{Z}^d$, so that the theorem applies in a larger range (*i.e.* if $p < d$ instead of $p < d/2$).

Thanks for your attention!

Based on 2 papers and a note:

1- Vanishing of ℓ^p-cohomology and transportation cost,
to appear in “Bull. London Math. Soc.”.

2- Boundary values of random walks and ℓ^p-cohomology in degree one, preprint.

3- Absence of non-constant harmonic functions with gradient in ℓ^p in wreath products, in progress.