Vanishing of $\ell^p$-cohomology via optimal transport

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Let $G$ be a discrete group of infinite cardinality which admits a presentation with $k$ generators and $r$ relations. If $1 + r - k < 0$ then any normal subgroup of $G$ is either

- finite,
- of finite index,
- not-finitely generated (as a group).

Corollary of a vanishing result for the $\ell^2$-cohomology in degree 1.

Hopf’s conjecture ($(-1)^n\chi(M^{2n}) > 0$ if the curvature of $M$ is negative) is true for Kähler manifolds, thanks to result on the vanishing of $\ell^2$-cohomology in all degrees (but the last). [Gromov 1991]
Why $\ell^p$-cohomology?

It is an invariant of quasi-isometry...
**Quasi-isometry**

**Definition**

Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. A map \(f : X \to Y\) is a quasi-isometry if there exists two constants \(L \geq 1\) and \(C \geq 0\) such that

\[
\forall x, x' \in X \quad \frac{1}{L}d(x, x') - C \leq d(f(x), f(x')) \leq Ld(x, x') + C,
\]

\[
\forall y \in Y, \exists x \in X \quad d(y, f(x)) \leq C.
\]

**Definition**

Let \(\Gamma\) be a finitely generated group. Let \(S\) be a finite generating set (i.e. \(s \in S \implies s^{-1} \in S\)). The Cayley graph of \(\Gamma\) (for \(S\)) \(G = Cay(\Gamma, S)\) is a graph with vertex set \(V = \Gamma\). Two vertices \(x\) and \(y\) are linked by an edge if \(\exists s \in S\) such that \(sx = y\).
Example: If $S$ and $S'$ are two [finite symmetric] generating sets, the identity $\text{Cay}(G, S) \rightarrow \text{Cay}(G, S')$ is a quasi-isometry. More precisely (if $d_S$ et $d_{S'}$ are, respectively, the distance on $\text{Cay}(G, S)$ and on $\text{Cay}(G, S')$)

$$d_{S'}(x, y) \leq K d_S(x, y)$$

where

$$K = \max_{s \in S} d_{S'}(e, s)$$

Theorem (Kanai, 1985)

Any Riemannian manifold of bounded geometry is quasi-isometric to a graph of bounded valency.
Example of quasi-isometries

Example: Let $M$ be a compact manifold, $\pi_1(M)$ its fundamental group and $\tilde{M}$ its universal covering. Then any Cayley graph of $\pi_1(M)$ is quasi-isometric to $\tilde{M}$. 

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Why $\ell^p$-cohomology?

1- Quasi-isometries. Pansu has computed and used it to show that some $\delta$-pinched homogeneous manifolds cannot be quasi-isometric to any $\delta'$-pinched manifold with $\delta' < \delta$. Gives many examples of rigidity (mostly for semi-direct products $N \rtimes \mathbb{R}$, where $N$ is nilpotent).

2- Hyperbolic boundary. In hyperbolic groups, there is a $p_c \geq 1$ so that the $\ell^p$-cohomology is non-trivial $\forall p > p_c$. This $p_c$ gives a critical exponent for groups acting on CAT($-1$)-spaces: $e(\Gamma) \geq p_c$ [Pansu + Bourdon, Martin & Valette]

$$e(\Gamma) = \inf \{ s \in \mathbb{R}_{>0} \mid \sum_{g \in \Gamma} e^{-s|g \cdot o - o|} < +\infty \}$$

Also, $p_c$ is less than the conformal dimension of the ideal boundary (least Hausdorff dimension of a metric in the natural conformal structure). [Bourdon & Pajot]
Why $\ell^p$-cohomology?

2- Hyperbolic boundary. In hyperbolic groups, there is a $p_c \geq 1$ so that the $\ell^p$-cohomology is non-trivial $\forall p > p_c$. This $p_c$ gives a critical exponent for groups acting on $CAT(-1)$-spaces: $e(\Gamma) \geq p_c$ [Pansu, Bourdon-Martin & Valette]

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Also, $p_c$ is less than the conformal dimension of the ideal boundary (least Hausdorff dimension of a metric in the natural conformal structure). [Bourdon & Pajot]

3- Poisson boundary. Under some condition on the isoperimetric profile, the reduced $\ell^p$-cohomology in degree 1 lives inside the Poisson boundary; giving a quasi-isometric invariant part of a quasi-isometry sensitive concept. (G.)
What is $\ell^p$-cohomology?

It is the cohomology of the complex of smooth forms with differential in $L^p$ (interesting for contractible non-compact manifold $M$ with bounded geometry)

$$L^p H^k(M) = \frac{\text{closed } k \text{- forms in } L^p}{d((k - 1) \text{- forms in } L^p)}$$

It turns out the definition is [even] simpler for graphs $G = (V, E)$ in degree 1. Take $E \subset V \times V$ symmetric, and let

$$\nabla : \{V \to \mathbb{R}\} \to \{E \to \mathbb{R}\}$$

$$f \mapsto \nabla f(x, y) = f(y) - f(x)$$

**Definition**

The $\ell^p$-cohomology (resp. reduced) in degree 1 of a graph is

$$\ell^p H^1(G) = \frac{\text{Im } \nabla \cap \ell^p(E)}{\nabla \ell^p(V)}$$

resp.

$$\ell^p H^1(G) = \frac{\text{Im } \nabla \cap \ell^p(E)}{\nabla \ell^p(V)}.$$
$\ell^p$-cohomology

The space of $p$-Dirichlet functions is $D^p(G) = \{ f : V \to \mathbb{R} \mid \nabla f \in \ell^p(E) \}$.

It is endowed with a semi-norm $\| f \|_{D^p} = \| \nabla f \|_{\ell^p}$. ("semi-" → constant functions).

In "integrated expression", the reduced $\ell^p$-cohomology in degree 1 is also

$$\ell^pH^1(G) = \frac{\text{Im} \nabla \cap \ell^p(E)}{\nabla \ell^p(V)} = \frac{D^p(G)}{\ell^p(V) + \text{cst}}$$

**Theorem (Élek 1998, Pansu ∅)**

Fix a bound on the geometry (valency, curvature and injectivity radius). Then the [reduced] $\ell^p$-cohomology [in degree 1] is an invariant of quasi-isometry.
A simple (yet important) example.

\[ \ell^p H^p(G) = D^p(G) / \ell^p(V) + \mathbb{R}^{D^p(G)} \]

**Example:** Let \( G = \text{Cay}(\mathbb{Z}, \{\pm 1\}) \) be the line.

\[
\begin{array}{cccccccc}
g: & 0 & 0 & 1 & 1 & \bullet & \bullet & \bullet & \bullet \\
g^*_n: & 0 & 0 & 1 & \frac{n-1}{n} & 0 & 0 & \bullet & \bullet \end{array}
\]

Then \( g^*_n \xrightarrow{D^p} g \) if \( p > 1 \). Thus \( \forall 1 < p < \infty, [g] = 0 \in \ell^p H^1(G) \).

But, \( [g] \neq 0 \in \ell^1 H^1(G) \).
Ends and $\ell^1 H^1$

$\ell^1 H^1(G)$ is intimately related to the ends of a graph.

**Definition**

Let $n \in \mathbb{Z}_{\geq 0}$. A graph $\Gamma = (V, E)$ has $\leq n$ ends if for all finite $F \subset V$, $V \setminus F$ has $\leq n$ infinite connected components.

**Examples:**

- A finite graph has 0 ends.
- $\text{Cay}(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\})$ (infinite grid) has 1 end.
- $\text{Cay}(\mathbb{Z}, \{\pm 1\})$ (infinite line) has 2 ends.
- $\text{Cay}(F_k, \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\})$ has $\infty$ many ends.
Lemma

The number of ends is a quasi-isometry invariant.

If \( f : X \to Y \) sends one end into two ends, there is a sequence of neighbours in \( X \), say \( x_n \) and \( x'_n \), which are sent into two distinct ends of \( Y \). Thus \( d_Y(f(x_n), f(x'_n)) \to \infty \) even if \( d_X(x_n, x'_n) = 1 \). So \( f \) is not a quasi-isometry.
**Ends and $\ell^1 H^1$**

**Lemma**
The number of ends is a quasi-isometry invariant.

**Theorem (Hopf, 1944)**
The number of ends of a Cayley graph is 0, 1, 2 or $\infty$.

**Idea:** 3 ends $\implies\infty$ ends

**Theorem (Stallings, 1971)**
[The Cayley graph of] a group has 2 ends iff it contains $\mathbb{Z}$ as a finite index subgroup. It has $\infty$ many ends iff it is a “non-trivial” amalgamated product or HNN extension.
**Ends and $\ell^1 H^1$**

\[ \ell^1 H^1(G) = D^1(G)/\ell^1(V) + \mathbb{R}D^1(G) \]

**Lemma ("well-known")**

If $G$ has finitely many ends, $\ell^1 H^1(G) \cong \mathbb{R}^{{\text{ends}(G)} - 1}$.

For $P$ a path from $x$ to $y$, $f(y) = f(x) + \sum_{e \in P} \nabla f(e)$, thus $D^1(G) \subset \ell^\infty(V)$. Furthermore $\|g\|_{D^1}$ is small outside a finite set $F$. This means $g$ is almost constant outside $F$. Thence $g \in D^1(G)$ has a value on each end.

The map “value on the ends” $\beta$ is linear and continuous on $D^1(G)$.

Consequently, $\ell^1(V) \subset \text{Ker} \beta \implies \ell^1(V)^{D^1(G)} \subset \text{Ker} \beta$.

If $g \in D^1(G)$ takes 0 value on the ends: let $g_t$ be the $g$ truncated to values $> t$, then $g - g_t$ has finite support and tends to $g$ if $t \to 0$. Thus, $g \in \ell^1(V)^{D^1(G)}$

Lastly, constant functions has the same value on all ends.
Link with $p$-harmonic functions

Given $f \in \overline{\ell^p H^p(G)} = D^p(G)/\ell^p(V) + \mathbb{R}^{D^p(G)}$, what does the element of minimal $D^p$-norm looks like in $[f]$?

In other words, what is a minimiser of the $p$-energy $\|\nabla g\|_{\ell^p}^p = \int |\nabla g|^p$?

A quick computation, if $f$ is of minimal norm then, $\forall g \in \ell^p(V)$,

$$0 = \left. \frac{d}{dt} \|f + tg\|_{D^p}^p \right|_{t=0} = p \sum_{e \in E} \nabla g(e) |\nabla f(e)|^{p-2} \nabla f(e) = p \langle \nabla g | \mu_{p,p'} \nabla f \rangle$$

where $p'$ is the Hölder conjugate of $p$ and $\mu_{p,q} : \ell^p(\mathbb{N}) \to \ell^q(\mathbb{N})$ is the [Mazur] map defined by $\mu_{p,q} F(n) = |F(n)|^{p' - 1} F(n)$.

The $p$-harmonic equations (for $p \in \mathbb{Z}_{\geq 2}$) originally came up when looking at conformal maps $\mathbb{R}^p \to \mathbb{R}^p$. 

Link with $p$-harmonic functions

$f$ minimal $D^p$-norm in $[f] \iff \forall g \in \ell^p(V), \langle \nabla g \mid \mu_{p,p'} \nabla f \rangle = 0$

For a countable $X$, there is a “canonical” pairing between finitely supported functions $f, g : X \rightarrow \mathbb{R}$, defined by

$$\langle f \mid g \rangle = \sum_{x \in X} f(x)g(x)$$

The dual of the gradient is the divergence:

$$\nabla^* : \{ E \rightarrow \mathbb{R} \} \rightarrow \{ G \rightarrow \mathbb{R} \}$$

$$f \mapsto \nabla^* f(y) = \sum_{x \sim y} f(x,y) - \sum_{x \sim y} f(y,x)$$

(Recall that, for a $k$-valent graph, $\nabla^* \nabla = \Delta = k(\text{Id} - P)$.)

Theorem (Puls 2006)

- $f$ minimises the $p$-energy in its reduced class iff it satisfies $\nabla^* \mu_{p,p'} \nabla f \equiv 0$.
- $\ell^p H^1(\Gamma)$ is also the reduced cohomology of the left-regular representation on $\ell^p \Gamma$. 

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A small trick

\[ f \in \ell^p H^p(G) = D^p(G)/\ell^p(V) + \mathbb{R}^{D^p(G)}. \]

Let \( G = \text{Cay}(\Gamma, S) \). The left-regular representation is the action of \( \Gamma \) on functions \( f : \Gamma \to \mathbb{R} \) defined by \( \lambda_\gamma f(x) = \delta_\gamma \ast f(x) = f(\gamma^{-1} x) \).

The gradient of \( f \) is made up of \( (\delta_s - \delta_1) \ast f \) as \( s \) runs over \( S \).

For \( s \in S \), \([\lambda_s f] = [f]\). Indeed,

\[ f(g) - \lambda_s f(g) = f(g) - f(s^{-1} g) = \nabla f(s^{-1} g, g). \]

By writing \( \gamma \) as a word in \( S \), \([\lambda_\gamma f] = [f]\). Then going to convex combinations, \([\mu \ast f] = [f]\) for any finitely supported probability measure on \( \Gamma \).
A small trick

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By writing \( \gamma \) as a word in \( S, [\lambda_\gamma f] = [f] \). Then going to convex combinations, \([\mu * f] = [f]\) for any finitely supported probability measure on \( \Gamma \).

Lemma (G.)

If there is a sequence \( \mu_n \) of finitely supported probability measures, and constants \( c_n \in \mathbb{R} \) with

- \( \mu_n * f - c_n \to 0 \) point-wise;
- \( \forall n, \|\mu_n * f\|_{D^p} \leq K; \)

then \([f] = 0 \in \ell^p H^1(\Gamma)\).
A small trick

**Lemma (G.)**

Let $1 < p < \infty$. If there is a sequence $\mu_n$ of finitely supported probability measures, and constants $c_n \in \mathbb{R}$ with

- $\mu_n \ast f - c_n \to 0$ point-wise
- $\forall n, \|\mu_n \ast f\|_{D^p} \leq K$

then $[f] = 0 \in \ell^p H^1(\Gamma)$.

**Proof:** in $\ell^p(\mathbb{N})$, boundedness + pointwise convergence implies weak* convergence.

In the reflexive regime, this implies weak convergence.

By Hahn-Banach, weakly closed convex sets are norm closed $\implies$ there is a sequence $\{h_n\}$ so that each $h_n$ is a convex combination of the $\{\mu_i \ast f_i - c_i\}_{i=1}^n$ and $h_n \to 0$ in norm.

But $[h_n] = [f]$, so $[f] = 0$. 
A simple result

Lemma (G.)

Let $1 < p < \infty$. If there is a sequence $\mu_n$ of finitely supported probability measures, and constants $c_n \in \mathbb{R}$ with

- $\mu_n * f - c_n \to 0$ point-wise
- $\forall n, \| \mu_n * f \|_{D^p} \leq K$

then $[f] = 0 \in \ell^p H^1(\Gamma)$.

This lemma applies easily to groups with infinitely many finite conjugation classes: take $\mu_n$ to be the uniform measure on $C_n$.

The gradient is made of $\{ \lambda_s f - f \}_{s \in S}$. Since $\mu_n * \lambda_s * f = \lambda_s * \mu_n * f$,

$\| \mu_n * f \|_{D^p(\Gamma)} \leq \| \mu_n \|_{\ell^1(\Gamma)} \| f \|_{D^p(\Gamma)}$.

Pick $c_n = \mu_n * f(1)$, then $\mu_n * f - c_n$ tends pointwise to 0 near 1.
Corollary

Theorem (Gromov, Puls, Martin-Valette, Kappos, ...)

Let $1 < p < \infty$. If $\Gamma$ has infinitely many finite conjugacy class (e.g. $\Gamma$ is nilpotent) then $\ell^p H^1(\Gamma) = 0$.

Passes to virtually such groups by quasi-isometry.

To go further, need:

Lemma (Holopainen & Soardi, 1994)

\[
\frac{D^p(G) \cap \ell^\infty(V)}{\ell^p(V) + \mathbb{R}^{D^p(G)}} = \{0\} \iff \frac{D^p(G)}{\ell^p(V) + \mathbb{R}^{D^p(G)}} = \{0\}
\]

In other words, if all bounded functions in $D^p(G)$ have a trivial class, then the reduced $\ell^p$-cohomology is trivial.
Amenable groups

Lemma (Holopainen & Soardi, 1994)

\[
\frac{D^p(G) \cap \ell^\infty(V)}{\ell^p(V) + \mathbb{R}^{D^p(G)}} = \{0\} \iff \frac{D^p(G)}{\ell^p(V) + \mathbb{R}^{D^p(G)}} = \{0\}
\]

This is interesting since

Definition (Følner 1955/Reiter)

A group is amenable if there is a sequence \(\xi_n\) of finitely supported probability measures with \(\|\xi_n - \lambda_s \xi_n\|_{\ell^1(G)} \to 0\). A Følner sequence is the specific case when \(\xi_n\) are uniform measures on some finite sets \(F_n\).
Historical note on amenability

Amenability was introduced by von Neumann to counter the (Hausdorff-)Banach-Tarski paradox. This paradox stems from the fact that \( SO_3(\mathbb{R}) \) contains a free subgroup (consider two irrational rotation with independant axis).

**Definition (Von Neumann 1929)**

A group is amenable if there is a linear functional \( m : \ell^\infty(\Gamma) \to \mathbb{R} \) such that 
\[
m(\lambda_\gamma f) = m(f), \quad m(f) \geq 0 \text{ if } f \geq 0.
\]

If \( \Gamma \) is not finite, the existence of such an element requires the axiom of choice, and is consequently completely non-explicit. However, the existence of sequence of almost-invariant probability measures can be very explicit: if the group does not have exponential growth, then a subsequence of \( 1_{B_n}/|B_n| \) (where \( B_n \) are balls of radius \( n \)) will do.
Amenable groups

Lemma (Holopainen & Soardi, 1994)

\[ \frac{D^p(G) \cap \ell^\infty(V)}{\ell^p(V) + \mathbb{R}^{D^p(G)}} = \{0\} \iff \frac{D^p(G)}{\ell^p(V) + \mathbb{R}^{D^p(G)}} = \{0\} \]

Definition (Følner 1955/Reiter)

A group is amenable if there is a sequence \( \xi_n \) of finitely supported probability measures with

\[ \| \xi_n - \lambda_s \xi_n \|_{\ell^1(G)} \to 0. \]

A Følner sequence is the specific case when \( \xi_n \) are uniform measures on some finite sets \( F_n \).

Indeed, if \( f \in D^p(\Gamma) \cap \ell^\infty(\Gamma) \), then

\[ \| \xi_n \ast f - \lambda_s \xi_n \ast f \|_{\ell^\infty(\Gamma)} \leq \| \xi_n - \lambda_s \xi_n \|_{\ell^1(\Gamma)} \| f \|_{\ell^\infty(\Gamma)} \]

So that \( \xi_n \ast f \) tends pointwise [even uniformly] to a constant.
Amenable groups

Indeed, if $f \in D^p(\Gamma) \cap \ell^\infty(\Gamma)$, then

$$\|\xi_n * f - \lambda_s \xi_n * f\|_{\ell^\infty(\Gamma)} \leq \|\xi_n - \lambda_s \xi_n\|_{\ell^1(\Gamma)} \|f\|_{\ell^\infty(\Gamma)}$$

So that $\xi_n * f$ tends pointwise [uniformly] to a constant.

It remains to check which condition comes up when one tries to verify the second condition in the “trick”: $\|\xi_n * f\|_{D^p} \leq K$ for all $n$.

$$(\delta_s - \delta_e) * \xi_n * f(\eta) = \sum_{\gamma \in \mathcal{S} \mathcal{X}_n} \xi_n(s^{-1} \gamma) \lambda_\gamma f(\eta) - \sum_{\gamma \in \mathcal{X}_n} \xi_n(\gamma) \lambda_\gamma f(\eta)$$

$$= \sum_{\gamma \in \mathcal{X}_n} \xi_n(\gamma) f(\gamma^{-1} s^{-1} \eta) - \sum_{\gamma \in \mathcal{X}_n} \xi_n(\gamma) f(\gamma^{-1} \eta),$$

where $\mathcal{X}_n$ is the support of $\xi_n$. 
Amenable groups

It remains to check which condition comes up when one tries to verify the second condition in the “trick”: \( \| \xi_n * f \|_{D^p} \leq K \) for all \( n \).

\[
(\delta_s - \delta_e) * \xi_n * f(\eta) = \sum_{\gamma \in sX_n} \xi_n(s^{-1} \gamma) \lambda_\gamma f(\eta) - \sum_{\gamma \in X_n} \xi_n(\gamma) \lambda_\gamma f(\eta) \\
= \sum_{\gamma \in X_n} \xi_n(\gamma) f(\gamma^{-1} s^{-1} \eta) - \sum_{\gamma \in X_n} \xi_n(\gamma) f(\gamma^{-1} \eta),
\]

where \( X_n \) is the support of \( \xi_n \).

Putting \( \phi^\vee(\gamma) = \phi(\gamma^{-1}) \), the expression

\[
(\delta_s - \delta_e) * \xi_n * f(\eta) = \int f d(\lambda_s \xi_n)^\vee - \int f d\xi_n^\vee
\]

indicates that estimates on transport cost would be welcome.
Transport cost

The pairing $\langle f \mid g \rangle = \sum_{x \in X} f(x)g(x)$ extends to more general spaces of functions (e.g. $f \in \ell^p(X)$ and $g \in \ell^p(X)$, e.g. $f$ arbitrary and $g$ finitely supported).

The adjoint of $\nabla$ (for this pairing) is the divergence:

$$\nabla^* : \{ E \to \mathbb{R} \} \to \{ G \to \mathbb{R} \}$$

$$f \mapsto \nabla^* f(y) = \sum_{x \sim y} f(x, y) - \sum_{x \sim y} f(y, x)$$

A “transport pattern” from $\xi$ to $\phi$ (two finitely supported probability measures) is a finitely supported function $m : E \to \mathbb{R}_{\geq 0}$ such that $\nabla^* m = \phi - \xi$.

It always exists since the graph is assumed connected. In pedantic terms, $\nabla^*$ is the boundary operator $\partial_1$ from 1-chains to 0-chains. In finite connected complexes the image of this operator is the kernel of $\partial_0$, i.e. functions which sum to 0.
Transport cost

The adjoint of $\nabla$ (for this pairing) is the divergence:

\[
\nabla^* : \{ E \to \mathbb{R} \} \to \{ G \to \mathbb{R} \}
\]
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f \mapsto \nabla^* f(y) = \sum_{x \sim y} f(x, y) - \sum_{x \sim y} f(y, x)
\]

A “transport pattern” from $\xi$ to $\phi$ (two finitely supported probability measures) is a finitely supported function $m : E \to \mathbb{R}_{\geq 0}$ such that $\nabla^* m = \phi - \xi$.

The assumption that $m$ is positive is also always true: if $m$ is not positive, consider $m'(x, y) = \max(m(x, y) - m(y, x), 0)$. Then, $m'$ is positive and $\nabla^* m' = \nabla^* m$.

Given a “price” on the edges (a function $f' : E \to \mathbb{R}$) the transport cost for $m$ is $TC_{f';m}(\xi, \phi) = \langle f' | m \rangle$, and the optimal transport cost $W_1(\xi, \mu) = \inf_m TC_{1;m}(\xi, \phi) = \inf_m \langle 1 | m \rangle$. 
Transport cost

A “transport pattern” from $\xi$ to $\phi$ (two finitely supported probability measures) is a finitely supported function $m : E \to \mathbb{R}_{\geq 0}$ such that $\nabla^* m = \phi - \xi$.

Given a “price” on the edges (a function $f' : E \to \mathbb{R}$) the transport cost for $m$ is $\text{TC}_{f';m}(\xi, \phi) = \langle f' \mid m \rangle$, and the optimal transport cost $\mathcal{W}_1(\xi, \mu) = \inf_m \text{TC}_{1,m}(\xi, \phi) = \inf_m \langle 1 \mid m \rangle$.

By design, $\text{TC}_{\nabla f;m}(\xi, \phi) = \langle \nabla f \mid m \rangle = \langle f \mid \nabla^* m \rangle = \langle f \mid \phi - \xi \rangle$.

Let’s rewrite

$$(\delta_s - \delta_e) * \xi_n * f(\eta) = \text{TC}_{\rho_\eta \nabla f,m}( (\lambda_s \xi_n)^\vee, \xi_n^\vee)$$

where $\rho_\eta$ is the right-regular representation and $\xi^\vee(\gamma) = \xi(\gamma^{-1})$.
Transport cost

\[(\delta_s - \delta_e) \ast \xi_n \ast f(\eta) = TC_{\rho_\eta \nabla f, m}((\lambda_s \xi_n)^\vee, \xi_n^\vee)\]

Define a map $T$ which sends a function $f' \in \ell^p(E)$ to 

$$(Tf')(\eta) = TC_{\rho_\eta f', m_n}((\lambda_s \xi_n)^\vee, \xi_n^\vee)$$

(a function on $\Gamma$). If one can show that $T$ is bounded from $\ell^1(E)$ to $\ell^1(\Gamma)$ and from $\ell^\infty(E)$ to $\ell^\infty(\Gamma)$ then, by Riesz-Thorin interpolation, this map is bounded for all $1 \leq p \leq \infty$.

A finitely generated group $G$ will be said transport amenable if there exists $S$ a finite generating set, a positive constant $K \in \mathbb{R}$ and a sequence of finitely supported probability measure $\xi_n$ converging to a left-invariant mean such that, $\forall s \in S$ and $\forall n$, the transport cost

$$TC(\rho_s \xi_n^\vee, \xi_n^\vee) \leq K,$$
\[ \ell^\infty \text{ bound} \]

Take \( m_n \) a transport pattern from \((\lambda_s \xi_n)^\vee\) to \( \xi_n^\vee\) so that \( \text{TC}_{1,m_n}( (\lambda_s \xi_n)^\vee, \xi_n^\vee) \leq 2K \).

\[
\| T f' \|_{\ell^\infty(\Gamma)} = \sup_{\eta \in \Gamma} | T f'(\eta) | = | \text{TC}_{\rho_n f', m_n}( (\lambda_s \xi_n)^\vee, \xi_n^\vee) |
\]

By replacing the price of each edge by the maximal price of an edge \( \| f' \|_{\ell^\infty} \), this last quantity is bounded by \( \| f' \|_{\ell^\infty(E)} \text{TC}_{1,m_n}( (\lambda_s \xi_n)^\vee, \xi_n^\vee) \leq 2K \| f' \|_{\ell^\infty(E)} \).
Given: \( m_n \) so that \( \text{TC}_{1,m_n}((\lambda_s \xi_n)^\vee, \xi_n^\vee) \leq 2K. \)

\[
\| Tf' \|_{\ell^1(\Gamma)} = \sum_{\eta \in \Gamma} |Tf'(\eta)| = \sum_{\eta \in \Gamma} |\langle \rho_{\eta} f' | m_n \rangle| \\
\leq \sum_{\eta \in \Gamma} \langle \rho_{\eta} | f' | m_n \rangle = \langle \sum_{\eta \in \Gamma} \rho_{\eta} | f' | m_n \rangle \\
= \| f' \|_{\ell^1(E)} \langle 1 | m_n \rangle = 2K \| f' \|_{\ell^1(E)}
\]

Thus \( T : \ell^p(E) \to \ell^p(\Gamma) \) is bounded.
**Theorem (G.)**

If $\Gamma$ is transport amenable then $\ell^p H^1(\Gamma) = \{0\}$.

If $F \subset V$ is a subset of vertices, $\partial F$ is the subset of edges incident with $F$ and $F^c$.

A group is CF (for controlled Følner) if there is a sequence of finite sets such that $\frac{|\partial F_n|}{|F_n|} \leq \frac{K}{\text{Diam} F_n}$.

Tessera showed virtually polycyclic groups, solvable Baumslag-Solitar, lampighter $\mathbb{Z}_2 \wr \mathbb{Z}$, ... have this property. In fact, Tessera showed it implies that $E(|W_n|) \sim n^{1/2}$ and (if it is also of exponential growth) $P_1(n)(1) \sim e^{-n^{1/3}}$. In particular, lamplighters $\mathbb{Z} \wr \mathbb{Z}$ or $\mathbb{Z}_2 \wr \mathbb{Z}^2$ do not have this property.
Examples

Recall: multiplication on the left defines edges, multiplication on the right is a Cayley graph automorphism.

**Example 0:** Abelian groups are transport amenable.

Take $F_n$ a sequence of balls of radius $n$ and $\xi_n = 1_{F_n}/|F_n|$. Must transport $F_n^{-1}$ to $F_n^{-1}s$.

Since the group law is commutative $F_n^{-1}s = sF_n^{-1}$. It suffice to move all elements by $s$.

This gives a transport cost of 1.

This strategy will always fail in a non-Abelian group: the action on the right (automorphism) may *a priori* distort the distance (defined by multiplication on the left) in an uncontrolled fashion.
Examples

CF groups: \[ \frac{|\partial F_n|}{|F_n|} \leq \frac{K}{\text{Diam } F_n} \]

**Example 1:** CF groups are transport amenable.
WLOG \( F_n \subset B_{2d_n}(1) \) with \( d_n = \text{Diam } F_n \).
Pick \( \xi_n = |F_n|^{-1} \mathbb{1}_{F_n} \).
Let \( \sigma \) be a bijection from \( F_n s \setminus F_n \) to \( F_n \setminus F_n s \).
Then \( m \) is defined by taking the weight at \( x \in F_n s \setminus F_n \) and sending it to \( \sigma(x) \).
There are \( |\partial F_n| \) such \( x \) to be transported, the weight of each one is \( |F_n|^{-1} \) and the distance they are carried along is at most \( \text{Diam } F_n \).
Thus the cost of the transport is \[ \leq \frac{|\partial F_n|}{|F_n|} \text{Diam } F_n \leq K, \forall n. \]
Example 2: Lamplighter groups $F \wr H$ where $H$ is of polynomial growth and $F$ finite.

$L \wr H = H \rtimes_\rho \left( \bigoplus_H L \right)$ where $H$ acts on the infinite sum $\bigoplus_H L$ by shifting indices: $\rho_h \phi(k) = \phi(kh)$. For $h, g \in H$ and $\phi, \psi : H \to L$ finitely supported, the product is

$$(h, \{\phi_k\}_{k \in H}) \cdot (g, \{\psi_k\}_{k \in H}) = (hg, \{\phi_k g \psi_k\}_{k \in H})$$

- Følner sequence: let $A_n$ be a CF sequence of diameter $d_n$ in $H$ and let $B_k$ be the functions $H \to L$ which are supported on $A_k$. Then $F_n := A_n \times B_n$ is a Følner sequence. Take $\xi_n = 1_{F_n}/|F_n|$.

  - $r \in \{e_H \to S_L\} \times \{e_H\}$ does not displace the Følner set at all.
  - For $z \in \{0\} \times S_H$, displace elements from $F_n^{-1} z$ to $F_n^{-1}$ with a cost which is bounded by a constant multiple of $|F_n|$.
  - The $F_n^{-1}$ are ungainly, invert everything: namely displace $z^{-1} F_n$ to $F_n$, where the cost of displacing $\gamma$ to $\gamma'$ is the length of the element $\eta$ such that $\gamma \eta = \gamma'$. 
Examples

Example 2: Lamplighter groups $F \wr H$ where $H$ is of polynomial growth and $F$ finite.

Goal: the set $F_n$ has been displaced to $zF_n$, and one must transport it back to $F_n$ by multiplying on the right.

Difficulty: For example, when $H$ is Abelian, when one looks at $zF_nz^{-1}$ then the lamplighter positions are OK, but there are going to be lamps states which might be “on” on the set $A_nz^{-1}$.

One must change the lamp states before going to the final position.

- Fix a bijection $\sigma : z^{-1}A_n \setminus A_n \rightarrow A_n \setminus z^{-1}A_n$.
- Given $i$ non-trivial lamp states, take the lamplighter (with the lamps) along a path so that the $i$ lamps (and their image by $\sigma$) pass at the identity. When these lamps are at the identity, interrupt this path so as to change their states. Then return the lamplighter to a neighbouring position. Make at most $2i(d_n + k) + d_n$ steps where the diameter of the “lamp states group” $F$ is $k$;
Example 2: Lamplighter groups $F \wr H$ where $H$ is of polynomial growth and $F$ finite.

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- For $i$ fixed, the number of such elements is $c|A_n| - |\Delta z A_n|(c - 1)^i\left(|\Delta z A_n|\right)$ where $c = |F|$;
- So the total of used edges is at most (recall $\sum_{i=1}^{N} (c - 1)^i \binom{N}{i} = Nc^{N-1}$)

$$c|A_n| - |\Delta z A_n| \sum_{i=0}^{\Delta z A_n} (c - 1)^i\left(|\Delta z A_n|\right)(2i(d_n + k) + d_n)$$

$$\leq c|A_n| - 1\left(2(d_n + k)|\Delta z A_n| + d_n c\right);$$

- As $|F_n| = c|A_n| |A_n|$, one gets a bound of $(2|\Delta z A_n|(d_n + k) + d_n c)/|A_n| c$. 
Examples

The essential point in the previous example was that the lamplighter position group has a sequence which is left-Følner and right-controlled-Følner (sequence of balls).

**Example 3:** Lamplighter groups $G \wr H$ where $H$ is of polynomial growth and $G$ amenable.

**Question:** Are there amenable groups which are not transport amenable?

Examples 2 & 3 are important as these groups have non-trivial bounded harmonic functions.

Under some assumptions on the isoperimetric profile, one has that the non-trivial bounded elements of $\ell^p H^1(G)$ can be mapped to bounded harmonic functions.
**Isoperimetric profiles**

**Note:** Though our upcoming examples and applications are mostly going to be about Cayley graphs, \( G \) is no longer necessarily such a graph.

**Definition**

Let \( d \in \mathbb{Z}_{\geq 1} \). A graph \( G \) satisfies a \( d \)-dimensional isoperimetric profiles (noted \( IS_d \)) if \( \exists K > 0 \) such that, \( \forall F \subset V \) finite,

\[
|F|^{1 - \frac{1}{d}} \leq K|\partial F|
\]

It has a strong isoperimetric profile (noted \( IS_\omega \)) if \( \exists K > 0 \) such that, \( \forall F \subset V \) finite, \( |F| \leq K|\partial F| \)

**Examples:** \( Cay(\mathbb{Z}^d, S) \) satisfait \( IS_d \).

A group is amenable iff its Cayley graph does not satisfy \( IS_\omega \). (Restatement of Følner)
Isoperimetric profiles

Satisfying $IS_\alpha$ (for $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\omega\}$) is invariant under quasi-isometries.

Hyperbolic $\Rightarrow IS_\omega \Rightarrow IS_d, \forall d$.

But $IS_d, \forall d \nRightarrow IS_\omega$. For example, Cayley graphs of $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ where $\alpha(1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Theorem (Varopoulos 1985 + Gromov 1981 + Wolf 1968)

$\Gamma$ has polynomial growth of degree $\leq d$ $\iff$ $\text{Cay}(\Gamma, S)$ does not have $IS_{d+1}$.

$\Rightarrow$ groups which are not virtually nilpotent have $IS_d$ for all $d$. 
Harmonic functions

Let $P_x^{(n)}$ be the measure defined by $P_x^{(n)}(y) = \text{the probability that a simple random walker starting at } x \text{ ends up in } y \text{ after } n \text{ steps.}$

This gives a kernel: $P^{(n)}g(x) := \int g(y) dP_x^{(n)}(y)$.

A function $g : V \rightarrow \mathbb{R}$ is harmonic if $P^{(1)}g = g$ (mean value property).

**Definition**

Here the “Poisson boundary” is the space of bounded harmonic functions. $\mathcal{H}_b(G)$ will denote this space modulo constant functions.
Harmonic functions

Definition

Here the “Poisson boundary” is the space of bounded harmonic functions. \( \mathcal{H}_b(G) \) will denote this space modulo constant functions.

\( \ell^1 H^1 \) is related to the ends [“well-known”]. In the hyperbolic case, there is a strong link between \( \ell^p \)-cohomology in degree 1 and some space of functions on the hyperbolic boundary [Bourdon & Pajot 2003]. A natural idea, to attack amenable groups, is to try to look at the “values” of this function on the “Poisson boundary”.

To define a value of \( g \) “on” the Poisson boundary, the natural idea is to look at \( \lim_{n \to \infty} P^{(n)} g \).
Transport again: to infinity and beyond!

\[ P^{(n)} g(x) \text{ Cauchy?} \rightarrow P^{(n)} g(x) - P^{(m)} g(x) = \int g \, dP^{(n)}_x - \int g \, dP^{(m)}_x. \]

With a transport pattern, one gets a bound:

\[
\int g \, d\xi - \int g \, d\phi = \langle g \mid \xi - \phi \rangle = \langle g \mid \nabla^* \tau_{\phi, \xi} \rangle \\
= \langle \nabla g \mid \tau_{\phi, \xi} \rangle \leq \| \nabla g \|_{\ell^p} \| \tau_{\phi, \xi} \|_{\ell^{p'}}
\]

Pointwise convergence of \( P^{(n)} g \) (in \( x \)) can be deduced from a \( \ell^{p'} \) bound on the transport from \( P^{(n)}_x \) to \( P^{(m)}_x \). There is a very natural transport patter:

take \( m - n \) random steps!

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So it essentially suffices to check that \( \sum_{n \geq 0} P_x^{(n)} \) is in \( \ell^{p'}(V) \).

**Theorem (Varopoulos 1985)**

If \( G \) has IS\(_d\), then for some \( K > 0 \), 
\[
\| P_x^{(n)} \|_{\ell^\infty(V)} \leq Kn^{-d/2}.
\]

Thus
\[
\| P_x^{(n)} \|_{\ell^q(V)} \leq \| P_x^{(n)} \|_{\ell^\infty(V)} \| P_x^{(n)} \|_{\ell^1(V)} \leq K'n^{-d(q-1)/2},
\]

and \( \sum P^{(n)} \) converges in \( \ell^q \) if \( q' < d/2 \).
Theorem (G.)

If $G$ satisfies $IS_d$ and $1 \leq p < d/2$. There is a map $\pi$ from $D^p(G)$ into harmonic functions modulo constants such that

- $[g] = [h] \in \ell^p H^1(G) \iff \pi(g) = \pi(h)$.
- if $g \in D^p(G) \cap \ell^\infty(V)$, then $\pi(g) \in \ell^\infty(V)$.
- if $g \in D^p(G)$, then $\pi(g) \in D^q(G)$ for all $q > \frac{dp}{d - 2p}$.

Using vanishing of reduced $\ell^2$-cohomology for all amenable groups (Cheeger-Gromov 1986):

Corollary (G.)

If $G$ is the Cayley graph of an amenable group $\Gamma$ and $1 < p < q < \infty$. Then the identity map $\ell^p H^1(G) \to \ell^q H^1(G)$ is injective. In particular, $\ell^p H^1(G) = \{0\}$ for all $1 < p \leq 2$. 
Corollary (G.)

If \( G \) satisfies \( IS_d \) and \( \mathcal{H}_b(G) = \{0\} \), then \( \ell^p H^1(G) = \{0\} \) for all \( p \in [1, \frac{d}{2}] \).

Indeed, \( \pi(g) = 0, \forall g \in D^p(G) \cap \ell^\infty(V) \), so the theorem gives \( [g] = 0 \).

Corollary (G.)

If \( \Gamma \) is of intermediate growth then \( \ell^p H^1(\Gamma) = \{0\} \) for all \( p \in [1, \infty] \).

Vanishing via this second theorem also contains strictly CF (except for polynomial growth).

However, transport amenability and this criterion are not contained in each other.
Corollaries

Using further techniques:

**Corollary (G.)**

An amenable wreath product $L \wr H$ may have non-constant bounded harmonic functions, but if $\ell^p H^1(H) = \{0\}$ their gradient is never in $\ell^p(E)$, $\forall p \in [1, \infty[.$

As mentioned at the beginning, isoperimetric profiles as well as reduced $\ell^p$-cohomology are quasi-isometry invariants. But Lyons (1987) showed the Poisson boundary (i.e. $\mathcal{H}_b(G)$) is not invariant under quasi-isometries.

**Corollary (G.)**

If $G$ satisfies $IS_d$ then $\bigcup_{p<d/2} \pi(D^p(G) \cap \ell^\infty(V))$ is a part of the Poisson boundary which is invariant under quasi-isometries.
Some questions

**Question 1:** Is there an amenable group which is not transport amenable?

**Question 2:** Is there an amenable group with a non-constant (bounded or not) harmonic functions whose gradient is in $c_0$?

It is easy to make harmonic functions with gradient in $\ell^\infty$ (*i.e.* Lipschitz harmonic functions) on ANY group. For example, in $\mathbb{Z}^d$, any linear map to $\mathbb{R}$ will be harmonic and Lipschitz.

More generally, any nilpotent groups has an infinite center, hence such a map. But, Kleiner showed that the space of Lipschitz harmonic functions is finite dimensional on [virtually] nilpotent groups. In particular, they may not have gradient in $c_0$.

However, even for other amenable groups, there are no known example of a harmonic function with gradient in $c_0$. 
Some questions

Question 1: Is there an amenable group which is not transport amenable?

Question 2: Is there an amenable group with a non-constant (bounded or not) harmonic functions whose gradient is in $c_0$?

In the group case, most of the arguments above carry if one has a sequence of measures $\xi_n$ which tends to a left-invariant mean and a bound $\|\tau_{\delta_e, \xi_n}\|_{\ell^p} \leq K, \forall n$.

$\Gamma$ has no bounded harmonic functions $\iff$ the $P^{(n)}$ tend to a left-invariant mean.

Question 3: Take an amenable group $\Gamma$ which has bounded harmonic functions and $\xi_n$ tending to a left-invariant mean. Can one find a transport pattern $\tau_{\delta_e, \xi_n}$ with uniform bound (in $n$) for its $\ell^q(E)$-norm, $q < 2$?

Indeed, the previous pattern is extremely bad. For example, one can easily improve the above in $\mathbb{Z}^d$, so that the theorem applies in a larger range (i.e. if $p < d$ instead of $p < d/2$).
Thanks for your attention!
Based on 2 papers and a note:


2- Boundary values of random walks and $\ell^p$-cohomology in degree one, preprint.

3- Absence of non-constant harmonic functions with gradient in $\ell^p$ in wreath products, in progress.