Benamou-Brenier curves on graphs

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Optimal transport on $\mathbb{Z}$

- Given two (finitely supported) measures $f_0, f_1 \in \mathcal{P}(\mathbb{Z})$, we can solve (for $p \geq 1$) the Monge-Kantorovitch problem

$$W_p(f_0, f_1) := \inf_{\pi \in \Pi(f_0, f_1)} \left(\sum_{k,l \in \mathbb{Z}\times\mathbb{Z}} |k-l|^p \pi(k,l) \right)^{1/p}$$

and W_p is a distance on $\mathcal{P}_p(\mathbb{Z})$.
- Problem: for $p > 1$, non-trivial curves have infinite length.
- When $p = 1$: a lot of optimal couplings and a lot of W_1-geodesics.
- Question: given f_0, f_1, can we find a curve $(f_t)_{t\in[0,1]}$ 'looking like' a W_2-geodesic and along which the entropy functional has nice convexity properties ?
- Entropy:

$$H(f) = \sum_{k\in\mathbb{Z}} f(k)\log(f(k)) \ , \ 0\log(0) = 0.$$

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Thinning curves on $\mathbb{Z}_+$

- f: probability measure supported on $\{0, \ldots N\}$.
- Thinning of f:

$$T_t f(k) = \sum_{l \geq 0} \text{Bin}_{l,t}(k) f(l),$$

  where

$$\text{Bin}_{l,t}(k) = \binom{l}{k} t^k (1-t)^{l-k} \; l \geq k.$$

- $(T_t f)_{t \in [0,1]}$: curve in $\mathcal{P}(\mathbb{Z}_+)$, joining $f_0 = \delta(\cdot = 0)$ to $f_1 = f$.
- Continuous analogue: $f_t(x) := \frac{1}{t} f\left(\frac{x}{t}\right)$: W_2-geodesic on $\mathcal{P}_2(\mathbb{R})$.

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Entropy along thinning curves (1)

- $(f_t)_{t\in[0,1]}$: thinning curve. Each $f_t(k)$ is smooth in t.
- We define

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- Entropy function $H(t) := H(f_t)$.

$$\begin{aligned} H''(t) &= \sum_k f_t''(k)\log(f_t(k)) + \sum_k \frac{(f_t'(k))^2}{f_t(k)} \\ &= \sum_k \nabla_2(h_t(k))\log(f_t(k)) + \sum_k \frac{(\nabla_1 g_t(k))^2}{f_t(k)} \\ &\quad + \sum_k \nabla_2[h_t(k)\log(h_t(k))] \\ &\quad -2\sum_k \nabla_1[(\nabla_1 h_t(k))\log(g_t(k))]. \end{aligned}$$

- The only inequality we will use is an elementary one:

$$\forall x > 0\ ,\ \log(x) \geq 1 - \frac{1}{x}.$$

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$$\begin{aligned}
H''(t) &= \sum_k h_t(k) \log \left(\frac{f_t(k) h_t(k)}{g_t(k)^2} \right) \\
&\quad + \sum_k h_t(k-2) \log \left(\frac{f_t(k) h_t(k-2)}{g_t(k-1)^2} \right) \\
&\quad + \sum_k \frac{(\nabla_1 g_t(k))^2}{f_t(k)} \\
&\geq \sum_k \left[h_t(k) - \frac{g_t(k)^2}{f_t(k)} \right] + \left[h_t(k-2) - \frac{g_t(k-1)^2}{f_t(k)} \right] \\
&\quad + \sum_k \frac{(\nabla_1 g_t(k))^2}{f_t(k)} \\
&= 2 \sum_k h_t(k-1) - \frac{g_t(k) g_t(k-1)}{f_t(k)} = 0. \square
\end{aligned}$$

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Monotonic BB-curves on $\mathbb{Z}$

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- We define

$$g_t(k) := -\sum_{l\le k} f_t'(l)\ ,\ h_t(k) := -\sum_{l\le k} g_t'(l).$$

- $(f_t)_{t\in[0,1]}$ is a monotonic BB-curve on $\{0,\dots N\}$ if:
  1. The support of f_t is included in $\{0,\dots N\}$.
  2. $g_t(k) > 0$ if $k \in \{0,\dots N-1\}$ and $g_t(k) = 0$ elsewhere.
  3. For every $k \in \{0,\dots N-1\}$ we have

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First properties

- $f_t(k) > 0$ (resp. $h_t(k) > 0$) iff $k \in \{0, \ldots N\}$ (resp. $k \in \{0, \ldots N-2\}$).
- If $s < t$ then $f_s << f_t$ (stochastic domination).
- The entropy functional is convex along BB-curves.
- The Renyi entropy functional

$$H_p(t) := -\sum_k f_t(k)^p,$$

where $0 < p < 1$, is also convex in t.

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Velocity fields.

- In continous case: $\frac{\partial f}{\partial t} = -\frac{\partial (vf)}{\partial x}$, so "$v = -g/f$".
- Benamou-Brenier : if $\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x}$, then $(f_t)_{t\in[0,1]}$ is a W_2-geodesic and

$$W_2(f_0, f_1)^2 = \int_{\mathbb{R}} \int_0^1 f_t(x) v_t(x)^2 dtdx.$$

- For a mBB-curve, we define

$$v_{+,t}(k) = \frac{g_t(k)}{f_t(k)} \ , \ v_{-,t}(k) = \frac{g_t(k-1)}{f_t(k)}.$$

- Differential equations:

$$v'_{+,t}(k) = -v_{+,t}(k)[v_{+,t}(k) - v_{+,t}(k-1)],$$

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Structure of monotonic BB-curves (1)

- If $(f_t)_{t\in[0,1]}$ is a BB-curve and $m \geq 1$ then

$$\frac{\partial^m}{\partial t^m} f_t(0) = (-1)^m \frac{g_t(0)\cdots g_t(i-1)}{f_t(1)\cdots f_t(i-1)}$$

and

$$\frac{\partial^m}{\partial t^m} f_t(N) = \frac{g_t(N-1)\cdots g_t(N-i)}{f_t(N-1)\cdots f_t(N-i+1)}.$$

- Consequences:
  1. $f_t(0)$ and $f_t(N)$ are polynomial in t, of degree at most N.
  2. The ratio

$$A := \frac{g_t(0)\cdots g_t(N-1)}{f_t(1)\cdots f_t(N-1)}$$

is constant.

Structure of mBB-curves (2)

- If $Q(t) := f_{1-t}(0)$ and $P(t) := \frac{1}{A} f_t(N)$ then

$$f_t(k) = P^{(N-k)}(t) Q^{(k)}(1-t).$$

- In particular $f_t(k)$ is polynomial in t.
- If $P(x) = \sum_{i=0}^{N} u(i) \frac{x^i}{i!}$ and $Q(x) = \sum_{j=0}^{N} v(j) \frac{x^{N-j}}{(N-j)!}$ then

$$f_t(k) = \sum_{i \le k \le j} \frac{u(i)v(j)}{(j-i)!} \mathrm{Bin}_{(i,j),t}(k),$$

where

$$\mathrm{Bin}_{(i,j),t}(k) = \mathrm{Bin}_{(j-i),t}(k-i) = \frac{(j-i)}{(k-i)(j-k)} t^{k-i} (1-t)^{j-k}.$$

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$$f_t(k) = P^{(N-k)}(t) Q^{(k)}(1-t).$$

- In particular $f_t(k)$ is polynomial in t.
- If $P(x) = \sum_{i=0}^{N} u(i) \frac{x^i}{i!}$ and $Q(x) = \sum_{j=0}^{N} v(j) \frac{x^{N-j}}{(N-j)!}$ then

$$f_t(k) = \sum_{i \le k \le j} \frac{u(i)v(j)}{(j-i)!} \, \mathrm{Bin}_{(i,j),t}(k),$$

where

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Existence and uniqueness of mBB-curves (1)

- The existence (uniqueness) of a mBB-curve joining two given measures f_0 and f_1 is equivalent to the existence (uniqueness) of a coupling $\pi \in \Pi(f_0, f_1)$ such that

$$\forall 0 \leq i \leq j \leq N \ , \ \pi(i,j) = \frac{u(i)v(j)}{(j-i)!}$$

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- Theorem: there exists a unique minimizer π to the problem

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Pseudo mBB-curves

- It is possible to deduce convexity of entropy from weaker assumptions.
- For instance if
  1. $f_t, g_t, h_t \geq 0$,
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  then $H(f_t)$ is convex.
- Used to prove a special case of Shepp-Olkin conjecture: if $p_1(t), \dots p_n(t) : [0,1] \to [0,1]$ with $p_i' \geq 0$ and S_t: sum of independent Bernoulli of parameters $p_1(t), \dots p_n(t)$, then $t \mapsto H(S_t)$ convex.

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BB-curves on graphs

- Question: is it possible to find a similar interpolation between two probability measures f_0, f_1 on a graph G?
- Done in the case where f_0 is a Dirac measure (Contraction of measures on graphs).
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The W_1-orientation

- γ_1, γ_2 two geodesics on G. We suppose $\exists\ k_1, k_2$ s.t.

  $$\gamma_1(k_1) = \gamma_2(k_2+1)\ ,\ \gamma_1(k_1+1) = \gamma_2(k_2).$$

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  Then π is not a W_1-optimal coupling.
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Calculus on the oriented graph

- $\mathcal{E}(x_1) := \{x_0 \in G : x_0 \to x_1\}$, $\mathcal{F}(x_1) := \{x_2 \in G : x_1 \to x_2\}$.
- Oriented edge graph: $(E(G), \to)$. The vertices are $(x_0 x_1)$ s.t. $x_0 \to x_1$, there is an oriented edge between $(x_0 x_1)$ and $(x_1 x_2)$.
- Graph $(E(E(G)), \to)$ vertices $(x_0 x_1 x_2)$ with $x_0 \to x_1 \to x_2$.
- Gradient operator : if $g : E(G) \to \mathbb{R}$, then $\nabla g : G \to \mathbb{R}$ defined by

$$(\nabla g)(x_1) := -\sum_{x_0 \in \mathcal{E}(x_1)} g(x_0 x_1) + \sum_{x_2 \in \mathcal{F}(x_1)} g(x_1 x_2).$$

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Definition of BB curves

Let $(f_t(x))_{t\in[0,1]}$ be a curve in $\mathcal{P}(G)$ such that f_0, f_1 are finitely supported. We orient G with the W_1-orientation relatively to f_0, f_1. We say that a family $(f_t)_{t\in[0,1]}$ of probability measures on G is a BB-curve if the following are satisfied:

1. There exists a family of positive functions g_t on $E(G)$ s.t.

$$\frac{\partial}{\partial t} f_t(x) = -\nabla g_t(x).$$

2. There exists a family of functions h_t on $E(E(G))$ s.t.

$$\frac{\partial}{\partial t} g_t(xy) = -\nabla h_t(xy).$$

3. The Benamou-Brenier condition holds:

$$h_t(x_0x_1x_2) f_t(x_1) = g_t(x_0x_1) g_t(x_1x_2).$$

Velocity fields

- We define two velocity fields on $E(G)$ by

$$v_{-,t}(x_1x_2) := \frac{g_t(x_1x_2)}{f_t(x_2)}\ ,\ v_{+,t}(x_1x_2) := \frac{g_t(x_1x_2)}{f_t(x_1)}.$$

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Geodesics on the oriented graph

- $\mathcal{A} := \{x \in G : \mathcal{E}(x) = \emptyset\}$: set of initial points for $(G, \rightarrow)$.
- $\mathcal{B} := \{x \in G : \mathcal{F}(x) = \emptyset\}$: set of final points for $(G, \rightarrow)$.
- $\mathcal{A}$ and $\mathcal{B}$ are non-empty.
- Extremal geodesics:
  $E\Gamma(G) := \{\gamma \in \Gamma(G) : e_0(\gamma) \in \mathcal{A}, e_1(\gamma) \in \mathcal{B}\}$.
- Semi-extremal geodesics of first type:
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  $SE\Gamma_1(x) = \{\gamma \in \Gamma(G) : e_0(\gamma) \in \mathcal{A}, e_1(\gamma) = x\}$.
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Geodesics on the oriented graph

- $\mathcal{A} := \{x \in G : \mathcal{E}(x) = \emptyset\}$: set of initial points for $(G, \to)$.
- $\mathcal{B} := \{x \in G : \mathcal{F}(x) = \emptyset\}$: set of final points for $(G, \to)$.
- $\mathcal{A}$ and $\mathcal{B}$ are non-empty.
- Extremal geodesics:
  $E\Gamma(G) := \{\gamma \in \Gamma(G) : e_0(\gamma) \in \mathcal{A}, e_1(\gamma) \in \mathcal{B}\}$.
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BB-curves and extremal geodesics

- For every
$$C_\gamma(t) := \frac{\prod_{i=0}^{N-1} g_t(\gamma(i)\gamma(i+1))}{\prod_{i=1}^{N-1} f_t(\gamma(i))}.$$
- If $\gamma \in SE\Gamma_1(x)$ (resp. $SE\Gamma_2(x)$) then $C_\gamma(t)$ is polynomial in t of degree at most $d(x, \mathcal{B})$ (resp. $d(x, \mathcal{A})$).
- If $\gamma := \gamma(0) \to \cdots \to \gamma(N)$ is in $E\Gamma(G)$ then $C_\gamma(t)$ is constant.
- Probability measure on $E\Gamma(G)$: $p(\gamma) = \frac{C(\gamma)}{\sum_{\gamma' \in E\Gamma(G)} C(\gamma')}$.
- Probability that an extremal geodesic visits a given set of vertices:
$$m(x_1, \dots x_k) := p(\{\gamma \in E\Gamma(G) : \exists i_1 \leq \cdots \leq i_k, \gamma(i_l) = x_l\}).$$

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- Theorem: $P_t(x) := \sum_{\gamma_2 \in SE\Gamma_2(x)} C_{\gamma_2}(t)$ then

$$\frac{\partial P_t(x)}{\partial t} = \sum_{y \in \mathcal{E}(x)} \frac{m(y,x)}{m(x)} P_t(y).$$

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- Similarly if $Q_t(x) := \sum_{\gamma_2 \in SE\Gamma_2(x)} C_{\gamma_2}(t)$, then

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- We have
$$f_t(x) = \frac{P_t(x)Q_t(x)}{\sum_{\gamma\in\Gamma(x)} C_\gamma}.$$
- Theorem: $(f_t)_{t\in[0,1]}$ is a BB-curve iff there exists a probability measure p on $E\Gamma(G)$ and two non-negative functions $U, V : G \to \mathbb{R}_+$ such that:
$$f_t(x) = \sum_{y\leq x\leq z} \frac{U(y)V(z)}{d(y,z)!} m(y,z) \left(\frac{m(y,x,z)}{m(y,z)} \mathrm{Bin}_{d(y,z),t}(d(y,x))\right).$$

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