Stability and separation in volume comparison problems.

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A typical comparison problem for the volume of convex bodies asks whether inequalities

$$f_K(\xi) \leq f_L(\xi), \qquad \forall \xi \in S^{n-1}$$

imply

$$|K| \leq |L|$$

for any K, L from a certain class of origin-symmetric convex bodies in $\mathbb{R}^n$, where f_K is a geometric characteristic of K and $|K|$ is volume of proper dimension.

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for any K, L from a certain class of origin-symmetric convex bodies in $\mathbb{R}^n$, where f_K is a geometric characteristic of K and $|K|$ is volume of proper dimension.

One can have in mind the *hyperplane section function*

$$f_K(\xi) = |K \cap \xi^{\perp}|,$$

or the *hyperplane projection function*

$$f_K(\xi) = |K|\xi^{\perp}|,$$

where $\xi^{\perp}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$, and $K|\xi^{\perp}$ is the orthogonal projection of K to $\xi^{\perp}$.

Stability

If the answer is affirmative, one can ask a stronger stability question. Suppose

$$f_K(\xi) \leq f_L(\xi) + \varepsilon, \qquad \forall \xi \in S^{n-1}.$$

Does there exist a constant c not dependent on ε and such that

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c\varepsilon?$$

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Suppose stability holds for both pairs K, L and L, K with the same constant c. Put $\varepsilon = \max_{\xi \in S^{n-1}} |f_K(\xi) - f_L(\xi)|$. Then one can switch K and L. The resulting inequality for volumes will be called a *volume difference inequality*:

$$\left| |K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} \right| \le c\varepsilon = c \max_{\xi \in S^{n-1}} |f_K(\xi) - f_L(\xi)|.$$

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Suppose now that the function f_L converges to zero uniformly with respect to ξ when L approaches the empty set. Then when $L \to \emptyset$ the volume difference inequality turns into what we call a *hyperplane inequality*:

$$|K|^{\frac{n-1}{n}} \le c \max_{\xi \in S^{n-1}} f_K(\xi).$$

Suppose

$$f_K(\xi) \le f_L(\xi) - \varepsilon, \qquad \forall \xi \in S^{n-1}.$$

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Suppose that separation holds and $\varepsilon = \min_{\xi \in S^{n-1}} f_L(\xi) - f_K(\xi) > 0$. We get another kind of a volume difference inequality:

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \ge c\varepsilon = c \min_{\xi \in S^{n-1}} \left(f_L(\xi) - f_K(\xi)\right).$$

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Again, if f_K converges to zero uniformly in ξ when K approaches the empty set, we get a hyperplane inequality:

$$|L|^{\frac{n-1}{n}} \ge c \min_{\xi \in S^{n-1}} f_L(\xi).$$

Hyperplane sections: $f_K(\xi) = |K \cap \xi^\perp|$

Busemann-Petty problem: K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ and $|K \cap \xi^\perp| \leq |L \cap \xi^\perp|, \forall \xi \in S^{n-1}$. Does it follow that $|K| \leq |L|$?

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Lutwak proved that if K is an *intersection body* and L is any origin-symmetric star body, then the answer is affirmative in every dimension. Following Lutwak we say that K is the *intersection body of a star body* L if for every $\xi \in S^{n-1}$

$$\rho_K(\xi) = |L \cap \xi^\perp|.$$

A more general class of *intersection bodies* is defined as the closure of the class of intersection bodies of star bodies in the radial metric: $\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|$.

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Intersection bodies are limits in the radial metric of radial sums of ellipsoids (Goodey, Weil), in particular, the radial sum of intersection bodies is an intersection body. Radial sum: $\rho_{K+_r L} = \rho_K + \rho_L$.

Hyperplane problem

Hyperplane problem: Does there exist an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.$$

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$$\frac{\max_{\xi\in S^{n-1}} |K\cap\xi^\perp|}{|K|^{\frac{n-1}{n}}} \ge \frac{|B_2^{n-1}|}{|B_2^n|^{\frac{n-1}{n}}}.$$

Suppose that $\varepsilon > 0$, K and L are origin-symmetric star bodies in $\mathbb{R}^n$, and K is an intersection body. If for every $\xi \in S^{n-1}$

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$$\left| |K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} \right| \le c_n \max_{\xi \in S^{n-1}} \left| |K \cap \xi^\perp| - |L \cap \xi^\perp| \right|.$$

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Putting $L = \delta B_2^n$, $\delta \to 0$, we get the well-known hyperplane inequality for intersection bodies:

$$|K|^{\frac{n-1}{n}} \le c_n \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.$$

If K is an intersection body, so is the radial sum $K +_r \varepsilon B_2^n$. Also, $(K +_r \varepsilon B_2^n) \cap \xi^\perp = (K \cap \xi^\perp) +_r \varepsilon B_2^{n-1}$. By the volume difference inequality applied to $K +_r \varepsilon B_2^n$ and K

$$\frac{|K +_r \varepsilon B_2^n|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}}{\varepsilon} \le c_n \max_{\xi \in S^{n-1}} \frac{|(K \cap \xi^\perp) +_r \varepsilon B_2^{n-1}| - |K \cap \xi^\perp|}{\varepsilon}.$$

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Sending $\varepsilon \longrightarrow 0$, we get

$$\mathrm{as}(K) \le \frac{|B_2^{n-1}|}{|B_2^{n-2}||B_2^n|^{\frac{1}{n}}} \max_{\xi \in S^{n-1}} \mathrm{as}(K \cap \xi^\perp)\ |K|^{\frac{1}{n}},$$

where

$$\mathrm{as}(K) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |K \cap \xi^\perp| d\xi$$

is the average volume of central hyperplane sections of K.

Equality for $K = B_2^n$.

The constant c in the separation result for sections does not depend on ε, but depends on the dimension and on the normalized inradius of K :

$$r(K) = \frac{\min_{\xi \in S^{n-1}} \rho_K(\xi)}{|K|^{1/n}}.$$

Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$ and $\varepsilon > 0$. Assume that K is an intersection body. If for every $\xi \in S^{n-1}$

$$|K \cap \xi^{\perp}| \leq |L \cap \xi^{\perp}| - \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} - \sqrt{\frac{2\pi}{n+1}}\, r(K)\varepsilon.$$

Hyperplane projections: $f_K(\xi) = |K|\xi^\perp|$

Shephard's problem (1964): Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ so that $|K|\xi^\perp| \leq |L|\xi^\perp|$ for every $\xi \in S^{n-1}$. Does it follow that $|K| \leq |L|$?

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The answer is affirmative if L is a projection body and K any origin-symmetric convex body in $\mathbb{R}^n$. The *projection body* ΠK of K is defined by

$$h_{\Pi K}(\theta) = |K|\theta^{\perp}|, \qquad \forall \theta \in S^{n-1},$$

where $h_K(x) = \max_{\{\xi \in \mathbb{R}^n : \|\xi\|_K = 1\}} (x, \xi)$ is the support function of K; $h_K = \|\cdot\|_{K^\circ}$ is the norm of the polar body. If L is the projection body of some convex body, we simply say that L is a projection body.

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Every projection body is the limit in the Hausdorff metric of Minkowski (vector) sums of ellipsoids centered at the origin. The Minkowski sum of projection bodies is also a projection body. An origin-symmetric convex body in $\mathbb{R}^n$ is a projection body if and only if the polar body is the unit ball of an n-dimensional subspace of L_1.

Separation for projections

Suppose that $\varepsilon > 0$, K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and L is a projection body. If for every $\xi \in S^{n-1}$

$$|K|\xi^\perp| \leq |L|\xi^\perp| - \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} - c_n\varepsilon.$$

Recall that $c_n > 1/\sqrt{e}$.

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$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \ge c_n \min_{\xi\in S^{n-1}}(|L|\xi^\perp| - |K|\xi^\perp|).$$

Sending K to the empty set, we get a well-known hyperplane inequality for projection bodies. If L is a projection body in $\mathbb{R}^n$, then

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For general symmetric convex bodies, K.Ball proved that c_n may and has to be replaced by $c/\sqrt{n}$.

It is also known and follows from the Cauchy projection formula for the surface area and the classical isoperimetric inequality that for any origin-symmetric convex body

$$|L|^{\frac{n-1}{n}} \leq c_n \max_{\xi \in S^{n-1}} |L|\xi^{\perp}|.$$

The volume difference inequality applied to $L+\varepsilon B_2^n$ and L leads to a hyperplane inequality for the surface area of projection bodies.

$$\frac{|L+\varepsilon B_2^n|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}}}{\varepsilon} \geq c_n \min_{\xi \in S^{n-1}} \frac{|(L|\xi^\perp) + \varepsilon B_2^{n-1}| - |L|\xi^\perp|}{\varepsilon}.$$

$$S(L) \geq \frac{n}{n-1} c_n \min_{\xi \in S^{n-1}} S(L|\xi^\perp)\ |L|^{\frac{1}{n}}.$$

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In the stability result for projections the constant depends on the dimension and body. Define the normalized circumradius of L by

$$R(L) = \frac{\max_{\xi\in S^{n-1}} \rho_L(\xi)}{|L|^{\frac{1}{n}}}.$$

Suppose that $\varepsilon > 0$, K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and L is a projection body. If for every $\xi \in S^{n-1}$

$$|K|\xi^\perp| \leq |L|\xi^\perp| + \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + \sqrt{\frac{2\pi}{n}}\ R(L)\varepsilon.$$

The Busemann-Petty problem for arbitrary measures; Zvavitch (2005).

Let f be an even continuous non-negative function on $\mathbb{R}^n$, and denote by μ the measure on $\mathbb{R}^n$ with density f. For every closed bounded set $B \subset \mathbb{R}^n$ define

$$\mu(B) = \int\limits_B f(x)\ dx, \qquad \mu(B \cap \xi^\perp) = \int_{B \cap \xi^\perp} f(x)\ dx.$$

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$$\mu(B) = \int_B f(x)\ dx, \qquad \mu(B\cap\xi^\perp) = \int_{B\cap\xi^\perp} f(x)\ dx.$$

Suppose that for convex origin-symmetric bodies K and L in $\mathbb{R}^n$

$$\mu(K\cap\xi^\perp) \le \mu(L\cap\xi^\perp), \qquad \forall\xi \in S^{n-1}.$$

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The answer is affirmative if K is an intersection body and L is any origin-symmetric star body in $\mathbb{R}^n$.

Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and let $\varepsilon > 0$. Suppose that K is an intersection body and that for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp) + \varepsilon.$$

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$$\mu(K) \leq \mu(L) + \frac{n}{n-1} c_n |K|^{1/n} \varepsilon.$$

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Interchanging K and L, we get the volume difference inequality. If both K and L are intersection bodies in $\mathbb{R}^n$ (in particular, any origin-symmetric convex bodies in $\mathbb{R}^3$ or $\mathbb{R}^4$), then

$$|\mu(K) - \mu(L)| \le \frac{nc_n}{n-1} \max_{\xi \in S^{n-1}} \left| \mu(K \cap \xi^\perp) - \mu(L \cap \xi^\perp) \right| \max\left\{ |K|^{\frac{1}{n}}, |L|^{\frac{1}{n}} \right\}.$$

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Sending L to the emply set, we get a hyperplane inequality for arbitrary measures. If K is an intersection body in $\mathbb{R}^n$, then

$$\mu(K) \leq \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}.$$

The inequality is sharp

Let $K = B_2^n$ and, for every $j \in N$, let f_j be a non-negative continuous function on $[0,1]$ supported in $(1-\frac{1}{j},1)$ and such that $\int_0^1 f_j(t)dt = 1$. Let μ_j be the measure on $\mathbb{R}^n$ with density $f_j(|x|_2)$.

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$$\mu_j(B_2^n) = |S^{n-1}| \int_0^1 r^{n-1} f_j(r)dr,$$

$$\mu_j(B_2^n \cap \xi^\perp) = |S^{n-2}| \int_0^1 r^{n-2} f_j(r)dr.$$

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Clearly,

$$\lim_{j\to\infty} \frac{\int_0^1 r^{n-1} f_j(r)dr}{\int_0^1 r^{n-2} f_j(r)dr} = 1,$$

so

$$\lim_{j\to\infty} \frac{\mu_j(B_2^n)}{\max_{\xi\in S^{n-1}} \mu_j(B_2^n \cap \xi^\perp)\ |B_2^n|^{1/n}} = \frac{|S^{n-1}|}{|S^{n-2}||B_2^n|^{1/n}} = \frac{n}{n-1} c_n.$$

Proof of stability for sections, part 1

If K is an intersection body, L is an origin-symmetric star body and

$$|K \cap \xi^\perp| \le |L \cap \xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1} \tag{1}$$

then

$$|K|^{\frac{n-1}{n}} \le |L|^{\frac{n-1}{n}} + c_n \varepsilon.$$

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$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c_n \varepsilon.$$

$$|K \cap \xi^{\perp}| = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^{\wedge}(\xi), \qquad \forall \xi \in S^{n-1},$$

so (1) can be written as

$$(\|x\|_K^{-n+1})^{\wedge}(\xi) \leq (\|x\|_L^{-n+1})^{\wedge}(\xi) + \pi(n-1)\varepsilon, \qquad \forall \xi \in S^{n-1}.$$

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so (1) can be written as

$$(\|x\|_K^{-n+1})^\wedge(\xi)\le(\|x\|_L^{-n+1})^\wedge(\xi)+\pi(n-1)\varepsilon,\qquad\forall\xi\in S^{n-1}.$$

An origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body iff $\|x\|_K^{-1}$ is a positive definite distribution, so the Fourier transform

$$(\|x\|_K^{-1})^\wedge(\xi)\ge0,\qquad\forall\xi\in S^{n-1}.$$

$$\int_{S^{n-1}} (\|x\|_K^{-n+1})^\wedge(\xi)(\|x\|_K^{-1})^\wedge(\xi)\ d\xi \leq \int_{S^{n-1}} (\|x\|_K^{-n+1})^\wedge(\xi)(\|x\|_K^{-1})^\wedge(\xi)\ d\xi +$$

$$\pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)\ d\xi.$$

Proof of stability for sections, part 2

$$\int_{S^{n-1}} (\|x\|_K^{-n+1})^\wedge(\xi)(\|x\|_K^{-1})^\wedge(\xi)\ d\xi \le \int_{S^{n-1}} (\|x\|_K^{-n+1})^\wedge(\xi)(\|x\|_K^{-1})^\wedge(\xi)\ d\xi +$$

$$\pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)\ d\xi.$$

By Parseval's formula on the sphere, polar formula for volume and Hölder's inequality,

$$(2\pi)^n n|K| = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-n+1} \|x\|_K^{-1} dx \le$$

$$(2\pi)^n \int_{S^{n-1}} \|x\|_L^{-n+1} \|x\|_K^{-1} dx + \pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) d\xi \le$$

$$(2\pi)^n n|K|^{\frac{1}{n}} |L|^{\frac{n-1}{n}} + \pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) d\xi.$$

$$|K| = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(x)\ dx = \frac{1}{n}\int_{S^{n-1}} \|x\|_K^{-n} dx.$$

Proof of stability for sections, part 3

To estimate the second summand, we use the formula for the Fourier transform

$$(|x|_2^{-n+1})^\wedge(\xi) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})}|\xi|_2^{-1}.$$

By Parseval's formula and Hölder's inequality,

$$\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)d\xi =$$

$$\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)(|x|_2^{-n+1})^\wedge(\xi)d\xi$$

$$= \frac{(2\pi)^n\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\int_{S^{n-1}} \|x\|_K^{-1}\ dx$$

$$\leq \frac{(2\pi)^n\Gamma(\frac{n-1}{2})|S^{n-1}|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}}\left(\int_{S^{n-1}} \|x\|_K^{-n}\ dx\right)^{\frac{1}{n}}$$

$$= \frac{(2\pi)^n\Gamma(\frac{n-1}{2})|S^{n-1}|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}}(n|K|)^{\frac{1}{n}}$$

Combining these estimates,

$$(2\pi)^n n|K| \le (2\pi)^n n|K|^{\frac{1}{n}}|L|^{\frac{n-1}{n}} + \frac{(2\pi)^n \pi(n-1)n^{\frac{1}{n}}\Gamma(\frac{n-1}{2})\left|S^{n-1}\right|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}}|K|^{\frac{1}{n}}\varepsilon.$$

Now to represent the coefficient in the required form use

$$|S^{n-1}| = n|B_2^n| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

We get

$$|K|^{\frac{n-1}{n}} \le |L|^{\frac{n-1}{n}} + \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|}\ \varepsilon.$$