Stability and separation in volume comparison problems.

Alexander Koldobsky

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A typical comparison problem for the volume of convex bodies asks whether inequalities
\[ f_K(\xi) \leq f_L(\xi), \quad \forall \xi \in S^{n-1} \]
imply
\[ |K| \leq |L| \]
for any \( K, L \) from a certain class of origin-symmetric convex bodies in \( \mathbb{IR}^n \), where \( f_K \) is a geometric characteristic of \( K \) and \( |K| \) is volume of proper dimension.
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One can have in mind the \textit{hyperplane section function}
\[ f_K(\xi) = |K \cap \xi^\perp|, \]

or the \textit{hyperplane projection function}
\[ f_K(\xi) = |K|\xi^\perp|, \]

where \( \xi^\perp \) is the central hyperplane perpendicular to \( \xi \in S^{n-1} \), and \( K|\xi^\perp \) is the orthogonal projection of \( K \) to \( \xi^\perp \).
If the answer is affirmative, one can ask a stronger stability question. Suppose

\[ f_K(\xi) \leq f_L(\xi) + \varepsilon, \quad \forall \xi \in S^{n-1}. \]

Does there exist a constant \( c \) not dependent on \( \varepsilon \) and such that

\[ |K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c\varepsilon? \]
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Suppose stability holds for both pairs \( K, L \) and \( L, K \) with the same constant \( c \).

Put \( \varepsilon = \max_{\xi \in S^{n-1}} |f_K(\xi) - f_L(\xi)|. \) Then one can switch \( K \) and \( L \). The resulting inequality for volumes will be called a volume difference inequality:

\[ \left| |K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} \right| \leq c\varepsilon = c \max_{\xi \in S^{n-1}} |f_K(\xi) - f_L(\xi)|. \]
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\]

Suppose now that the function \( f_L \) converges to zero uniformly with respect to \( \xi \) when \( L \) approaches the empty set. Then when \( L \to \emptyset \) the volume difference inequality turns into what we call a \textit{hyperplane inequality}:

\[
|K|^{\frac{n-1}{n}} \leq c \max_{\xi \in S^{n-1}} f_K(\xi).
\]
Suppose

\[ f_K(\xi) \leq f_L(\xi) - \varepsilon, \quad \forall \xi \in S^{n-1}. \]

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\[ \left| K \right|^{\frac{n-1}{n}} \leq \left| L \right|^{\frac{n-1}{n}} - c\varepsilon? \]

Suppose that separation holds and \( \varepsilon = \min_{\xi \in S^{n-1}} f_L(\xi) - f_K(\xi) > 0 \). We get another kind of a volume difference inequality:
\[ \left| L \right|^{\frac{n-1}{n}} - \left| K \right|^{\frac{n-1}{n}} \geq c\varepsilon = c \min_{\xi \in S^{n-1}} (f_L(\xi) - f_K(\xi)). \]
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Does there exist a constant \( c \) not dependent on \( \varepsilon \) and such that
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\[ |L|^\frac{n-1}{n} - |K|^\frac{n-1}{n} \geq c\varepsilon = c \min_{\xi \in S^{n-1}} (f_L(\xi) - f_K(\xi)). \]

Again, if \( f_K \) converges to zero uniformly in \( \xi \) when \( K \) approaches the empty set, we get a hyperplane inequality:
\[ |L|^\frac{n-1}{n} \geq c \min_{\xi \in S^{n-1}} f_L(\xi). \]
Hyperplane sections: $f_K(\xi) = |K \cap \xi^\perp|

Busemann-Petty problem: $K, L$ are origin-symmetric convex bodies in $\mathbb{IR}^n$ and $|K \cap \xi^\perp| \leq |L \cap \xi^\perp|$, $\forall \xi \in S^{n-1}$. Does it follow that $|K| \leq |L|$?
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Yes if \( n \leq 4 \), no if \( n \geq 5 \); solution completed in the end of the 90’s.

Ball, Bourgain, Gardner, Giannopoulos, K., Larman, Lutwak, Papadimitrakis, Rogers, Schlumprecht, Zhang
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Lutwak proved that if \( K \) is an intersection body and \( L \) is any origin-symmetric star body, then the answer is affirmative in every dimension. Following Lutwak we say that \( K \) is the intersection body of a star body \( L \) if for every \( \xi \in S^{n-1} \)

\[
\rho_K(\xi) = |L \cap \xi^\perp|.
\]

A more general class of intersection bodies is defined as the closure of the class of intersection bodies of star bodies in the radial metric:

\[
\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.
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Every origin-symmetric convex body in $\mathbb{R}^3$ or $\mathbb{R}^4$ is an intersection body (Gardner, Zhang).
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Intersection bodies are limits in the radial metric of radial sums of ellipsoids (Goodey, Weil), in particular, the radial sum of intersection bodies is an intersection body. Radial sum: \( \rho_{K+L} = \rho_K + \rho_L \).
Hyperplane problem: Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^\frac{n-1}{n} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.$$ 

The best-to-date estimate $C \sim n^{1/4}$ is due to Klartag, who improved the previous estimate of Bourgain.
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If \( K \) is an intersection body

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|K|^{\frac{n-1}{n}} \leq c_n \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,
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where \( c_n := |B_2^n|^{\frac{n-1}{n}} / |B_2^{n-1}| \in (\frac{1}{\sqrt{e}}, 1) \).
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Suppose $K$ is an intersection body, and $|K| = |B_2^n|$. Then it is not possible that $|K \cap \xi^\perp| < |B_2^{n-1}|$ for every $\xi \in S^{n-1}$. 

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Suppose $K$ is an intersection body, and $|K| = |B_2^n|$. Then it is not possible that $|K \cap \xi^\perp| < |B_2^{n-1}|$ for every $\xi \in S^{n-1}$. So, $\max_{\xi \in S^{n-1}} |K \cap \xi^\perp| \geq |B_2^{n-1}|$. Now divide both sides by equal numbers

$$\frac{\max_{\xi \in S^{n-1}} |K \cap \xi^\perp|}{|K|^{\frac{n-1}{n}}} \geq \frac{|B_2^{n-1}|}{|B_2^n|^{\frac{n-1}{n}}}.$$
Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric star bodies in $IR^n$, and $K$ is an intersection body. If for every $\xi \in S^{n-1}$

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp| + \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c_n \varepsilon,$$

where $c_n := |B_2^n|^{\frac{n-1}{n}} / |B_2^{n-1}| \in \left(\frac{1}{\sqrt{e}}, 1\right)$.
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If $K$ and $L$ are intersection bodies in $\mathbb{R}^n$, then

$$\left| |K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} \right| \leq c_n \max_{\xi \in S^{n-1}} \left| |K \cap \xi^\perp| - |L \cap \xi^\perp| \right|.$$
Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric star bodies in $\mathbb{R}^n$, and $K$ is an intersection body. If for every $\xi \in S^{n-1}$

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$$\left| |K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} \right| \leq c_n \max_{\xi \in S^{n-1}} \left| K \cap \xi^\perp \right| - \left| L \cap \xi^\perp \right|.$$

Putting $L = \delta B_2^n$, $\delta \to 0$, we get the well-known hyperplane inequality for intersection bodies:

$$|K|^{\frac{n-1}{n}} \leq c_n \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.$$
If $K$ is an intersection body, so is the radial sum $K + r \varepsilon B_2^n$. Also, $(K + r \varepsilon B_2^n) \cap \xi^\perp = (K \cap \xi^\perp) + r \varepsilon B_2^{n-1}$. By the volume difference inequality applied to $K + r \varepsilon B_2^n$ and $K$

$$\frac{|K + r \varepsilon B_2^n|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}}{\varepsilon} \leq c_n \max_{\xi \in S^{n-1}} \frac{|(K \cap \xi^\perp) + r \varepsilon B_2^{n-1}| - |K \cap \xi^\perp|}{\varepsilon}.$$
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$$\frac{|K + r \varepsilon B_2^n|^{n-1} - |K|^{n-1}}{n} \leq c_n \max_{\xi \in S^{n-1}} \frac{|(K \cap \xi^\perp) + r \varepsilon B_2^{n-1}| - |K \cap \xi^\perp|}{\varepsilon}.$$ 

Sending $\varepsilon \to 0$, we get

$$\text{as}(K) \leq \frac{|B_2^{n-1}|}{|B_2^{n-2}||B_2^n|^{1/n}} \max_{\xi \in S^{n-1}} \text{as}(K \cap \xi^\perp) |K|^{1/n},$$

where

$$\text{as}(K) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |K \cap \xi^\perp| d\xi$$

is the average volume of central hyperplane sections of $K$. Equality for $K = B_2^n$. 


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The constant $c$ in the separation result for sections does not depend on $\varepsilon$, but depends on the dimension and on the normalized inradius of $K$:

$$r(K) = \min_{\xi \in S^{n-1}} \frac{\rho_K(\xi)}{|K|^{1/n}}.$$

Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$ and $\varepsilon > 0$. Assume that $K$ is an intersection body. If for every $\xi \in S^{n-1}$

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp| - \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} - \sqrt{\frac{2\pi}{n+1}} r(K)\varepsilon.$$
Shephard’s problem (1964): Suppose that $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$ so that $|K|_\perp \leq |L|_\perp$ for every $\xi \in S^{n-1}$. Does it follow that $|K| \leq |L|$?
Hyperplane projections: $f_K(\xi) = |K|\xi^\perp$.

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Yes if $n = 2$, no if $n \geq 3$; solved by Petty and Schneider, independently, in 1966.
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The answer is affirmative if $L$ is a projection body and $K$ any origin-symmetric convex body in $\mathbb{R}^n$. The projection body $\Pi K$ of $K$ is defined by

$$h_{\Pi K}(\theta) = |K|\theta^\perp|, \quad \forall \theta \in S^{n-1},$$

where $h_K(x) = \max\{\xi \in \mathbb{R}^n : \|\xi\|_K = 1\}(x, \xi)$ is the support function of $K$; $h_K = \|\cdot\|_{K^\circ}$ is the norm of the polar body. If $L$ is the projection body of some convex body, we simply say that $L$ is a projection body.
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Every projection body is the limit in the Hausdorff metric of Minkowski (vector) sums of ellipsoids centered at the origin. The Minkowski sum of projection bodies is also a projection body. An origin-symmetric convex body in $\mathbb{R}^n$ is a projection body if and only if the polar body is the unit ball of an $n$-dimensional subspace of $L_1$.
Suppose that \( \varepsilon > 0 \), \( K \) and \( L \) are origin-symmetric convex bodies in \( \mathbb{R}^n \), and \( L \) is a projection body. If for every \( \xi \in S^{n-1} \)

\[
|K|\xi \perp \leq |L|\xi \perp - \varepsilon,
\]

then

\[
|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} - c_n \varepsilon.
\]

Recall that \( c_n > 1/\sqrt{e} \).
Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $L$ is a projection body. If for every $\xi \in S^{n-1}$

$$|K|\frac{\xi^\perp}{\|\xi^\perp\|} \leq |L|\frac{\xi^\perp}{\|\xi^\perp\|} - \varepsilon,$$

then

$$|K|\frac{n-1}{n} \leq |L|\frac{n-1}{n} - c_n\varepsilon.$$

Recall that $c_n > 1/\sqrt{e}$.

If $L$ is a projection body in $\mathbb{R}^n$ and $K$ is an arbitrary origin-symmetric convex body in $\mathbb{R}^n$ so that $\min_{\xi \in S^{n-1}} (|L|\xi^\perp - |K|\xi^\perp) > 0$, then we get a volume difference inequality

$$|L|\frac{n-1}{n} - |K|\frac{n-1}{n} \geq c_n \min_{\xi \in S^{n-1}} (|L|\xi^\perp - |K|\xi^\perp).$$
Sending $K$ to the empty set, we get a well-known hyperplane inequality for projection bodies. If $L$ is a projection body in $\mathbb{R}^n$, then

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For general symmetric convex bodies, K. Ball proved that $c_n$ may and has to be replaced by $c/\sqrt{n}$.

It is also known and follows from the Cauchy projection formula for the surface area and the classical isoperimetric inequality that for any origin-symmetric convex body

$$|L|^\frac{n-1}{n} \leq c_n \max_{\xi \in S^{n-1}} |L|\xi^\perp|.$$
The volume difference inequality applied to $L + \varepsilon B^n_2$ and $L$ leads to a hyperplane inequality for the surface area of projection bodies.

$$\frac{|L + \varepsilon B^n_2|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}}}{\varepsilon} \geq c_n \min_{\xi \in S^{n-1}} \frac{|(L|\xi) + \varepsilon B^{n-1}_2| - |L|\xi|}{\varepsilon}.$$

$$S(L) \geq \frac{n}{n-1} c_n \min_{\xi \in S^{n-1}} S(L|\xi) |L|^\frac{1}{n}.$$
The volume difference inequality applied to $L + \varepsilon B^n_2$ and $L$ leads to a hyperplane inequality for the surface area of projection bodies.

$$\frac{|L + \varepsilon B^n_2|^{\frac{n-1}{n}}}{\varepsilon} - |L|^{\frac{n-1}{n}} \geq c_n \min_{\xi \in S^{n-1}} \frac{|(L|\xi^\perp) + \varepsilon B^n_2| - |L|\eta|\xi^\perp|}{\varepsilon}.$$ 

In the stability result for projections the constant depends on the dimension and body. Define the normalized circumradius of $L$ by

$$R(L) = \frac{\max_{\xi \in S^{n-1}} \rho_L(\xi)}{|L|^{\frac{1}{n}}}.$$ 

Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $L$ is a projection body. If for every $\xi \in S^{n-1}$

$$|K|\xi^\perp| \leq |L|\xi^\perp| + \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + \sqrt{\frac{2\pi n}{n}} R(L)\varepsilon.$$
The Busemann-Petty problem for arbitrary measures; Zvavitch (2005).
Let $f$ be an even continuous non-negative function on $\mathbb{R}^n$, and denote by $\mu$ the measure on $\mathbb{R}^n$ with density $f$. For every closed bounded set $B \subset \mathbb{R}^n$ define

$$
\mu(B) = \int_B f(x) \, dx, \quad \mu(B \cap \xi^\perp) = \int_{B \cap \xi^\perp} f(x) \, dx.
$$

Suppose that for convex origin-symmetric bodies $K$ and $L$ in $\mathbb{R}^n$

$$
\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp), \quad \forall \xi \in S^{n-1}.
$$

Does it follow that $\mu(K) \leq \mu(L)$?

Yes, if $n \leq 4$; no, if $n \geq 5$ in the sense that for every strictly positive $f$ there exist $K$, $L$ providing a counterexample.

The answer is affirmative if $K$ is an intersection body and $L$ is any origin-symmetric star body in $\mathbb{R}^n$. 

Alexander Koldobsky

Stability and separation in volume comparison problems.
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Yes, if $n \leq 4$; no, if $n \geq 5$ in the sense that for every strictly positive $f$ there exist $K, L$ providing a counterexample.

The answer is affirmative if $K$ is an intersection body and $L$ is any origin-symmetric star body in $\mathbb{R}^n$. 
Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$, and let $\varepsilon > 0$. Suppose that $K$ is an intersection body and that for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp) + \varepsilon.$$ 

Then

$$\mu(K) \leq \mu(L) + \frac{n}{n-1} c_n |K|^{1/n} \varepsilon.$$
Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$, and let $\varepsilon > 0$. Suppose that $K$ is an intersection body and that for every $\xi \in S^{n-1}$,

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Interchanging $K$ and $L$, we get the volume difference inequality. If both $K$ and $L$ are intersection bodies in $\mathbb{R}^n$ (in particular, any origin-symmetric convex bodies in $\mathbb{R}^3$ or $\mathbb{R}^4$), then

$$|\mu(K) - \mu(L)| \leq \frac{n c_n}{n-1} \max_{\xi \in S^{n-1}} \left| \mu(K \cap \xi^\perp) - \mu(L \cap \xi^\perp) \right| \max \left\{ |K|^{1/n}, |L|^{1/n} \right\}.$$
Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$, and let $\varepsilon > 0$. Suppose that $K$ is an intersection body and that for every $\xi \in S^{n-1}$,

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Sending $L$ to the empty set, we get a hyperplane inequality for arbitrary measures. If $K$ is an intersection body in $\mathbb{R}^n$, then

$$\mu(K) \leq \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) |K|^{1/n}.$$
Let $K = B_2^n$ and, for every $j \in N$, let $f_j$ be a non-negative continuous function on $[0,1]$ supported in $(1 - \frac{1}{j},1)$ and such that $\int_0^1 f_j(t) dt = 1$. Let $\mu_j$ be the measure on $\mathbb{R}^n$ with density $f_j(|x|_2)$.
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$$
\mu_j(B_2^n) = |S^{n-1}| \int_0^1 r^{n-1} f_j(r)dr,
$$

$$
\mu_j(B_2^n \cap \xi^\perp) = |S^{n-2}| \int_0^1 r^{n-2} f_j(r)dr.
$$
The inequality is sharp

Let $K = B_2^n$ and, for every $j \in N$, let $f_j$ be a non-negative continuous function on $[0,1]$ supported in $(1 - \frac{1}{j}, 1)$ and such that $\int_0^1 f_j(t)dt = 1$. Let $\mu_j$ be the measure on $IR^n$ with density $f_j(|x|_2)$. We have

$$\mu_j(B_2^n) = |S^{n-1}| \int_0^1 r^{n-1} f_j(r)dr,$$

$$\mu_j(B_2^n \cap \xi^\bot) = |S^{n-2}| \int_0^1 r^{n-2} f_j(r)dr.$$

Clearly,

$$\lim_{j \to \infty} \frac{\int_0^1 r^{n-1} f_j(r)dr}{\int_0^1 r^{n-2} f_j(r)dr} = 1,$$

so

$$\lim_{j \to \infty} \frac{\mu_j(B_2^n)}{\max_{\xi \in S^{n-1}} \mu_j(B_2^n \cap \xi^\bot) |B_2^n|^{1/n}} = \frac{|S^{n-1}|}{|S^{n-2}| |B_2^n|^{1/n}} = \frac{n}{n-1} c_n.$$
If $K$ is an intersection body, $L$ is an origin-symmetric star body and

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp| + \varepsilon, \quad \forall \xi \in S^{n-1}$$

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c_n \varepsilon.$$
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so (1) can be written as

$$\left(\|x\|_K^{-n+1}\right)^\wedge(\xi) \leq \left(\|x\|_L^{-n+1}\right)^\wedge(\xi) + \pi(n-1)\varepsilon, \quad \forall \xi \in S^{n-1}.$$
If $K$ is an intersection body, $L$ is an origin-symmetric star body and

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp| + \varepsilon, \quad \forall \xi \in S^{n-1}$$

(1)

then

$$|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c_n \varepsilon.$$

$$|K \cap \xi^\perp| = \frac{1}{\pi(n-1)}(\|x\|^{-n+1}_K)^\wedge(\xi), \quad \forall \xi \in S^{n-1},$$

so (1) can be written as

$$(\|x\|^{-n+1}_K)^\wedge(\xi) \leq (\|x\|^{-n+1}_L)^\wedge(\xi) + \pi(n-1) \varepsilon, \quad \forall \xi \in S^{n-1}.$$

An origin-symmetric star body $K$ in $\mathbb{R}^n$ is an intersection body iff $\|x\|^{-1}_K$ is a positive definite distribution, so the Fourier transform

$$(\|x\|^{-1}_K)^\wedge(\xi) \geq 0, \quad \forall \xi \in S^{n-1}.$$
\[
\int_{S^{n-1}} (\|x\|^{-n+1}_K)^\xi (\|x\|^{-1}_K)^\xi \, d\xi \leq \int_{S^{n-1}} (\|x\|^{-n+1}_K)^\xi (\|x\|^{-1}_K)^\xi \, d\xi + \pi (n-1) \varepsilon \int_{S^{n-1}} (\|x\|^{-1}_K)^\xi \, d\xi.
\]
Proof of stability for sections, part 2

$$\int_{S^{n-1}} (\|x\|_{K}^{n+1})^\wedge (\xi)(\|x\|_{K}^{-1})^\wedge (\xi) \, d\xi \leq \int_{S^{n-1}} (\|x\|_{K}^{n+1})^\wedge (\xi)(\|x\|_{K}^{-1})^\wedge (\xi) \, d\xi + \pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_{K}^{-1})^\wedge (\xi) \, d\xi.$$  

By Parseval’s formula on the sphere, polar formula for volume and Hölder’s inequality,

$$(2\pi)^n n|K| = (2\pi)^n \int_{S^{n-1}} \|x\|_{K}^{-n+1}\|x\|_{K}^{-1} dx \leq (2\pi)^n \int_{S^{n-1}} \|x\|_{L}^{-n+1}\|x\|_{K}^{-1} dx + \pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_{K}^{-1})^\wedge (\xi) d\xi \leq (2\pi)^n n|K| \frac{1}{n} \frac{n-1}{n} + \pi(n-1)\varepsilon \int_{S^{n-1}} (\|x\|_{K}^{-1})^\wedge (\xi) d\xi.$$

$$|K| = \frac{1}{n} \int_{S^{n-1}} \rho_{K}^{n}(x) \, dx = \frac{1}{n} \int_{S^{n-1}} \|x\|_{K}^{-n} dx.$$
To estimate the second summand, we use the formula for the Fourier transform

\[ (|x|_2^{-n+1})^\wedge (\xi) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})} |\xi|_2^{-1}. \]

By Parseval’s formula and Hölder’s inequality,

\[
\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge (\xi) d\xi = \\
\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge (\xi)(|x|_2^{-n+1})^\wedge (\xi) d\xi \\
= \frac{(2\pi)^n \Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{S^{n-1}} \|x\|_K^{-1} dx \\
\leq \frac{(2\pi)^n \Gamma(\frac{n-1}{2}) |S^{n-1}|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}} \left( \int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{\frac{1}{n}} \\
= \frac{(2\pi)^n \Gamma(\frac{n-1}{2}) |S^{n-1}|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}} (n|K|)^{\frac{1}{n}}
\]
Combining these estimates,

\[(2\pi)^n n |K| \leq (2\pi)^n n |K| \frac{1}{n} |L| \frac{n-1}{n} + \frac{(2\pi)^n \pi (n-1) n \frac{1}{n} \Gamma \left( \frac{n-1}{2} \right) |S^{n-1}| \frac{n-1}{n}}{2\pi \frac{n+1}{2}} |K| \frac{1}{n} \varepsilon.\]

Now to represent the coefficient in the required form use

\[|S^{n-1}| = n |B^n_2| = \frac{2\pi \frac{n}{2}}{\Gamma \left( \frac{n}{2} \right)}.\]

We get

\[|K| \frac{n-1}{n} \leq |L| \frac{n-1}{n} + \frac{|B^n_2| \frac{n-1}{n}}{|B^{n-1}_2|} \varepsilon.\]