# Hessian metrics with application to functional inequalities

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based on joint works with **Emanuel Milman** (Haifa) and **Bo'az Klartag** (Tel Aviv)

2015

## Hessian metrics

These are metrics on $\mathbb{R}^n$ which have the form

$$g = D^2\Phi,$$

where $\Phi$ is a convex function. Motivation comes from the **optimal transportation** theory and the theory of the **Monge-Ampère** equation.

## Optimal transportation problem

$$\mu = e^{-V}dx, \; \nu = e^{-W}dx$$

are probability measures on $\mathbb{R}^n$

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# Monge-Ampére equation

$$V = W(\nabla\Phi) - \log\det D^2\Phi$$

Differentiating along $e \in \mathbb{R}^n$ we get

$$V_e = -L\Phi_e$$

Where

$$Lf = \mathrm{Tr}(D^2\Phi)^{-1}D^2f - \langle\nabla f, \nabla W(\nabla\Phi)\rangle$$

$L$ is the generator of the Dirichlet energy form $\mathcal{E}$

$$\mathcal{E}(f,h) = \int\langle\nabla_g f, \nabla_g h\rangle_g \, d\mu = \int\langle(D^2\Phi)^{-1}\nabla f, \nabla h\rangle \, d\mu$$

on the metric-measure space

$$(\mu, g = D^2\Phi, \mathbb{R}^n)$$

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# Geometric motivation

**E. Calabi** (1958) applied Hessian metrics to regularity problem for the Monge-Ampère equation

$$\det D^2\Phi = 1.$$

**S.-T. Yau** developed idea of Calabi in his work on the complex Monge-Ampère equation

**Kähler-Einstein equation**

$$e^{-\Phi} = f(\nabla\Phi)\det D^2\Phi$$

**B. Klartag, D. Cordero-Erausquin**, Moment measures (2015)
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# Probabilistic motivation

1. **Caffarelli** contraction theorem: optimal transportation mapping
$$T = \nabla \Phi$$
is a contraction provided $D^2 W \geq 1$, $D^2 V \leq 1$.
More generally: uniform and $L^p$ estimates for the eigenvalues of the Monge-Ampère equation.
2. Regularity of optimal transportation on the Wiener space
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# Generalized Bakry-Emery tensor

Metric measure space $(g, m)$

$$\mathrm{Ric}_{g,m,N} = \mathrm{Ricci}_g + \mathrm{Hess} P - \frac{1}{N-n} \nabla P \otimes \nabla P,$$

where

$$m = e^{-P} dvol_g.$$

Space $(g, m)$ is called $CD(K, N)$ space if

$$\mathrm{Ric}_{g,m,N} \geq K \cdot g.$$

## When our space is $CD(0, \infty)$?

Notations:

$$V_i = V_{x_i}, V_{ij} = V_{x_i x_j}, \ldots \quad W^i = W_{x_i}(\nabla\Phi), W^{ij} = W_{x_i x_j}(\nabla\Phi), \ldots$$

$$\Phi_i = \Phi_{x_i}, \Phi_{ij} = \Phi_{x_i x_j}, \Phi_{ijk} = \Phi_{x_i x_j x_k}, \ldots$$

Standard notation agreements of Riemannian geometry...
Bakry-Emery tensor

$$\mathrm{Ric}_{g,\mu,\infty}(i,j) = \frac{1}{2}V_{ij} + \frac{1}{2}W_{ij} + \frac{1}{4}\Phi^l_{ik}\Phi^k_{jl}$$

If $\mu$ and $\nu$ are log-concave, then $\mathrm{Ric}_{g,\mu,\infty} \geq 0$.

# When our space is $CD(0,N)$?

$N > n$

$$\mathrm{Ric}_{g,\mu,N}(i,i) \geq \frac{1}{2}\left(V_{ii} - \frac{V_i^2}{N-n} + W_{ii} - \frac{W_i^2}{N-n}\right) + \frac{1}{4}\Phi^l_{ik}\Phi^k_{il}.$$

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Assume that $\mu$, $\nu$ satisfy (standard Euclidean) $CD(0,N)$-condition, $\frac{1}{N} \in (-\infty, \frac{1}{n}]$, then our metric measure space satisies the $CD(0,N)$-condition.

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## Functional inequalities under $CD(K, N)$ condition for $N < 0$

**S. Ohta–A. Takatsu (2013); K.–E.Milman (2013)**

Brascamp-Lieb inequality for $CD(0, N)$ spaces $\frac{1}{N} \in (-\infty, \frac{1}{n}]$:

$$\frac{N}{N-n}\mathrm{Var}_\mu f \leq \int \langle \mathrm{Ric}_{g,\mu,N}^{-1} \nabla f, \nabla f \rangle d\mu.$$

**E. Milman (2014)** Beyond traditional Curvature-Dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension.

## Application: entropic inequalities

When the "entropic" generalization of the Brascamb-Lieb inequality holds?

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu \leq 2C \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle^2 d\mu,$$

where $\mu = e^{-V} dx$ is a log-concave probability measure on $\mathbb{R}^d$.

**S. Bobkov, M. Ledoux** : $V$ is convex, $x \to V_{hh}(x)$ is concave $\forall h \in \mathbb{R}^n$. Then

$$\mathrm{Ent}_\mu f^2 \leq 3 \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle \, d\mu.$$

**K., E. Milman.**

$V$ is convex, $V^*$ is the Legendre transform of $V$:

$$V^*(y) = \sup_x \Big( \langle x, y \rangle - V(x) \Big),$$

$$F = \langle x, \nabla V^* \rangle - \log \det D^2 V^*$$

. Assume that for some $C > 0$

$$D^2 F + \frac{1}{d} \nabla \log \det D^2 V^* \otimes \nabla \log \det D^2 V^* \ge 2C \cdot D^2 V^*(x).$$

Then

$$\int f^2 \log \Big( \frac{f^2}{\int f^2 \ d\mu} \Big) \ d\mu \le \frac{2}{C} \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle \ d\mu.$$

# Product case

$$\Phi(x) = \sum_{i=1}^{n} \Phi_i(x_i).$$

$$\mathrm{Ric}_{g,\mu,\infty} = D^2 V + \mathrm{diag}\Big\{ V_{x_i} \frac{u_i'(x_i)}{u_i(x_i)} - \frac{u_i''(x_i)}{u_i(x_i)} \Big\}$$

$$u_i = \frac{1}{\sqrt{\Phi_i''}}.$$

The support $\Omega$ of $\mu$ is supposed to be orthant unconditional, i.e.

$$\Omega \subset (0,\infty)^n,$$

the outer normal of $\partial\Omega \cap (0,\infty)^n$ has non-negative coordinates. Under this assumption $\Omega$ is geodesically convex in metric $g$ provided $\Phi'(x_i) \le 0, \forall i$.

- Testing different types of $\Phi_i$ we deduce various inequalities of the Poincare and log-Sobolev type from the Brascamp-Lieb and log-Sobolev inequalities for our metric measure space. Assume that $V$ is convex on orthant unconditional support $\Omega$ of $\mu = e^{-V}dx$
- (Recover Klartag's result) Set $\Phi_i = \frac{1}{x_i^2}$. Assume $V_{x_i} \geq 0$. Then

$$\mathrm{Var}_\mu \leq 4 \int \sum_{i=1}^{n} x_i^2 f_{x_i}^2 d\mu$$

- (Entropic estimates for exponential growth) Set $\Phi_i = \frac{1}{x_i}$. Assume $V_{x_i} \geq \lambda > 0$. Then

$$\mathrm{Ent}_\mu \leq \frac{4}{\lambda} \int \sum_{i=1}^{n} x_i f_{x_i}^2 d\mu$$

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- (Variance estimates for exponential growth) Set $\Phi_i = e^{-2\lambda x_i}$. Assume $V_{x_i} \geq \lambda > 0$. Then

$$\mathrm{Var}_\mu \leq \int \sum_{i=1}^{n} \frac{f_{x_i}^2}{\lambda_i (V_{x_i} - \lambda_i)} \, d\mu.$$

- Let $\mu$ be the normalized Lebesgue measure on orthant unconditional $\Omega$ and the normal $n$ of $\partial\Omega$ satisfies

$$0 < \lambda \leq \frac{\langle n, e_i \rangle}{\langle n, x \rangle}, \quad \forall i.$$

Then the Poincaré constant $C(\Omega)$ of $\Omega$ satisfies

$$C(\Omega) \leq \frac{C}{n^2} \Big( \int_\Omega |x|^2 d\mu + \frac{n}{\lambda^2} \int_{\partial\Omega} \frac{|x|^2}{\langle x, n \rangle^2} d\sigma_\Omega \Big),$$

where $\sigma_\Omega$ is the probability cone measure on $\partial\Omega$.

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# Dimension-free estimates for eigenvalues of the Monge-Ampère operator

Theorem

**(Klartag, K.)** *(Distribution of eigenvalues of $D^2\Phi$) Let $\mu, \nu$ be absolutely-continuous, log-concave probability measures on $\mathbb{R}^n$. Write*

$$0 < \lambda_1(x) \leq \ldots \leq \lambda_n(x)$$

*for the eigenvalues of the matrix $D^2\Phi(x)$, repeated according to their multiplicity. Then, for $i = 1, \ldots, n$,*

$$\mathrm{Var}_\mu [\log \lambda_i] \leq 4.$$

**Generalization:** assume that $\mu$ and $\nu$ satisfy $CD(0, N)$ condition, $N < 0$. Then

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# Perspectives

- Reverse Hölder inequalities for eigenvalues

$$\|\lambda_i\|_{L^p(\mu)} \leq C_{p,q}\|\lambda_i\|_{L^q(\mu)}, \;\; q < p$$

- Reverse estimate for Kantorovich distances

$$W_p(\mu,\nu) \leq F(W_2(\mu,\nu)), \;\; p \geq 2$$

- Application of the Kähler-Einstein equation to the log-Sobolev and transportational inequalities
- Analysis for high order derivatives of the transportation potential ($\geq 3$) (in particular in the KE case).

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