Sudakov-Type Minoration Principle

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Introduction

Let X be a random vector in $\mathbb{R}^d$. How to estimate $\mathbb{E}\sup_{t\in T}\langle t, X\rangle$ for $T\subset\mathbb{R}^d$?

Suppose that $\mathbb{E}X = 0$, $\#T \le e^p$ and t_0 is a vector in $\mathbb{R}^d$. Then

$$\begin{aligned}\mathbb{E}\sup_{t\in T}\langle t, X\rangle &= \mathbb{E}\sup_{t\in T}\langle t-t_0, X\rangle \le \mathbb{E}\sup_{t\in T}|\langle t-t_0, X\rangle| \\ &\le \Big(\mathbb{E}\sup_{t\in T}|\langle t-t_0, X\rangle|^p\Big)^{1/p} \le \Big(\mathbb{E}\sum_{t\in T}|\langle t-t_0, X\rangle|^p\Big)^{1/p} \\ &\le e\sup_{t\in T}\|\langle t-t_0, X\rangle\|_p.\end{aligned}$$

Here and in the sequel we denote for $p\ge 1$,

$$\|Y\|_p := \|Y\|_{L_p} = (\mathbb{E}|Y|^p)^{1/p}.$$

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Formulation of the problem

Problem

May the above estimate be reversed for regular vectors? And which vectors are regular?

To be more precise, assume that $T \subset \mathbb{R}^d$, $p \geq 2$ and

$$\|\langle t-s, X\rangle\|_p = \Big(\mathbb{E}\Big(\sum_{i=1}^{d}(t_i-s_i)X_i\Big)^p\Big)^{1/p} \geq A \quad \text{for all } s,t \in T,\ s \neq t,$$

and $\#T$ is large enough (hopefully $\#T \geq e^p$ or $\#T \geq e^{C'p}$).
What should we assume about X so that this would imply

$$\mathbb{E}\sup_{t\in T}\langle t, X\rangle = \mathbb{E}\sup_{t\in T}\sum_{i=1}^{d} t_i X_i \geq \frac{1}{C}A?$$

If such implication holds (for $\#T \geq e^p$) than we will say that X satisfies *(generalized) Sudakov minoration*.
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Classical Sudakov Minoration Principle

Suppose that $X = G = (g_1, \ldots, g_d)$ is a canonical Gaussian vector in $\mathbb{R}^d$, i.e. g_i are independent $\mathcal{N}(0,1)$ random variables. Let $d_2(x, y) = |x - y|$ be the standard Euclidean distance on $\mathbb{R}^d$.

Theorem (Sudakov)

For any $\varepsilon > 0$ and $T \subset \mathbb{R}^d$,

$$\mathbb{E} \sup_{t \in T} \langle t, G \rangle = \mathbb{E} \sup_{t \in T} \sum_{i=1}^{d} t_i g_i \geq \frac{1}{C} \varepsilon \sqrt{\ln N(T, d_2, \varepsilon)}.$$

Equivalently, if $T \subset \mathbb{R}^d$ satisfies $|t - s| \geq \varepsilon$ for all $t, s \in T$, $t \neq s$ then

$$\mathbb{E} \sup_{t \in T} \langle t, G \rangle \geq \frac{1}{C} \varepsilon \sqrt{\ln(\# T)}.$$

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Generalized and Classical Sudakov Minoration

For canonical Gaussian vectors G, one has $\langle t, G\rangle \sim \mathcal{N}(0, |t|^2)$ and

$$\|\langle t-s, G\rangle\|_p \sim \sqrt{p}|t-s| \quad \text{for } p \geq 2.$$

Let p be such that $\#T \geq e^p$, if $\|\langle t-s, G\rangle\|_p \geq A$ for all $t \neq s \in T$ then $|t-s| \geq cAp^{-1/2}$ and classical Sudakov minoration gives

$$\mathbb{E}\sup_{t\in T}\langle t, X\rangle \geq \frac{1}{C}cAp^{-1/2}\sqrt{\ln \#T} \geq \frac{1}{C'}A.$$

In fact choosing p such that $\#T = \lfloor e^p \rfloor$ it is not hard to see that for Gaussian vectors generalized Sudakov minoration is formally equivalent to the classical one.

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Rademachers

Now let $X = (\varepsilon_1, \ldots, \varepsilon_d)$, where (ε_i) are independent symmetric ± 1 random variables. Then for $p \geq 2$,

$$\|\langle t, X\rangle\|_p = \|\sum_{i=1}^{d} t_i \varepsilon_i\|_p \sim \sum_{i \leq p} t_i^* + \sqrt{p}(\sum_{i>p} |t_i^*|^2)^{1/2},$$

where (t_i^*) denotes nonincreasing rearrangement of $(|t_i|)$.
Condition $\|\langle t-s, X\rangle\|_p \geq A$ implies that $t-s \notin cA(B_1^d + \sqrt{p}B_2^d)$ and Sudakov Minoration in this case follows by

Theorem (Talagrand)

For any $\varepsilon > 0$ and $T \subset \mathbb{R}^d$,

$$\varepsilon\sqrt{\ln N(T, C(r(T)B_1^d + \varepsilon B_2^d))} \leq r(T) := \mathbb{E}\sup_{t\in T}\sum_{i=1}^{d} t_i\varepsilon_i.$$

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Independent variables with log-concave tails

We say that a symmetric random variables X_i have *log-concave* tails if functions $N_i\colon [0,\infty) \mapsto [0,\infty]$ are convex, where

$$N_i(t) := -\ln \mathbb{P}(|X_i| \geq t), \quad t \geq 0.$$

Theorem (Talagrand'94 for $N_i(t) = t^r$, $r \geq 1$, L'97)

Suppose that $X = (X_1, \ldots, X_d)$ and X_i are independent symmetric with log-concave tails. Then X satisfies Sudakov minoration principle, i.e. whenever $T \subset \mathbb{R}^d$, $p \geq 2$ satisfy

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More general independent case

Theorem (L.-Tkocz)

Let $X = (X_1, \ldots, X_d)$, where X_i are independent centered and

$$\|X_i\|_r \leq \alpha \frac{r}{q} \|X_i\|_q \quad \text{for all } i \text{ and } r \geq q \geq 2 \tag{1}$$

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Remark. R.v's with logconcave tails satisfy (1) with $\alpha = 1$.

Example

Let $X = (X_1, X_2, \ldots, X_d)$ where X_i are i.i.d. symmetric. Take

$$T := \Big\{ t \in \{0,1\}^d \colon \; \sum_{i=1}^{d} t_i \leq m \Big\}.$$

It is not hard to see that for $t, s \in T$, $t \neq s$ and $p \geq 2$

$$\|\langle t - s, X\rangle\|_p \geq \|X_1\|_p \|t - s\|_\infty = \|X_1\|_p.$$

Moreover,

$$\mathbb{E} \sup_{t \in T} \langle t, X\rangle \leq m \mathbb{E} \max_{i \leq d} |X_i| \leq m (\mathbb{E} \sum_{i \leq d} |X_i|^q)^{1/q} = m d^{1/q} \|X_1\|_q.$$

If we take $d = e^{Cq}$ and $m = \frac{p}{q}$ we see that $\# T \geq e^p$ if $e^q > p$, hence for Sudakov's minoration in this case one should have

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Log-concave vectors

Sudakov minoration for vectors with independent coordinates seems to be quite well understood. It is then natural to ask what happens without independence assumption. Obviously we need to assume some regularity. One of possible natural assumptions is log-concavity.

We say that a random vector X in $\mathbb{R}^d$ is *logarithmically concave* (*log-concave* in short) if for any compact nonempty sets $A, B \subset \mathbb{R}^d$ and $\lambda \in [0,1]$,

$$\mathbb{P}(X \in \lambda A + (1-\lambda)B) \geq \mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda}.$$

Theorem (Borell)

A random vector X with a full dimensional support is log-concave iff it has a log-concave density, i.e a density of the form $e^{-h(x)}$ with $h\colon \mathbb{R}^d \to (-\infty, \infty]$ convex.

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Examples of log-concave vectors

- Gaussian vectors
- Vectors with independent log-concave coordinates (in particular vectors with the product exponential distribution)
- Vectors uniformly distributed on convex bodies
- Affine images of log-concave vectors
- Sums of independent log-concave vectors

It may be shown that the class of log-concave distributions is the smallest class that contains uniform distributions on convex bodies and is closed under affine transformations and weak limits. Moreover, real, symmetric log-concave random variables have log-concave tails. This gives for symmetric log-concave vectors,

$$\|\langle t, X\rangle\|_p \leq \frac{p}{q}\|\langle t, X\rangle\|_q \quad \text{for } p \geq q \geq 2.$$

Conjecture

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Centered log-concave vectors X satisfy Sudakov minoration with absolute constant, i.e. if $T \subset \mathbb{R}^d$, $p \geq 2$ satisfy

$$\|\langle t-s, X\rangle\|_p = \Big(\mathbb{E}\Big(\sum_{i=1}^{d}(t_i-s_i)X_i\Big)^p\Big)^{1/p} \geq A \quad \text{for all } s,t \in T,\ s \neq t$$

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Considering $X - X'$ with X' independent copy of X immediately reduces the conjecture to the case of symmetric vectors.
From previously stated results this is true under the additional assumption of independence of coordinates of X.

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Disjoint supports

We say that a random vector $X = (X_1, \ldots, X_n)$ is *isotropic* if $\mathbb{E}X_i = 0$ and $\mathbb{E}X_iX_j = \delta_{i,j}$.

Proposition

Let X be isotropic and log-concave. Suppose that $T \subset \mathbb{R}^d$, $p \geq 2$ satisfy

$$\|\langle t-s, X\rangle\|_p = \Big(\mathbb{E}\Big(\sum_{i=1}^{d}(t_i-s_i)X_i\Big)^p\Big)^{1/p} \geq A \quad \text{for all } s,t \in T,\ s \neq t,$$

$\# T \geq e^p$ and vectors in T have disjoint supports then

$$\mathbb{E}\sup_{t\in T}\langle t, X\rangle = \mathbb{E}\sup_{t\in T}\sum_{i=1}^{d} t_iX_i \geq \frac{1}{C}A.$$

Main Lemma

Proposition

Suppose that X is isotropic log-concave, vectors $t_1, \dots, t_n \in \mathbb{R}^d$ have disjoint supports, $n \geq e^p$ and $\|\langle t_i, X\rangle\|_p \geq A$ for all $1 \leq i \leq n$. Then

$$\mathbb{E} \max_{i \leq n} |\langle t_i, X\rangle| \geq \frac{1}{C} A.$$

Sketch of the proof. Let $a_i = |t_i|$, $Y_i := \langle t_i, X\rangle / |a_i|$, then $Y = (Y_1, \dots, Y_n)$ is isotropic log-concave in $\mathbb{R}^n$ and $\|a_i Y_i\|_p \geq A$ for all i. We need to show that $\mathbb{E} \max_i |a_i Y_i| \geq \frac{1}{C} A$.
The crucial fact is that Y as isotropic log-concave vector satisfies Poincaré inequality with a constant $Cn^{1/2-\varepsilon}$, where $\varepsilon > 0$ (Klartag). Thus if $\mathbb{P}(Y \in B) \geq 1/2$ then $\mathbb{P}(Y \in B + tn^{1/2-\varepsilon} B_2^n) \geq 1 - e^{t/C}$. We apply this with

$$B = \{x \in \mathbb{R}^n \colon \sum_{i=1}^n (|a_i x_i|^{p'} \wedge A') \leq 2 \sum_{i=1}^n \mathbb{E}(|a_i Y_i|^{p'} \wedge A')\}$$

where $p' = \lambda p$, $A' = \lambda A/2$, $\lambda = \lambda(\varepsilon) \in (0, 1)$.

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where $p' = \lambda p$, $A' = \lambda A/2$, $\lambda = \lambda(\varepsilon) \in (0,1)$.

Main Lemma

Proposition

Suppose that X is isotropic log-concave, vectors $t_1, \ldots, t_n \in \mathbb{R}^d$ have disjoint supports, $n \geq e^p$ and $\|\langle t_i, X\rangle\|_p \geq A$ for all $1 \leq i \leq n$. Then

$$\mathbb{E} \max_{i \leq n} |\langle t_i, X\rangle| \geq \frac{1}{C} A.$$

Sketch of the proof. Let $a_i = |t_i|$, $Y_i := \langle t_i, X\rangle / |a_i|$, then $Y = (Y_1, \ldots, Y_n)$ is isotropic log-concave in $\mathbb{R}^n$ and $\|a_i Y_i\|_p \geq A$ for all i. We need to show that $\mathbb{E} \max_i |a_i Y_i| \geq \frac{1}{C} A$.
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Comparison of weak and strong moments

Corollary

Let X be an isotropic logconcave vector in $\mathbb{R}^d$. Then for $p \geq 2$ and any sequence (a_i),

$$(\mathbb{E} \max_{i \leq d} |a_i X_i|^p)^{1/p} \leq C\Big(\mathbb{E} \max_{i \leq d} |a_i X_i| + \max_{i \leq d} (\mathbb{E}|a_i X_i|^p)^{1/p}\Big).$$

Sketch of the proof. Let $A = \mathbb{E} \max_{i \leq d} |a_i X_i|$, from the main Lemma, $\|a_i X_i\|_p > CA$ for at most e^p coordinates. So we may (after rearranging coordinates) assume that $\|a_i X_i\|_n \leq CA$ for $i \geq 2e^n$ and conclude by an easy integration by parts argument.

Conjecture

For all norms on $\mathbb{R}^d$ and all log-concave vectors for $p \geq 2$,

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Unconditional case

We say that a random vector $X = (X_1, \ldots, X_d)$ is *unconditional* if $(\eta_1 X_1, \eta_2 X_2, \ldots, \eta_d X_d)$ has the same distribution as X for any choice of signs $\eta_1, \ldots, \eta_d \in \{-1, 1\}$.
In such case much weaker Sudakov-type estimate holds

Proposition

Let X be unconditional, log-concave. Suppose that $T \subset \mathbb{R}^d$ satisfy

$$\|\langle t-s, X\rangle\|_p = \Big(\mathbb{E}\Big(\sum_{i=1}^{d}(t_i-s_i)X_i\Big)^p\Big)^{1/p} \geq A \quad \text{for all } s,t \in T,\ s \neq t,$$

and $\#T \geq e^{p^2}$, $p \geq 2$. Then

$$\mathbb{E}\sup_{t\in T}\langle t, X\rangle = \mathbb{E}\sup_{t\in T}\sum_{i=1}^{d} t_i X_i \geq \frac{1}{C}A.$$

In fact what plays a crucial role in the proof is not log-concavity but Ψ_1 condition, i.e. $\|\langle t, X\rangle\|_p \leq Cp\|\langle t, X\rangle\|_2$ for $p \geq 2$.

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Uniform distributions on B_p^d-balls

There are examples of nonproduct log-concave distributions with Sudakov minoration property.

Proposition

If X is uniformly distributed on B_p^d then X satisfies Sudakov minoration.

What is crucial is that concentration properties of such vectors are well understood.

It is reasonable to believe (work in progress) that unconditional log-concave vectors such that $|X_i|$ are negatively associated satisfy Sudakov minoration. By the work of Pilipczuk and Wojtaszczyk uniform distributions on Orlicz balls have these properties.

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Application I - comparison of weak and strong moments

Sudakov minoration implies several interesting properties of random vectors. One of them is comparison of weak and strong moments up to log factor.

Proposition

Suppose that a random vector X in $\mathbb{R}^d$ satisfies Sudakov minoration. Then for any $p \geq 2$ and any norm on $\mathbb{R}^d$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big(\ln(1+\frac{d}{p})\mathbb{E}\|X\| + \sup_{\|t\|_* \leq 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}\Big).$$

Proof is based on an easy chaining argument.

Application II - Paouris-type estimate

Paouris showed that for isotropic log-concave vectors $(\mathbb{E}|X|^p)^{1/p} \leq C(\sqrt{n} + p)$. In fact from his paper follows a more general result - comparison of weak and strong moments of $|X|$. Similar result holds under Sudakov minoration assumption.

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Application III - comparison of processes

One of important consequences of Fernique-Talagrand majorizing measure theorem is that suprema of subgaussian processes may be estimated by suprema of Gaussian processes. Sudakov minoration yields similar fact up to log factor.

Proposition

Suppose that a random vector X in $\mathbb{R}^d$ satisfies Sudakov minoration. Then for any bounded nonempty $T \subset \mathbb{R}^d$ and any process $(Y_t)_{t \geq 0}$ such that for some $A < \infty$,

$$\|Y_t - Y_s\|_p \leq A \|\langle t - s, X\rangle\|_p \quad \text{for all } s, t \in T, p \geq 2$$

we have

$$\mathbb{E} \sup_{t \in T} |Y_t| \leq CA \log d \, \mathbb{E} \sup_{t \in T} |\langle t, X\rangle|.$$

Proof is based on chaining.

Thank you for your atttention

All the best Alain!