Invertibility of adjacency matrices of random digraphs

Alexander Litvak

University of Alberta

based on a joint work with

A. Lytova, K. Tikhomirov, N. Tomczak-Jaegermann, and P. Youssef

Marne-La-Vallée 2015

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Remark 3. The cases $d = d_0$ and $d = n - d_0$ are essentially the same (by interchanging zeros and ones).

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Theorem (LLTTP)

The conjecture holds for $C \le d \le n/\ln^2 n$ *with probability* $1 - C\ln^3 d/\sqrt{d}$.

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Then we treat the remaining "good" matrices.

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In this case we find a row R_i that has exactly one 1 in the first m columns and zeros in columns $m+1, ..., k$. Then

$$|\langle R_i, x^* \rangle| \geq |x_m^\star| - (d-1)|x_k^\star| > 0.$$

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To find such a row, let S be the support of the first m columns. Then with high probability

$$|S| \geq (1 - \varepsilon)dm.$$

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The case $x_m^\star = 0$ is similar.

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Finally we need a probability bound on individual terms of the net, which fits the size of the net. This is a (non-trivial) consequence of an anti-concentration result with "frozen" columns in $I_0 \cup I$.

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b. To fix such a minor and to "play" with two remaining rows ("shuffling").

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In Step **b** (the shuffling) we show that the event $v(M) \in \mathrm{Ker} M$ is small in $\mathcal{E}^{ij}$.

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Thus, in both cases $v(M)$ is in the kernel, which occurs with a small probability.

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In the classical setting there is no restriction $|B| = d$.

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Then A has the same distribution as a randomly chosen set B of cardinality d, hence

$$\mathbb{P}\Big(\sum_{i \in B} v_i = a \Big) = \mathbb{P}\Big(\sum_{i \in A} v_i = a \Big) = \mathbb{P}\Big(\sum_{i \leq d} \xi_i = a \Big)$$

and for every "good" π one can apply the Erdos anti-concentration.