On the Brunn-Minkowski inequality for convex measures with applications to new isoperimetric inequalities

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$$\gamma_n((1-\lambda)A+\lambda B)^{1/n} \geq (1-\lambda)\gamma_n(A)^{1/n}+\lambda\gamma_n(B)^{1/n}$$

holds for every symmetric convex set $A, B \subset \mathbb{R}^n$ *and every* $\lambda \in [0,1]$, *where* γ_n *denotes the standard Gaussian measure.*

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The conjecture cannot hold for arbitrary set:
$\rightarrow$ Take $A = [0,1]$, $B = \{b\}$ and let b go to $+\infty$.

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<u>unconditional:</u> $A \subset \mathbb{R}^n$ is unconditional if for every $(x_1, \ldots, x_n) \in A$ and every $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, one has

$$(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in A.$$

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Let A be an unconditional convex set in $\mathbb{R}^n$ (or a symmetric convex set in $\mathbb{R}^2$). Let $r > 0$ such that $\gamma_n(A) = \gamma_n(rB_2^n)$. Then,

$$r\gamma_n^+(\partial A) + \int_A |x|^2 \mathrm{d}\gamma_n(x) \geq r\gamma_n^+(\partial(rB_2^n)) + \int_{rB_2^n} |x|^2 \mathrm{d}\gamma_n(x).$$

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In other words, Euclidean balls minimize the quantity

$$r\gamma_n^+(\partial A) + \int_A |x|^2 \mathrm{d}\gamma_n(x)$$

among unconditional convex sets in $\mathbb{R}^n$ (or symmetric convex sets in $\mathbb{R}^2$) with prescribed measure.

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Support function of a convex set:

Let $A \subset \mathbb{R}^n$ be a convex set. The support function of A is

$$h_A(u) = \max_{x \in A} \langle x, u \rangle, \quad u \in S^{n-1}.$$

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It is known that

$$A = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_A(u), \text{ for all } u \in S^{n-1}\}.$$

and

$$h_{A+B} = h_A + h_B, \qquad h_{\lambda A} = \lambda h_A.$$

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L^p-Minkowski sum:

Let $A, B \subset \mathbb{R}^n$ be convex sets, and let $\lambda \in [0, 1]$.

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Let us define, for $a, b \geq 0$, $\lambda \in [0,1]$, and $p \in [-\infty, +\infty]$,

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Let $A, B \subset \mathbb{R}^n$ be convex sets, let $\lambda \in [0,1]$, and $p \in [-\infty, +\infty]$. We define their L^p-Minkowski sum by

$$(1-\lambda)\cdot A +_p \lambda \cdot B = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq M_p^{\lambda}(h_A(u), h_B(u)), \text{ for all } u \in S^{n-1}\}.$$

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Remark

For every $p\ge q$, $\quad (1-\lambda)\cdot A+_p\lambda\cdot B\supset(1-\lambda)\cdot A+_q\lambda\cdot B.$

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The log-Brunn-Minkowski conjecture holds in $\mathbb{R}^2$.

Theorem (Saroglou (2014))

The log-Brunn-Minkowski conjecture holds in $\mathbb{R}^n$ in the unconditional case.

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holds for symmetric log-concave measure μ and symmetric convex sets A, B in $\mathbb{R}^2$; and for unconditional log-concave measure μ and unconditional convex sets A, B in $\mathbb{R}^n$.

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3. *Let μ be an unconditional product measure with decreasing density. Then μ satisfies the Brunn-Minkowski inequality in the class of unconditional convex sets in $\mathbb{R}^n$.*

Proof:
Let $p \in (0,1)$.

$$\mu((1-\lambda)A+\lambda B) \quad = \quad \mu\left((1-p)\frac{1-\lambda}{1-p}A+p\frac{\lambda}{p}B\right)$$

Proof:
Let $p \in (0,1)$.

$$\begin{aligned}\mu((1-\lambda)A+\lambda B) &= \mu\left((1-p)\frac{1-\lambda}{1-p}A + p\frac{\lambda}{p}B\right) \\ &\geq \mu\left((1-p)\cdot\left(\frac{1-\lambda}{1-p}A\right) +_0 p\cdot\left(\frac{\lambda}{p}B\right)\right)\end{aligned}$$

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&= \mu\left(\left[\left(\frac{1-\lambda}{1-p}\right)^{1-p}\left(\frac{\lambda}{p}\right)^{p}\right](1-p)\cdot A+_0 p.B\right) \\
&\geq \left[\left(\frac{1-\lambda}{1-p}\right)^{1-p}\left(\frac{\lambda}{p}\right)^{p}\right]^{n}\mu((1-p)\cdot A+_0 p.B)
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Optimizing in $p \in (0,1)$, i.e. taking

$$p = \frac{\lambda\mu(B)^{1/n}}{(1-\lambda)\mu(A)^{1/n}+\lambda\mu(B)^{1/n}} \tag{1}$$

yields

$$\mu((1-\lambda)A+\lambda B)^{1/n} \geq (1-\lambda)\mu(A)^{1/n}+\lambda\mu(B)^{1/n}.$$

As a consequence:

Theorem (Livshyts, M., Nayar, Zvavitch)

The inequality

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holds *when:*

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Theorem (Livshyts, M., Nayar, Zvavitch)

1. *Let μ be a measure on $\mathbb{R}^2$ with an even log-concave density. Then μ satisfies the Brunn-Minkowski inequality in the class of symmetric convex sets in $\mathbb{R}^2$.*
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Fix $\lambda, p \in (0,1)$. Suppose that m, f, g are unconditional decreasing non-negative functions and let μ be an unconditional product measure with decreasing density on $\mathbb{R}^n$.

Theorem (Livshyts, M., Nayar, Zvavitch)

Fix $\lambda, p \in (0,1)$. Suppose that m, f, g are unconditional decreasing non-negative functions and let μ be an unconditional product measure with decreasing density on $\mathbb{R}^n$. Assume that for any $x, y \in \mathbb{R}^n$ we have

$$m((1-\lambda)x+\lambda y) \geq f(x)^{1-p} g(y)^p.$$

Theorem (Livshyts, M., Nayar, Zvavitch)

Fix $\lambda, p \in (0,1)$. Suppose that m, f, g are unconditional decreasing non-negative functions and let μ be an unconditional product measure with decreasing density on $\mathbb{R}^n$. Assume that for any $x, y \in \mathbb{R}^n$ we have

$$m((1-\lambda)x + \lambda y) \geq f(x)^{1-p} g(y)^p.$$

Then

$$\int m \, \mathrm{d}\mu \geq \left[\left(\frac{1-\lambda}{1-p} \right)^{1-p} \left(\frac{\lambda}{p} \right)^p \right]^n \left(\int f \, \mathrm{d}\mu \right)^{1-p} \left(\int g \, \mathrm{d}\mu \right)^p.$$

IV. Functional versions of the Brunn-Minkowski inequality

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$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

holds for every $x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)$*, then*

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$\to$ (Almost) Yields the Brunn-Minkowski inequality by taking indicator of sets ($f = 1_A$, $g = 1_B$, $h = 1_{(1-\lambda)A+\lambda B}$).

Theorem (Borell-Brascamp-Lieb inequality)

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Let $\gamma \geq -\frac{1}{n}$, $\lambda \in [0,1]$ *and* $f, g, h : \mathbb{R}^n \to [0,+\infty)$ *be measurable functions. If the inequality*

$$h((1-\lambda)x + \lambda y) \geq M_\gamma^\lambda(f(x), g(y))$$

holds for every $x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)$, *then*

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$\rightarrow$ (Exactly) Yields the Brunn-Minkowski inequality by taking indicator of sets and $\gamma = +\infty$.

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Let $\lambda \in [0,1]$ *and* $f, g, h : [0,+\infty)^n \to [0,+\infty)$ *be measurable functions. If the inequality*

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Theorem (nonlinear extension of the Brunn-Minkowski inequality)

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Let $\mathbf{p} = (p_1, \ldots, p_n) \in [0,1]^n$, $\gamma \geq -(\sum_{i=1}^n p_i^{-1})^{-1}$, $\lambda \in [0,1]$, *and* $f, g, h : [0,+\infty)^n \to [0,+\infty)$ *be measurable functions. If the inequality*

$$h(M_{\mathbf{p}}^{\lambda}(x,y)) \geq M_{\gamma}^{\lambda}(f(x), g(y))$$

holds for every $x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)$, *then*

$$\int_{[0,+\infty)^n} h \geq M^{\lambda}_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}} \left(\int_{[0,+\infty)^n} f, \int_{[0,+\infty)^n} g \right).$$

Theorem (Borell inequality (1974))

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Let $f, g, h : \mathbb{R}^n \to [0, +\infty)$ *be measurable functions. Let* $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathrm{supp}(f) \times \mathrm{supp}(g) \to \mathbb{R}^n$ *be a continuously differentiable function with positive partial derivatives, such that* $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ *for every* $x = (x_1, \ldots, x_n) \in \mathrm{supp}(f)$, $y = (y_1, \ldots, y_n) \in \mathrm{supp}(g)$. *Let* $\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$h(\varphi(x, y)) \Pi_{k=1}^n \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k)$$

holds for every $x \in \mathrm{supp}(f)$, *for every* $y \in \mathrm{supp}(g)$, *for every* $\rho_1, \ldots, \rho_n > 0$ *and for every* $\eta_1, \ldots, \eta_n > 0$, *then*

$$\int h \geq \Phi \left(\int f, \int g \right).$$

[Sketch of proof]

By induction on the dimension (the inequality tensorizes). To prove the inequality in dimension 1, we use a mass transportation technique:

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[Sketch of proof]

By induction on the dimension (the inequality tensorizes). To prove the inequality in dimension 1, we use a mass transportation technique:

We may assume that $\int f = \int g = 1$, and that f, g are compactly supported positive Lipschitz functions. Thus there exists a non-decreasing map $T : \mathrm{supp}(f) \to \mathrm{supp}(g)$ such that for every $x \in \mathrm{supp}(f)$,

$$f(x) = g(T(x))T'(x).$$

Hence one has,

$$\int h(z)\mathrm{d}z \quad \geq \quad \int_{\mathrm{supp}(f)} h(\varphi(x, T(x))) \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} T'(x) \right) \mathrm{d}x$$

Hence one has,

$$\begin{aligned}\int h(z)\mathrm{d}z &\geq \int_{\mathrm{supp}(f)} h(\varphi(x,T(x)))\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}T'(x)\right)\mathrm{d}x \\ &\geq \int_{\mathrm{supp}(f)} \Phi(f(x),g(T(x))T'(x))\mathrm{d}x\end{aligned}$$

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\end{aligned}$$

Using homogeneity of Φ, one deduces that

$$\int h \geq \Phi(1,1)\int f(x)\mathrm{d}x = \Phi\left(\int f, \int g\right).$$

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The log-Brunn-Minkowski conjecture holds in $\mathbb{R}^n$ in the unconditional case.

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Saroglou's proof tells us that the geometric prekopa-Leindler inequality implies the log-Brunn-Minkowski inequality for unconditional sets.

$$\text{geometric-PL} \Longrightarrow \text{log-BM for unconditional sets}$$

The log-BM for convex measures:

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Conjecture (M. (2015))

Let $p \in [0,1]$. Let μ be a symmetric measure in $\mathbb{R}^n$ that has an α-concave density function, with $\alpha \geq -\frac{p}{n}$. Then for every symmetric convex set $A, B \subset \mathbb{R}^n$ and for every $\lambda \in [0,1]$,

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Recall that

$$(1-\lambda)\cdot A +_p \lambda \cdot B = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq M^{\lambda}_p(h_A(u), h_B(u)), \text{ for all } u \in S^{n-1}\}.$$

If α or p is equal to 0, then $(n/p + 1/\alpha)^{-1}$ is defined by continuity and is equal to 0. The log-Brunn-Minkowski conjecture is obtained by taking μ to be Lebesgue measure and $p = 0$.

Theorem (M. 20++)

If the log-Brunn-Minkowski inequality for Lebesgue measure holds then the inequality

$$\mu((1-\lambda)\cdot A +_p \lambda \cdot B) \geq M^{\lambda}_{\left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}}(\mu(A),\mu(B)).$$

holds for every symmetric measure μ that has an α-concave density function, with $\alpha \geq -\frac{p}{n}$, and for all symmetric convex sets $A, B \subset \mathbb{R}^n$.

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BM $\implies$ PL (Folk.)

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$$\Longrightarrow \quad r\gamma_n^+(\partial A) + \int_A |x|^2 \mathrm{d}\gamma_n(x) \geq r\gamma_n^+(\partial(rB_2^n)) + \int_{rB_2^n} |x|^2 \mathrm{d}\gamma_n(x)$$

for unconditional sets $A \subset \mathbb{R}^n$ such that $\gamma_n(A) = \gamma_n(rB_2^n)$.

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Conclusion: In the unconditional case, the Brunn-Minkowski inequality is very strong!

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Summary:
In the symmetric case, the log-Brunn-Minkowski inequality is very strong!

$$\text{log-BM} \implies \text{gaussian-BM} \quad \text{(Livshyts, M., Nayar, Zvavitch 2015+)}$$
$$\implies r\gamma_n^+(\partial A) + \int_A |x|^2 \mathrm{d}\gamma_n(x) \geq r\gamma_n^+(\partial(rB_2^n)) + \int_{rB_2^n} |x|^2 \mathrm{d}\gamma_n(x)$$

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The inequality

$$|(1-\lambda)\cdot A +_0 \lambda \cdot B| \geq |A|^{1-\lambda}|B|^{\lambda}$$

holds for every symmetric convex set $A, B \subset \mathbb{R}^n$, $n \geq 3$.

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Thank you for your attention !!!