Convex discretization of functionals involving the Monge-Ampère operator

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Joint work with J.D. Benamou, G. Carlier and É. Oudet

GeMeCod Conference

October 26-30, 2015, Marne-la-Vallée

1. Discretization of the space of convex functions

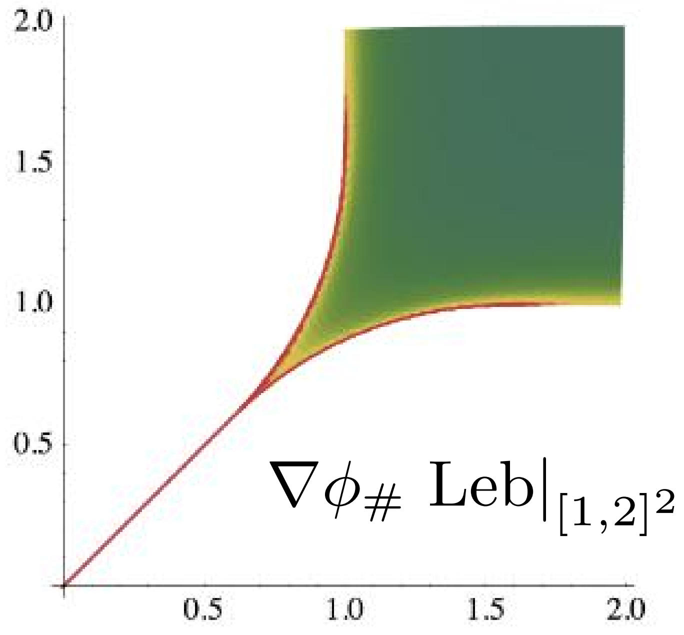
Motivation: variational problems over convex functions

- Principal-agent problem in economy

$$\min_{\phi\in\mathcal{K},\phi\geq 0} \int_{[1,2]^2} \frac{1}{2}(\|\nabla\phi(x) - x\|^2 - \phi(x))\,\mathrm{d}\,x$$

$\mathcal{K} := \{\phi$ convex $\}$

[Rochet-Choné '98]



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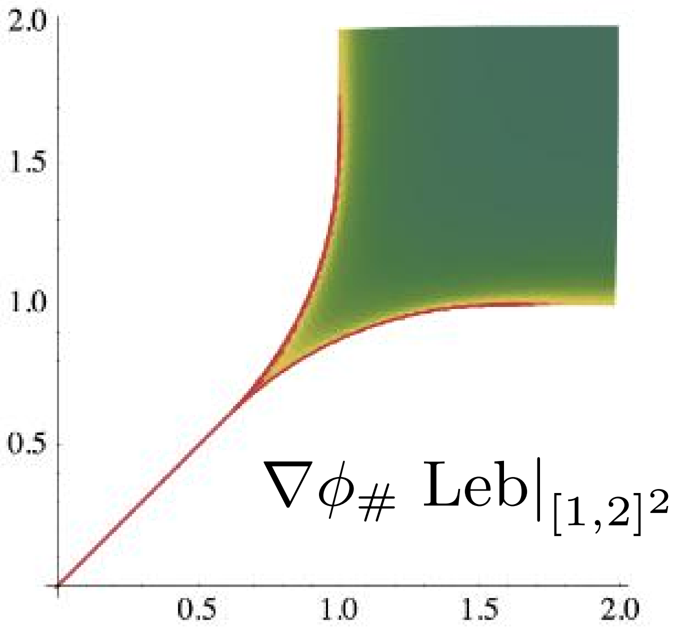

- Gradient flows in Wasserstein space

[Otto]
[Jordan-Kinderlehrer-Otto '98]

$$\min_{\phi\in\mathcal{K}_X} \frac{1}{2\tau}\,\mathrm{W}_2^2(\rho, \nabla\phi_{\#}\rho) + \mathcal{E}(\nabla\phi_{\#}\rho)$$

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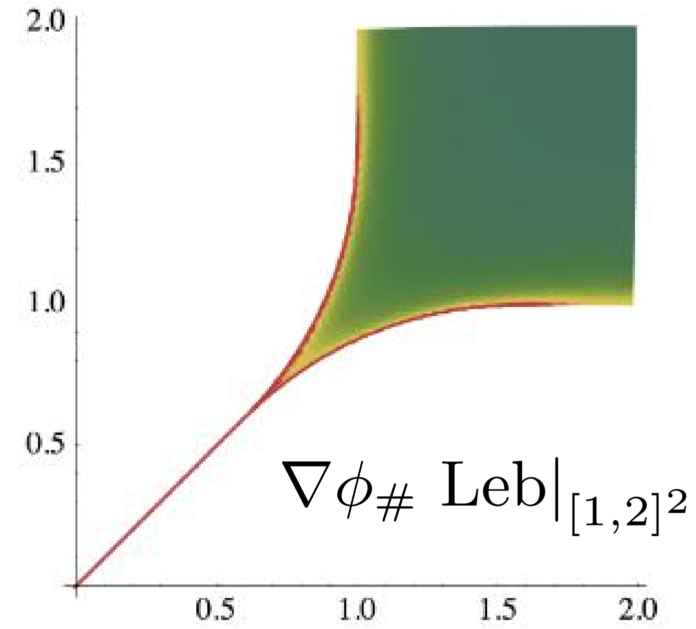

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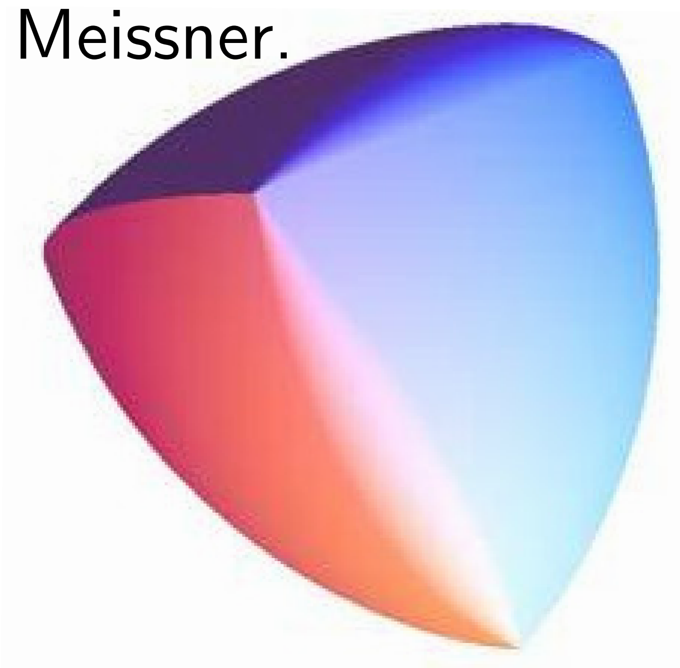
$\mathcal{K}_* := \{$ support function of convex bodies $\}$



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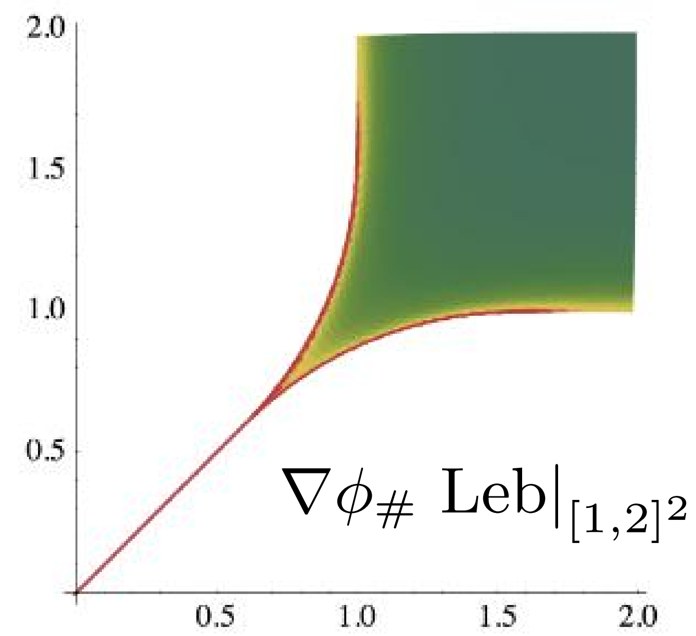

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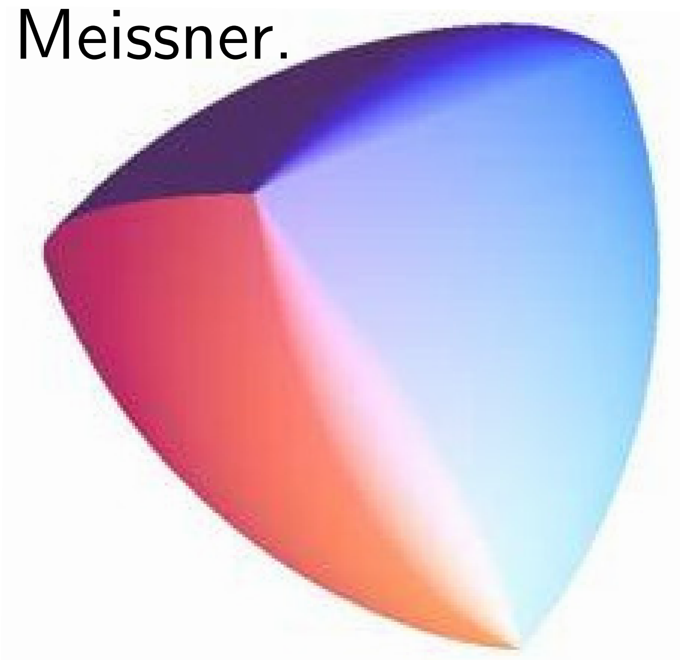
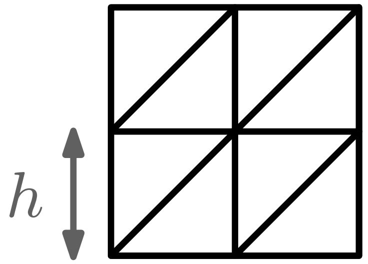


Prior work: Discretizing the Space of Convex Functions

- Finite elements: piecewise-linear convex functions over a **fixed** mesh



standard grid

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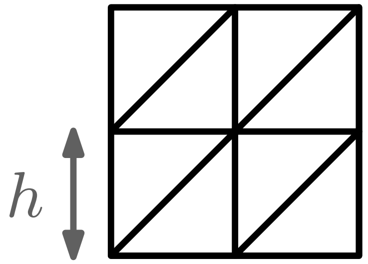

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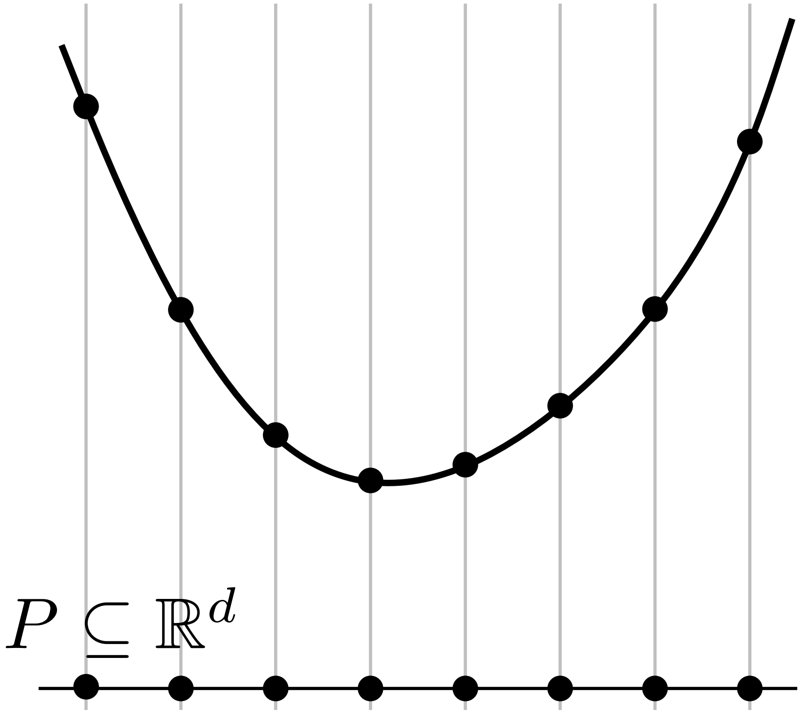

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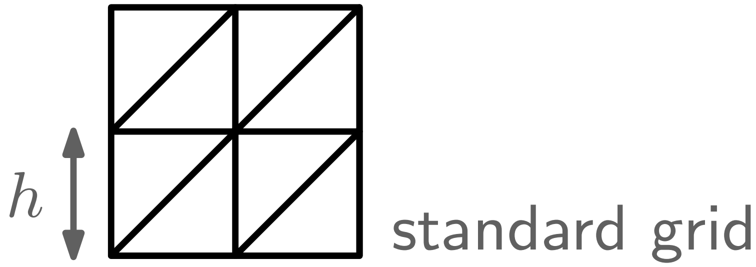

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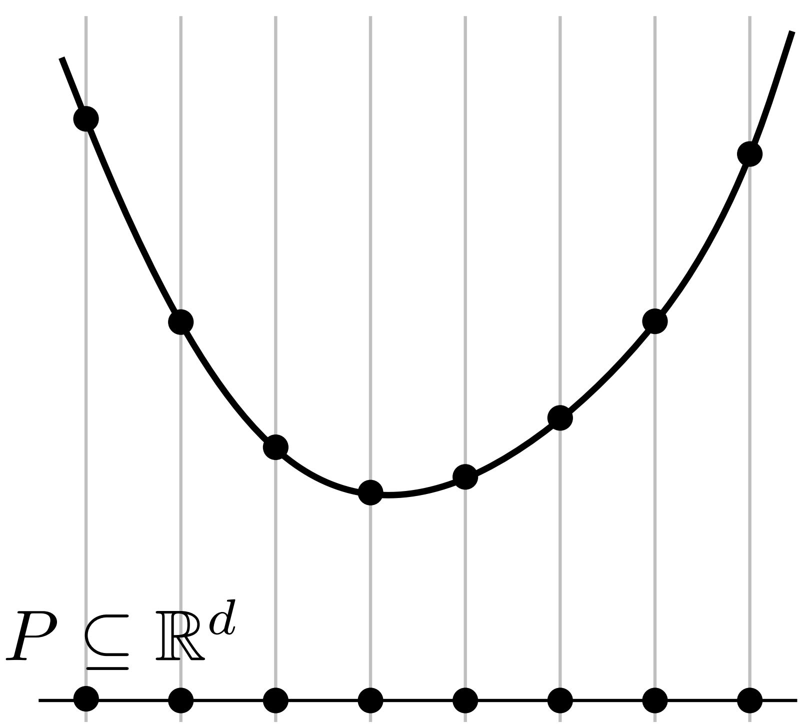

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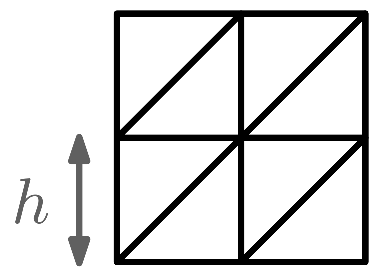

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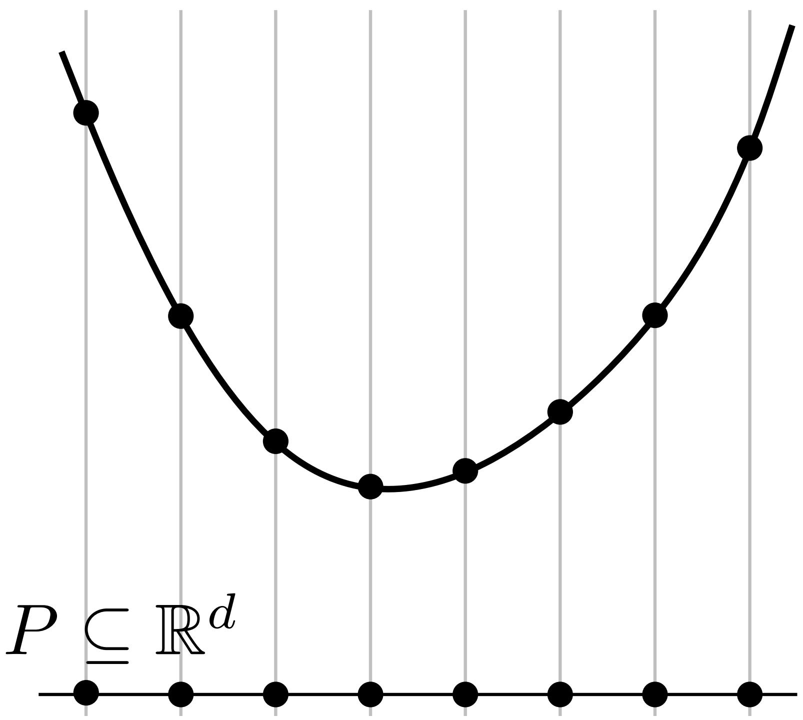

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adaptive method [Mirebeau '14]

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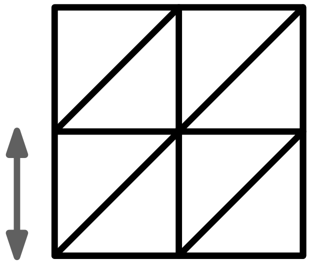

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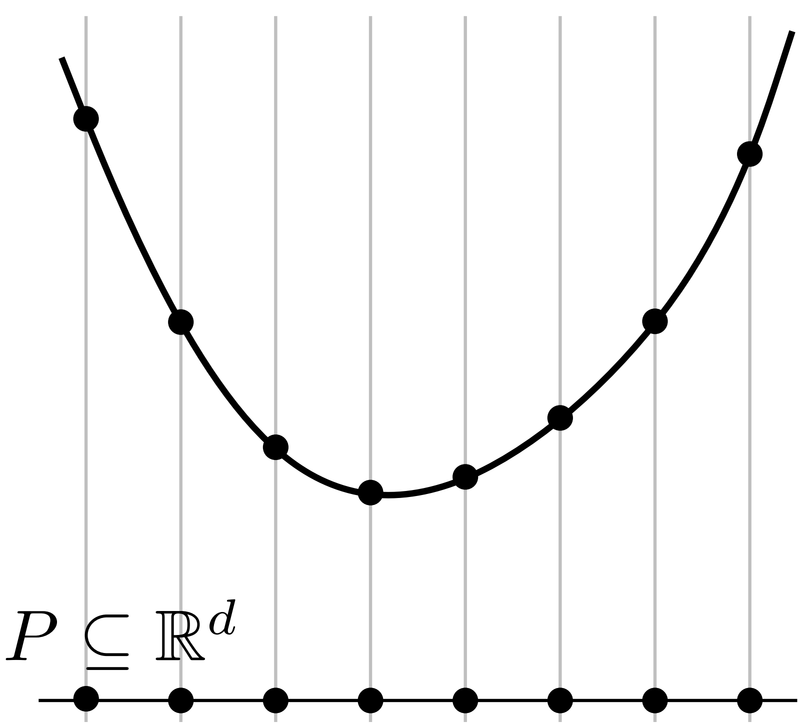

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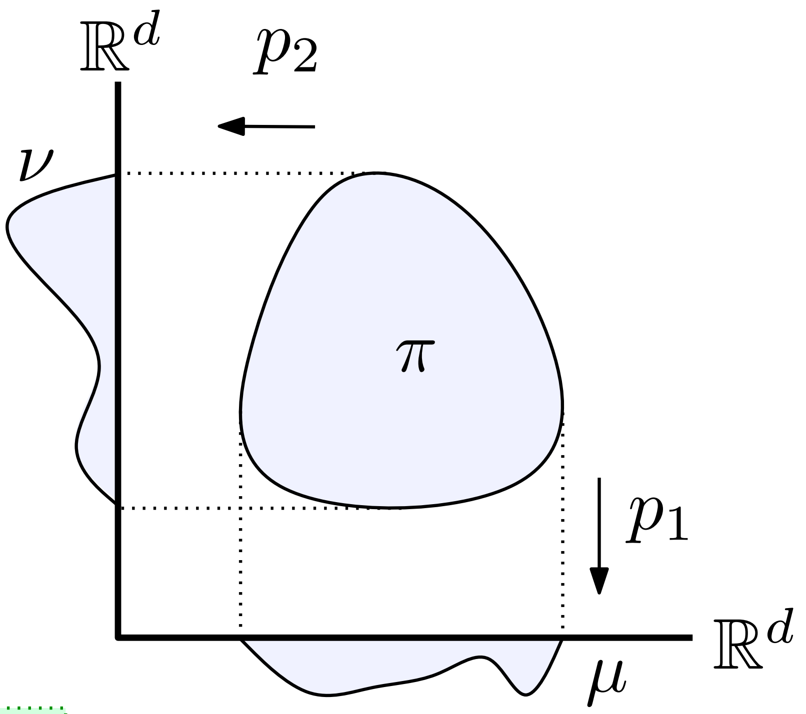
This talk: Using the Monge-Ampère operator to describe $\mathcal{K}(P)$ more conveniently.

2. Space-discretization of Wasserstein gradient flows

Background: Optimal transport

$\mathcal{P}_2(\mathbb{R}^d)$ = prob. measures with finite second moment

$\mathcal{P}_{\mathrm{ac}}^2(\mathbb{R}^d) = \mathcal{P}_2(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d)$



- Wasserstein distance between $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\Gamma(\mu,\nu) := \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d); p_{1\#}\pi = \mu, p_{2\#}\pi = \nu\}$$

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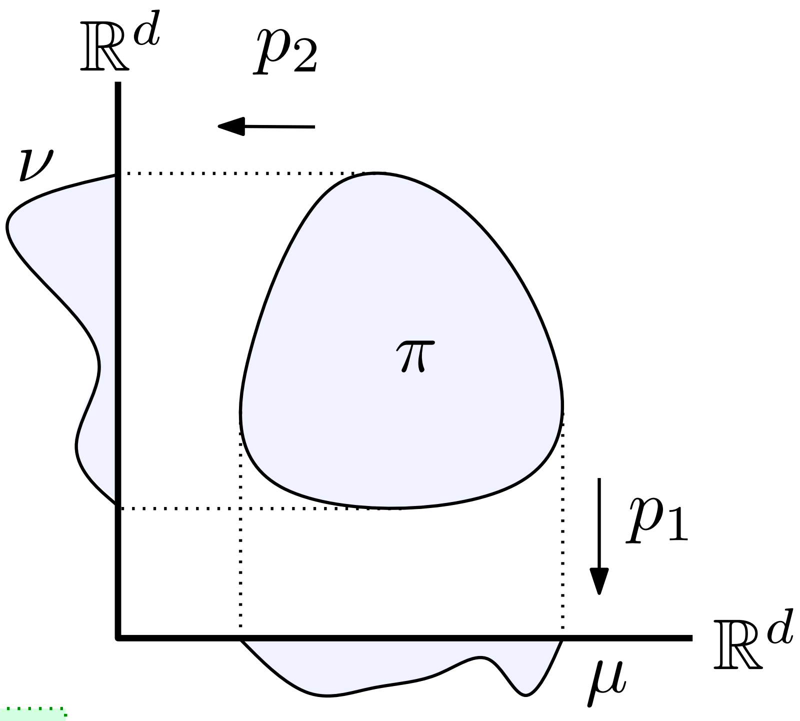

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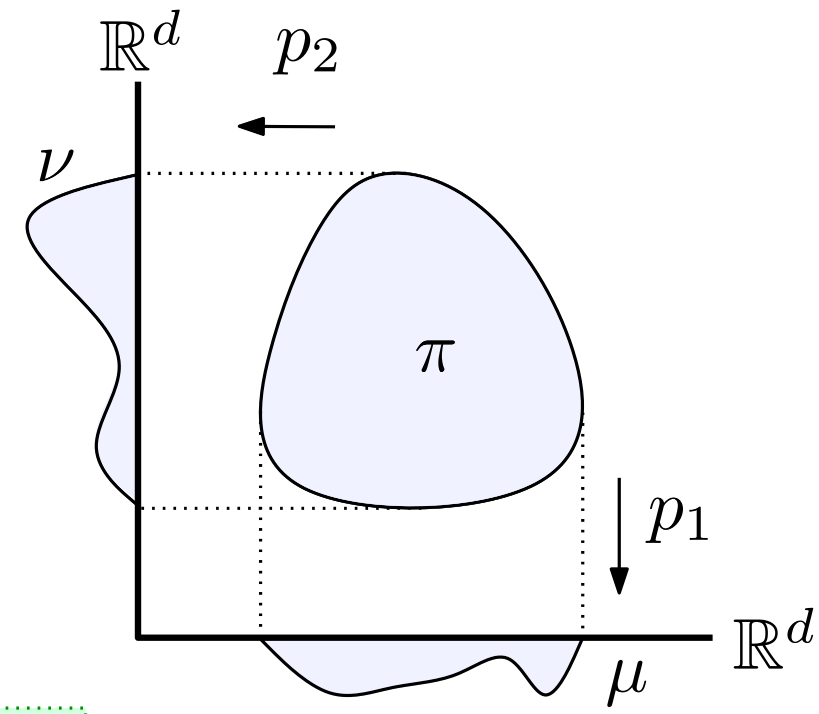

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Theorem (Brenier): Given $\mu \in \mathcal{P}^2_{ac}(\mathbb{R}^d)$ the map $\phi \in \mathcal{K} \mapsto \nabla\phi_{\#}\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is **surjective** and moreover,

$$\mathrm{W}_2^2(\mu, \nabla\phi_{\#}\mu) = \int_{\mathbb{R}^d} \|x - \nabla\phi(x)\|^2 \,\mathrm{d}\,\mu(x)$$

[Brenier '91]

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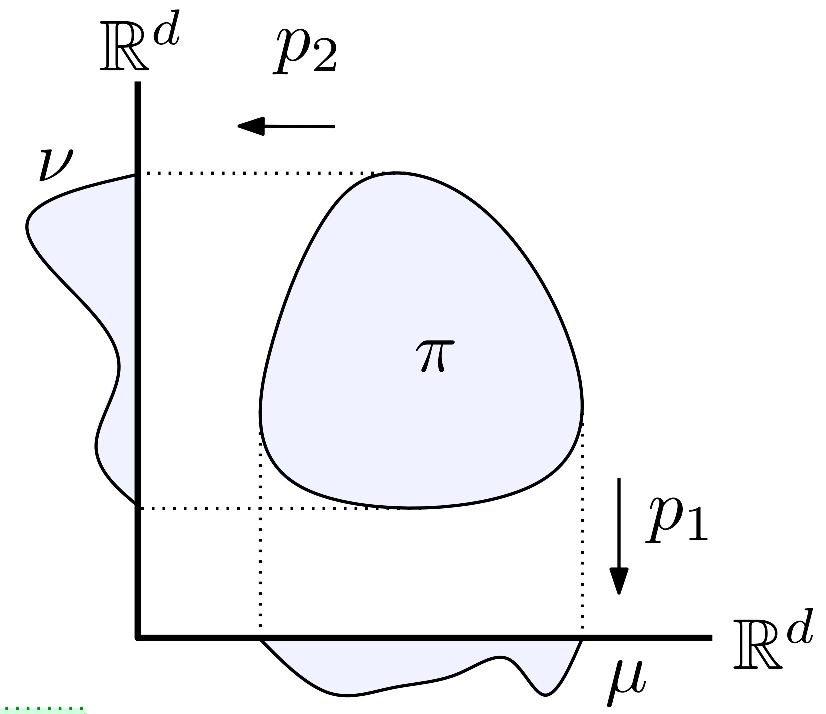

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$\longrightarrow \simeq$ Lagrangian parameterization of $\mathcal{P}_2(\mathbb{R}^d)$, as "seen" from $\mu \in \mathcal{P}_2^{\rm ac}(\mathbb{R}^d)$.

Motivation 1: Crowd Motion Under Congestion

- JKO scheme for crowd motion with hard congestion:
  [Maury-Roudneff-Chupin-Santambrogio 10]

$$\boxed{\rho_{k+1}^{\tau} = \min_{\sigma \in \mathcal{P}_2(X)} \tfrac{1}{2\tau} \mathrm{W}_2^2(\rho_k^{\tau}, \sigma) + \mathcal{E}(\sigma) + \mathcal{U}(\sigma)} \qquad (*)$$

where $X \subseteq \mathbb{R}^d$ is convex and bounded, and

$$\mathcal{U}(\nu) := \begin{cases} 0 \text{ if } \nu \in \mathcal{P}_{\mathrm{ac}} \text{ and } \dfrac{\mathrm{d}\,\nu}{\mathrm{d}\,x} <= 1 \\ +\infty \text{ if not} \end{cases} \qquad \mathcal{E}(\nu) := \int_{\mathbb{R}^d} V(x)\,\mathrm{d}\,\nu(x)$$

congestion — potential energy

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potential energy

- Assuming $\sigma = \nabla\phi_\# \rho^\tau_k$ with ϕ convex, the Wasserstein term becomes explicit:

$$(*) \iff \boxed{\min_\phi \frac{1}{2\tau} \int_{\mathbb{R}^d} \|x - \nabla\phi(x)\|^2 \rho^\tau_k(x) \,\mathrm{d}\,x + \mathcal{E}(\nabla\phi_\# \rho^\tau_k) + \mathcal{U}(\nabla\phi_\# \rho^\tau_k)}$$

On the other hand, the constraint becomes strongly nonlinear:

$$\mathcal{U}(\nabla\phi_\# \rho^\tau_k) < +\infty \iff \det \mathrm{D}^2\phi(x) \geq \rho_k(x)$$

Motivation 2: Nonlinear Diffusion

$$(*) \qquad \boxed{\frac{\partial \rho}{\partial t} = \mathrm{div}\left[\rho \nabla (U'(\rho) + V + W * \rho)\right]} \qquad \begin{array}{l} \rho(0,.) = \rho_0 \\ \rho(t,.) \in \mathcal{P}^{\mathrm{ac}}(\mathbb{R}^d) \end{array}$$

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- JKO time discrete scheme: for $\tau > 0$, [Jordan-Kinderlehrer-Otto '98]

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$\longrightarrow$ Many applications: porous medium equation, cell movement via chemotaxis, generalization to higher order PDEs, etc.

Displacement Convex Setting

$$\min_{\sigma \in \mathcal{P}(\mathbb{R}^d)} \frac{1}{2\tau} \mathrm{W}_2^2(\mu, \sigma) + \mathcal{E}(\sigma) + \mathcal{U}(\sigma)$$

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$\longrightarrow$ Using concavity of $\det^{1/d}$ over SDP, this gives:

Theorem: Given $\mu \in \mathcal{P}_{\mathrm{ac}}^2(\mathbb{R}^2)$, for the functional $\phi \in \mathcal{K} \mapsto \mathcal{U}(\nabla\phi_{\#\mu})$ to be convex, it suffices that $r > 0 \mapsto r^d U(r^{-d})$ is convex non-increasing and $U(0) = 0$.

[McCann '94]

Discretizing the Space of Convex Functions

We replace $\mathcal{K}$ with $\mathcal{K}_X := \{\phi \text{ convex} \,; \nabla\phi \in X\}$

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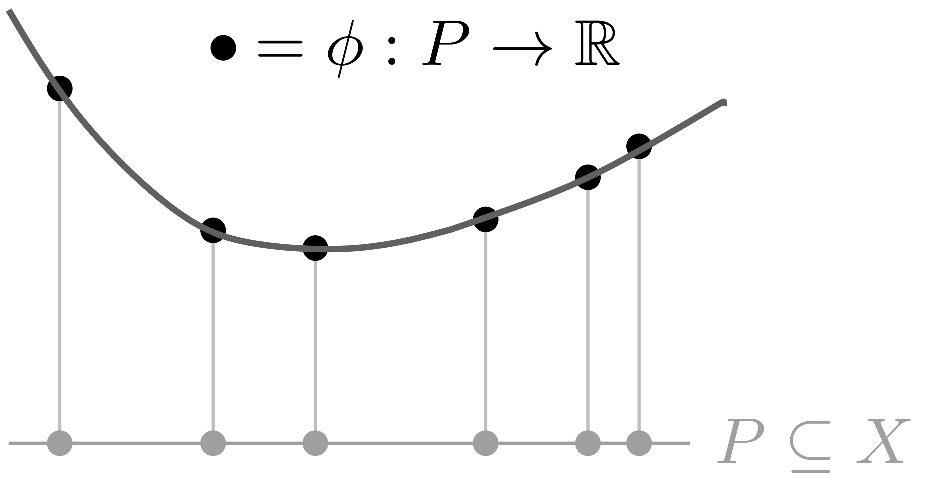

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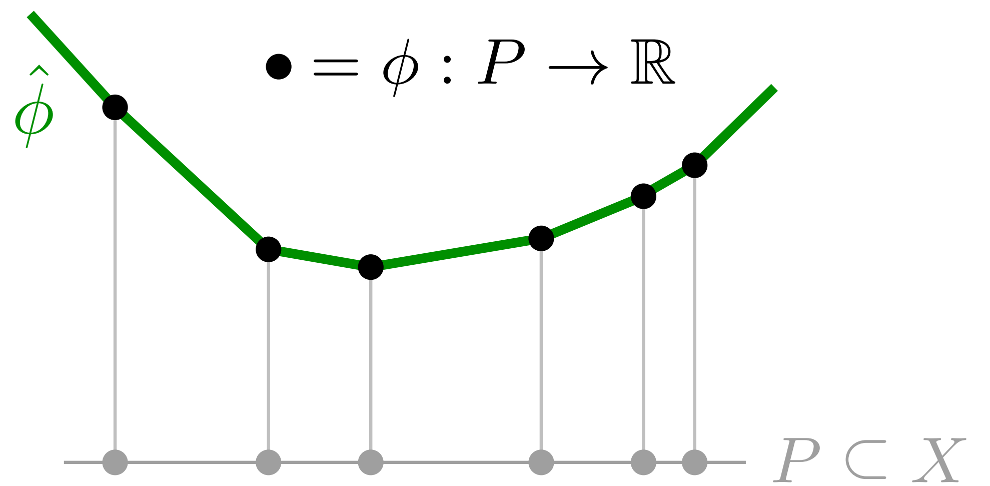
$$\hat{\phi} := \max\{\psi ; \psi \in \mathcal{K}_X \text{ and } \psi|_P \leq \phi\} \in \mathcal{K}_X$$



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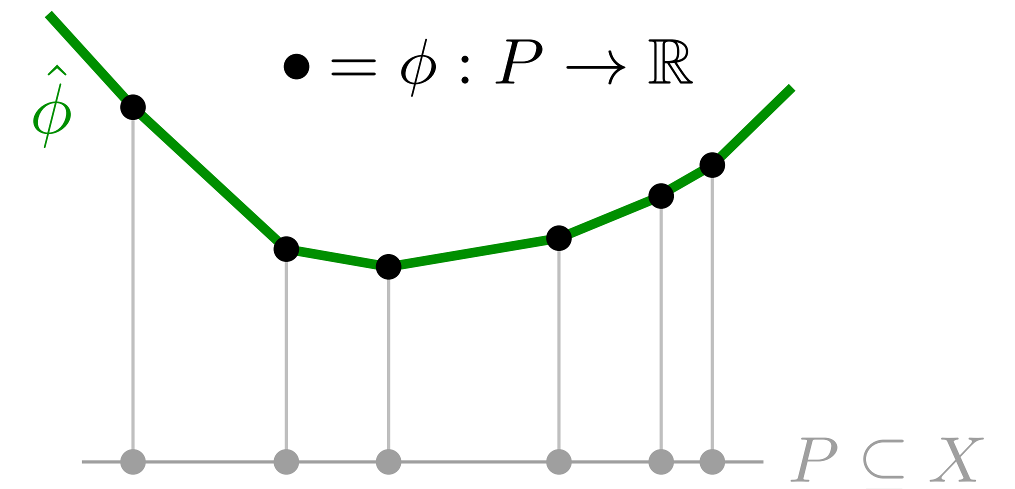

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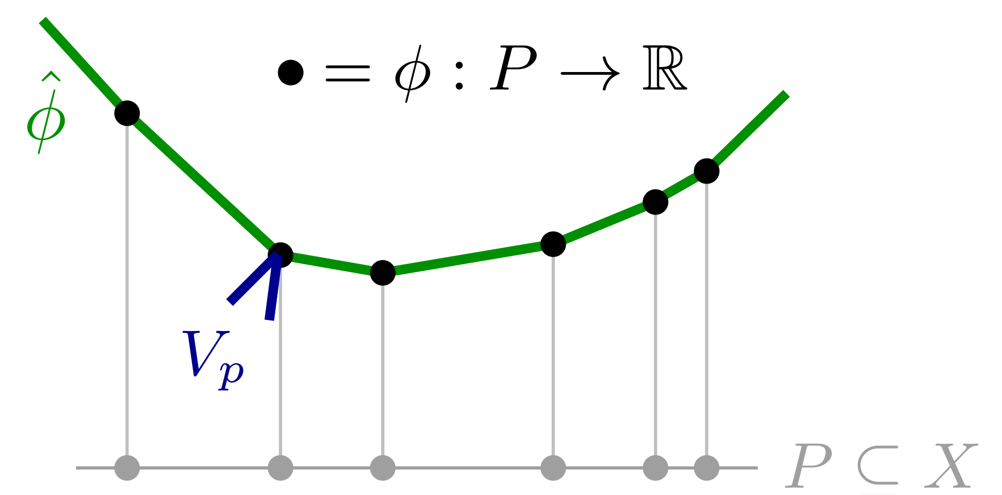

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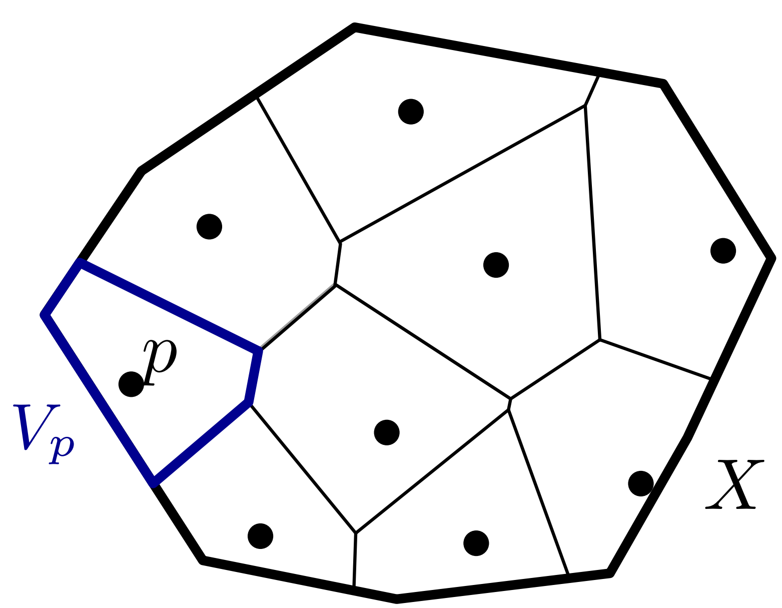

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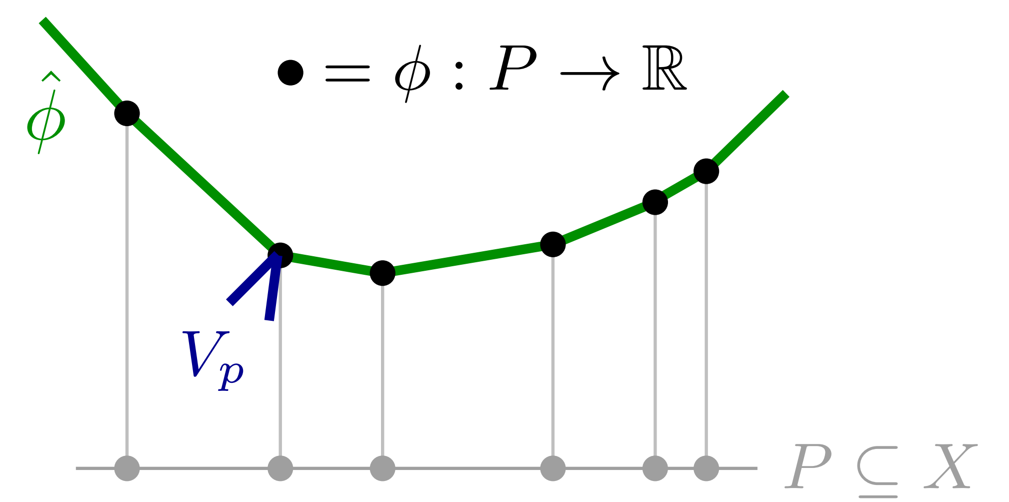

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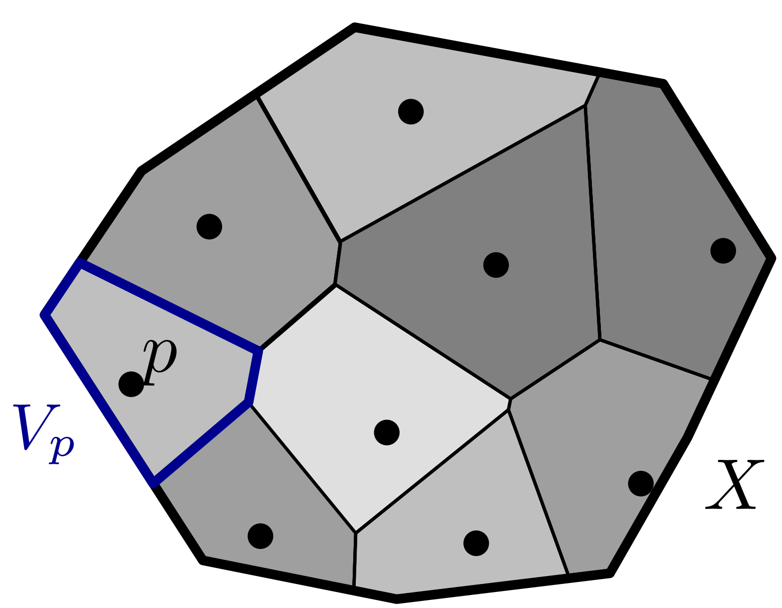

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Proposition: Under McCann's assumption, $\phi \in \mathcal{K}_X(P) \mapsto \mathcal{U}(G\phi_{\#}\mu_P)$ is convex.

Convex Space-Discretization of One Gradient Step

Theorem: Under McCann's hypotheses the discrete problem

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[Benamou, Carlier, M., Oudet '14]

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- Unfortunately, two different definitions of push-forward seem necessary to get a convex discretisation (i.e. $\phi \mapsto \mathcal{E}(G\phi_\#\mu)$ is non-convex).
- If $\lim_{r\to\infty} U(r)/r = +\infty$, the internal energy $\mathcal{U}$ is a barrier for convexity, i.e.

$$\begin{aligned}\mathcal{U}(G\phi_\#\mu_P) < +\infty &\Longrightarrow \forall p \in P, \mathrm{MA}[\phi](p) = \mathrm{Leb}(\partial\hat{\phi}(p)) > 0\\ &\Longrightarrow \phi \text{ is in the interior of } \mathcal{K}_X(P).\end{aligned}$$

NB: $|P|$ non-linear constraints vs $|P|^2$ linear constraints

Convergence of the Space-Discretization

Setting: X bounded and convex, $\mu \in \mathcal{P}_{\mathrm{ac}}(X)$ with density $c^{-1} \leq \rho \leq c$

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- Proof relies on Caffarelli's regularity theorem.
- When P_n is a regular grid, there is an alternative (and quantitative) argument

3. Numerical results

Monge-Ampère operator and Computational Geometry

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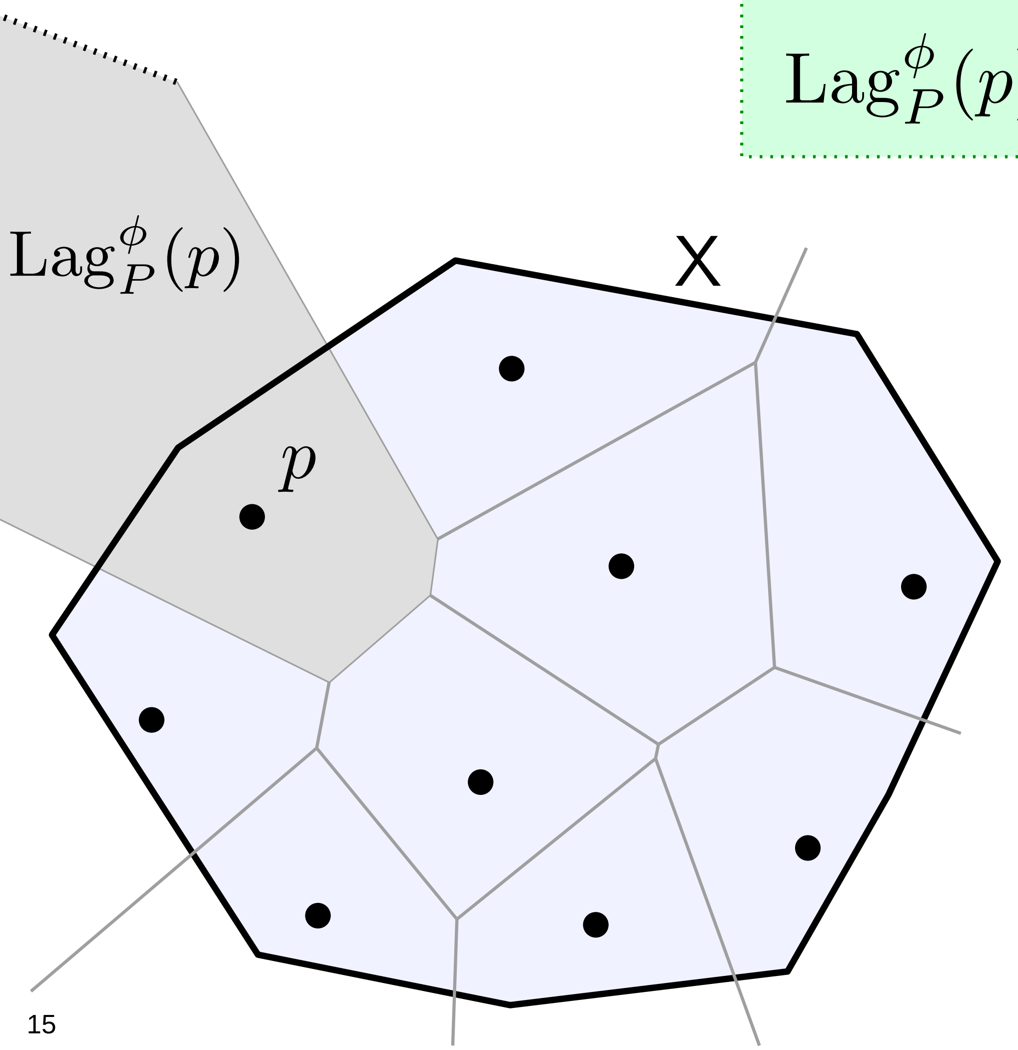

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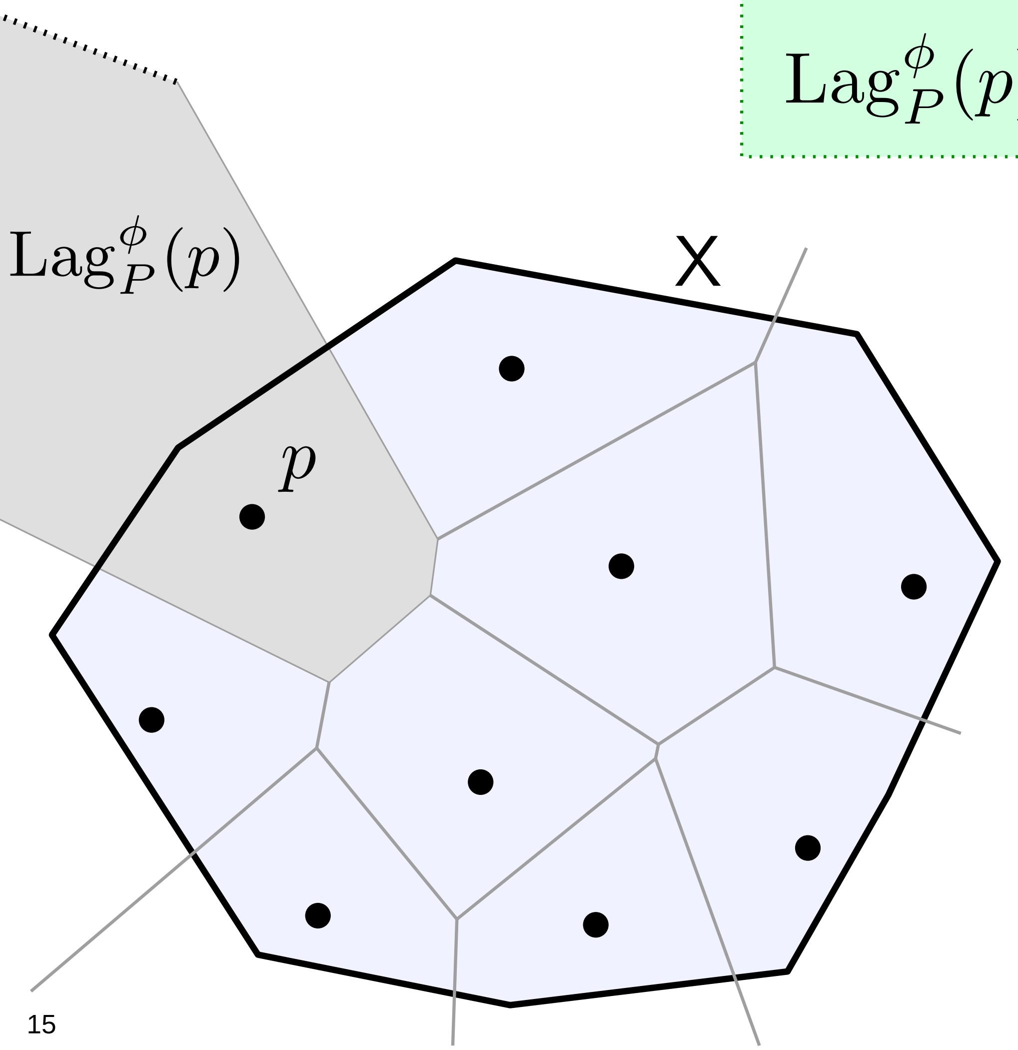

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Monge-Ampère operator and Computational Geometry

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with $\mathrm{MA}[\phi](p) := \mathrm{Leb}(\partial\hat{\phi}(p))$.

- **Goal:** fast computation of $\mathrm{MA}[\phi]$ and its 1st/2nd derivatives w.r.t ϕ
- We rely on the notion of **Laguerre diagram** in computational geometry

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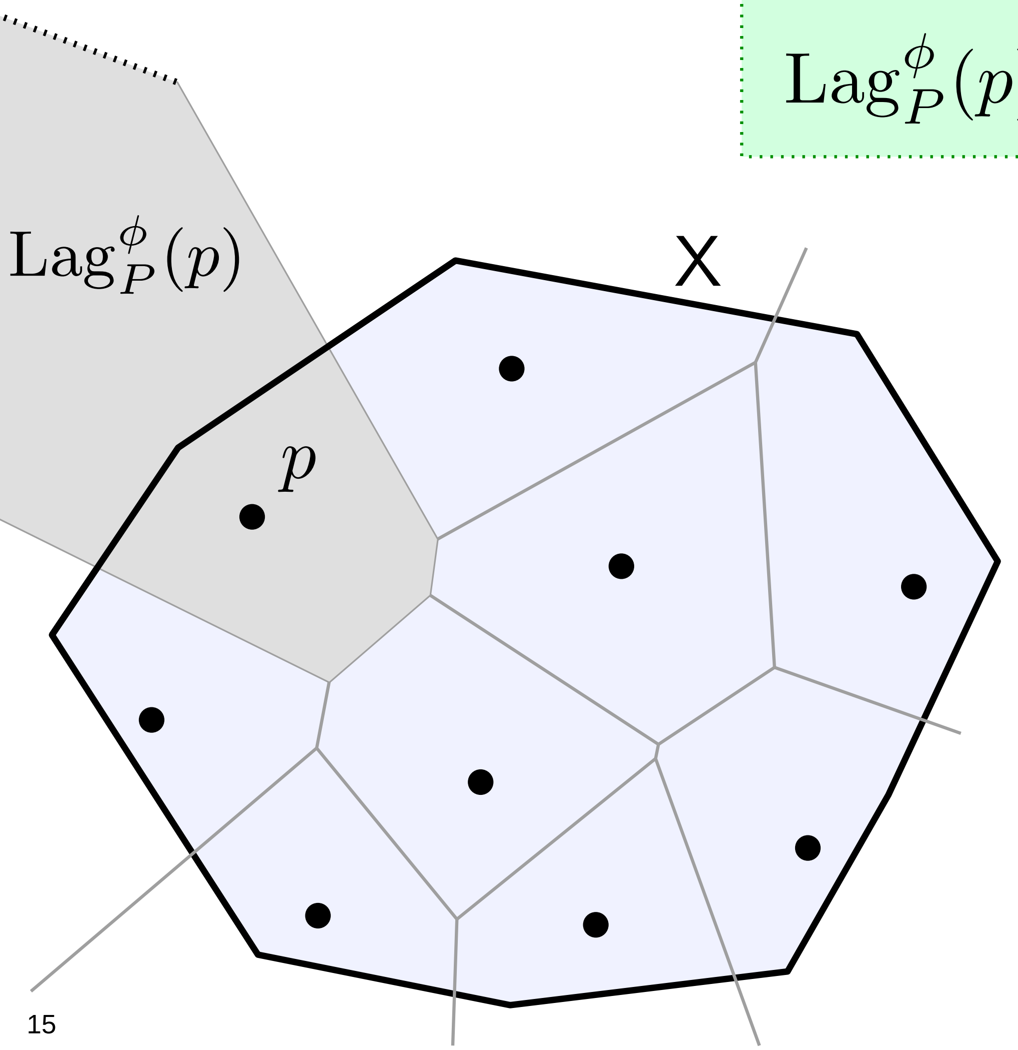

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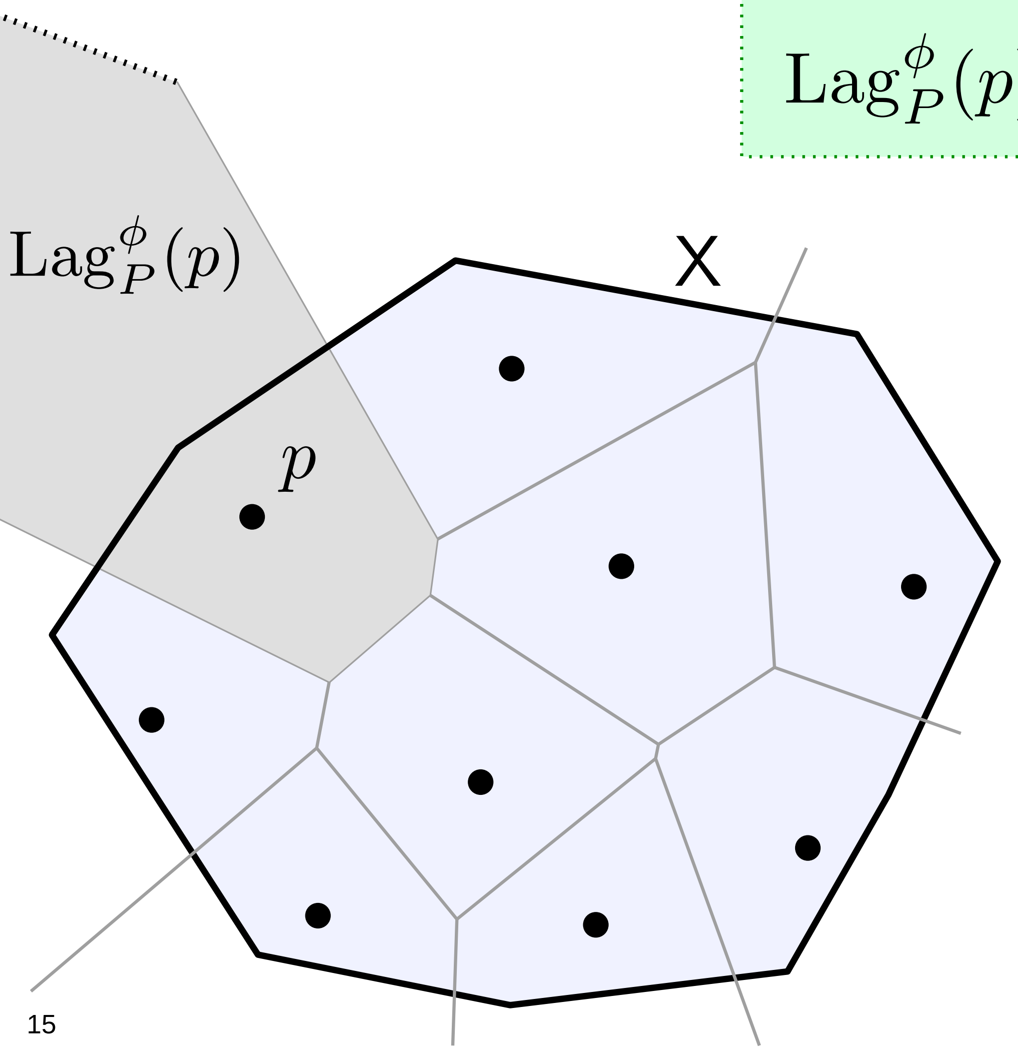

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$\longrightarrow$ Computation in time $O(|P| \log |P|)$ in 2D

Example: semi-Lagrangian scheme for crowd motion

- Gradient flow model of crowd motion with congestion, with a JKO scheme: [Maury-Roudneff-Chupin-Santambrogio 10]

$$\mu_{k+1} = \min_{\nu \in \mathcal{P}(X)} \frac{1}{2\tau} \mathrm{W}_2^2(\mu_k, \nu) + \mathcal{E}(\nu) + \mathcal{U}(\nu)$$

$$\mathcal{E}(\nu) := \int_X V(x) \,\mathrm{d}\,\nu(x)$$

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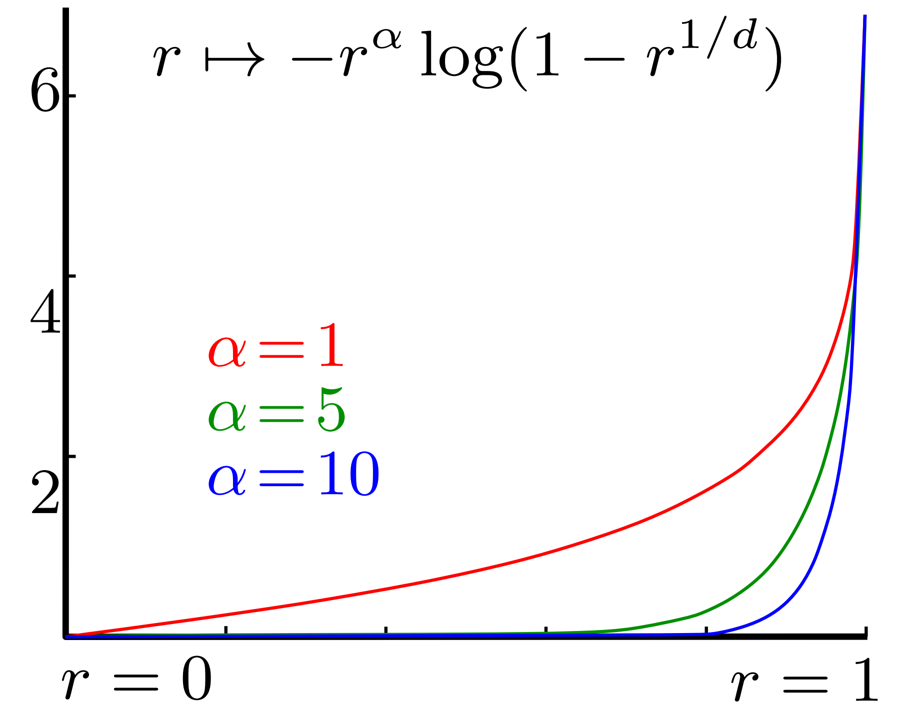
- Convex optimization problem if V is λ-convex ($V + \lambda\|.\|^2$ convex) and $\tau \leq \lambda/2$.

  We solve this problem with a relaxed hard congestion term:

$$\mathcal{U}_\alpha(\rho) := -\int \rho(x)^\alpha \log(1 - \rho(x)^{1/d}) \,\mathrm{d}\,x$$



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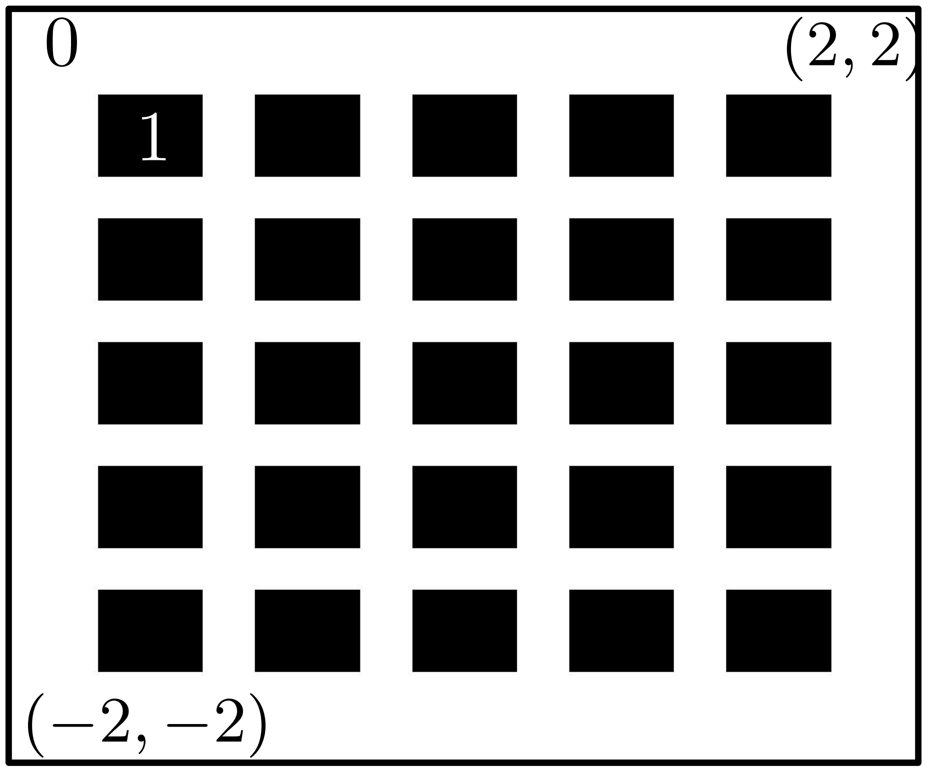
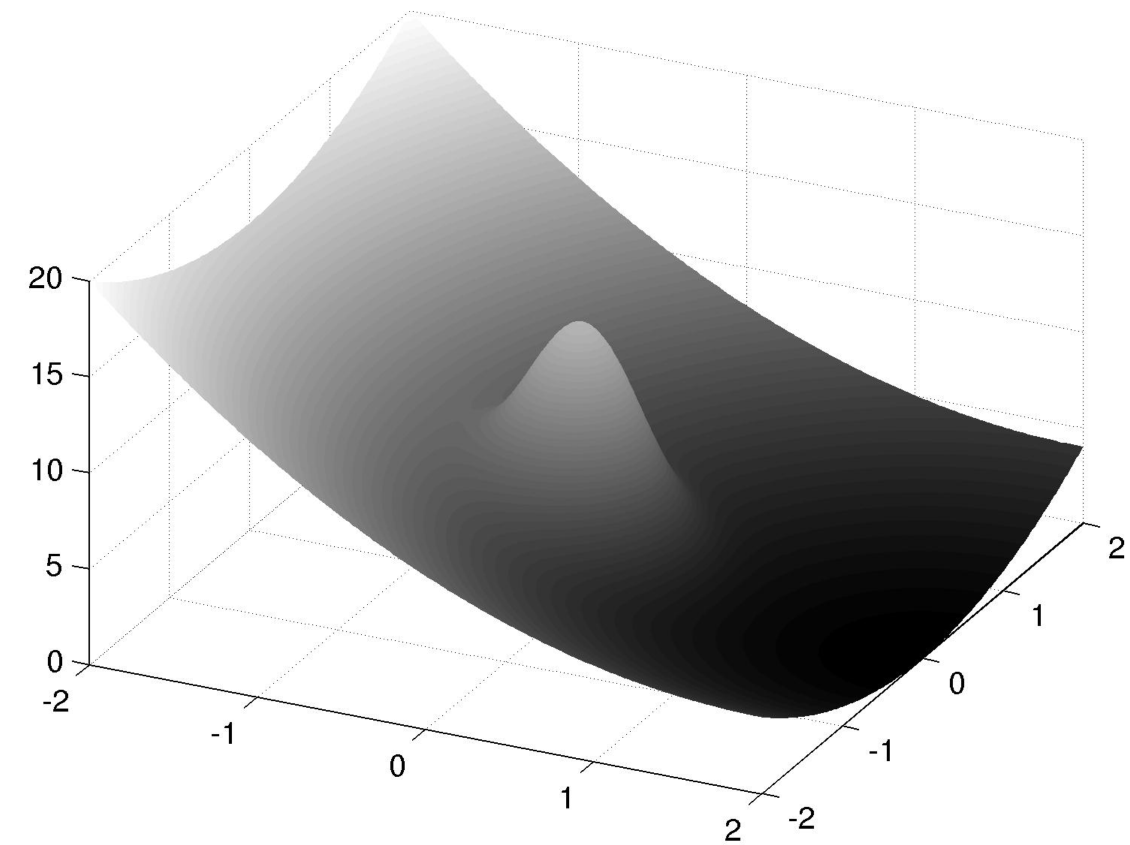
Initial density on $X = [-2, 2]^2$



$P = 200 \times 200$ regular grid.

Potential



$V(x) = \|x - (2,0)\|^2 + 5\exp(-5\|x\|^2/2)$

Algorithm: Input: $\mu_0 \in \mathcal{P}(P), \tau > 0$, $\alpha > 0$, $\beta \geq 1$.

For $k \in \{0, ..., T\}$

$$\phi \leftarrow \arg\min_{\phi \in \mathcal{K}_X^G(P_k)} \frac{1}{2\tau} \mathrm{W}_2^2(\mu_k, H\phi_\# \mu_k) + \mathcal{E}(H\phi_\# \mu_k) + \alpha \mathcal{U}_\beta(G\phi_\# \mu_k)$$

$\nu \leftarrow G_{\phi\#}\mu_k$; $\mu_{k+1} \leftarrow$ projection of $\nu|_{[-2,2)\times[-2,2]}$ on P.

4. Application to Geometry?

Ongoing work with J.M. Mirebeau

Support Functions of Convex Bodies

Objective: Logarithmic barrier for the space of support functions of convex bodies.

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$\longrightarrow$ interior point method for shape optimization problem: Minkowski, Meissner, etc.

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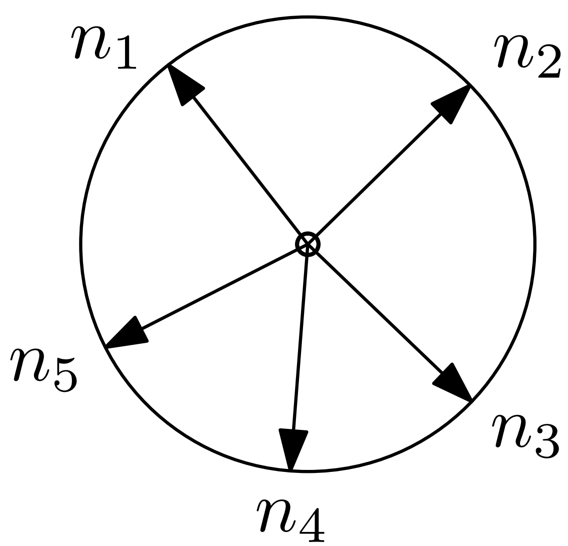

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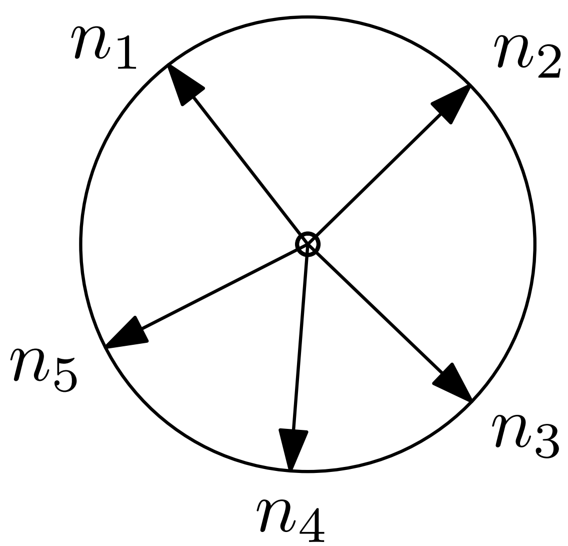

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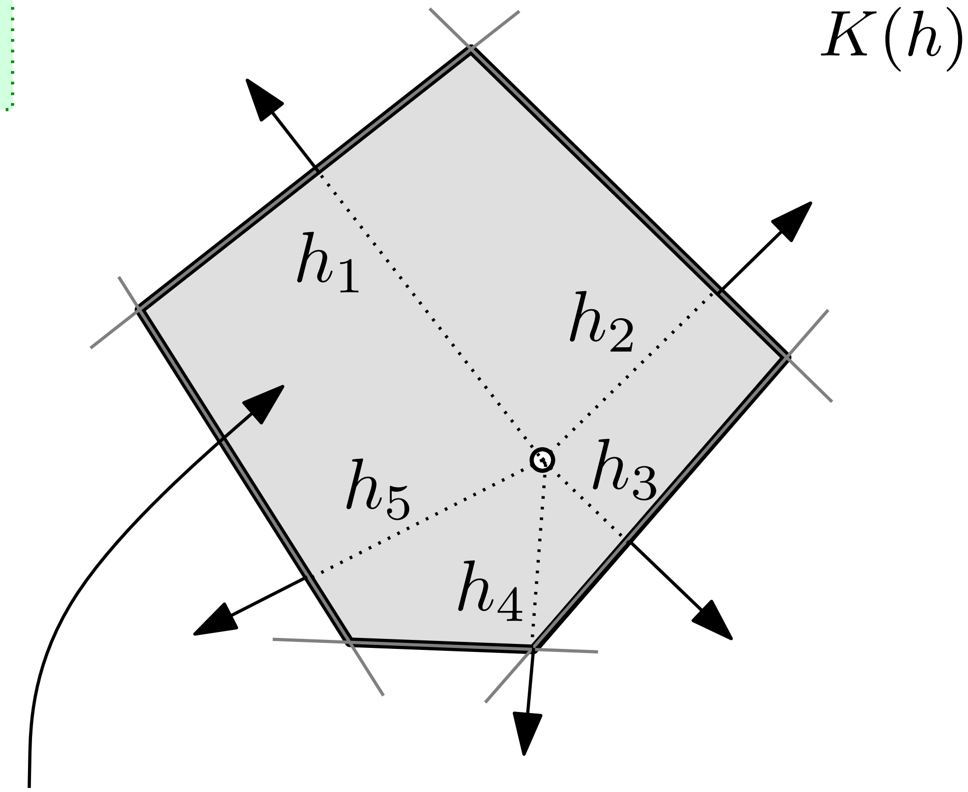

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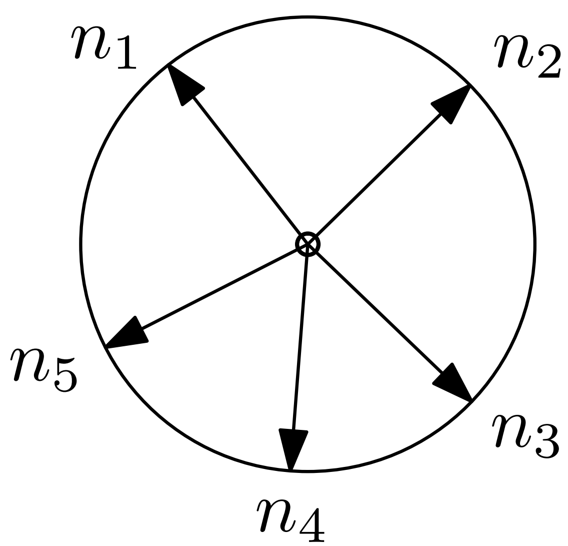

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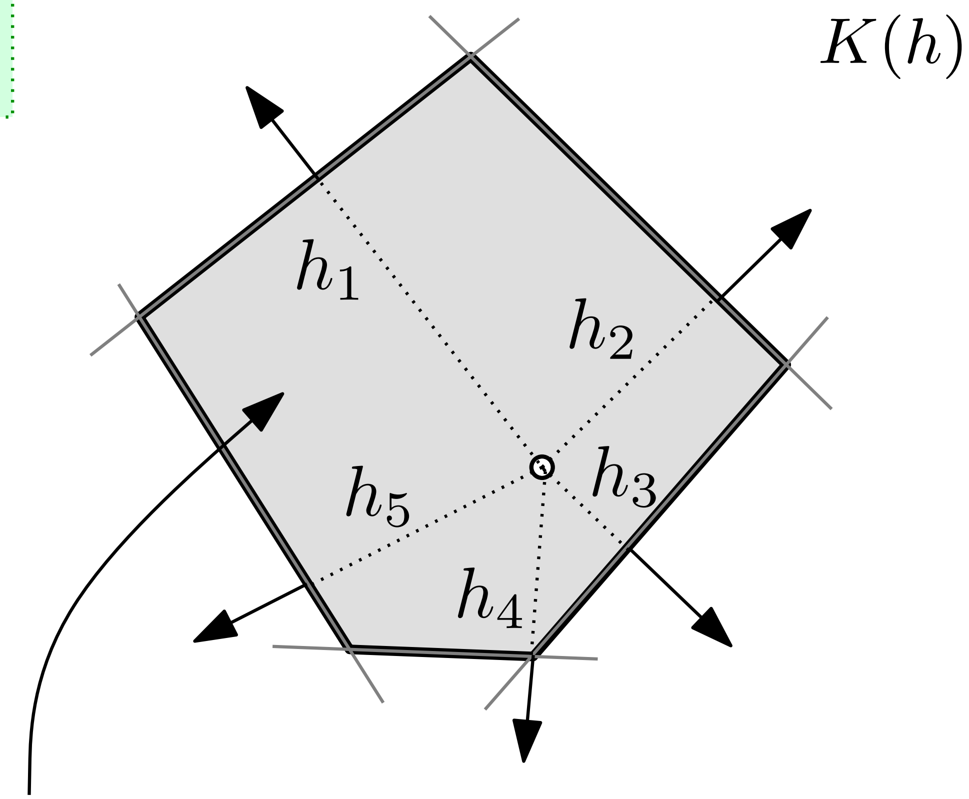

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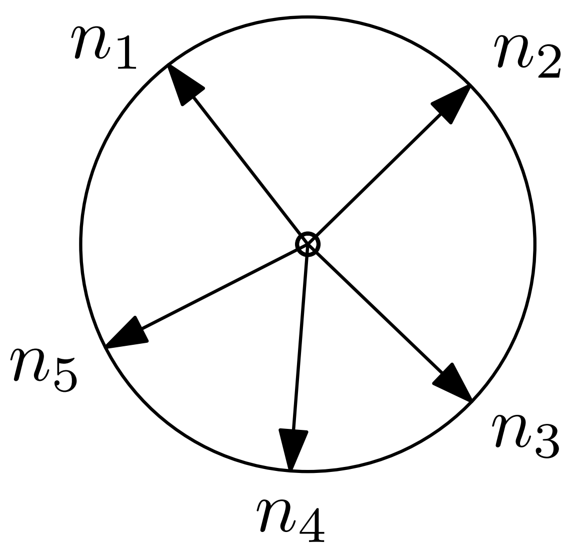

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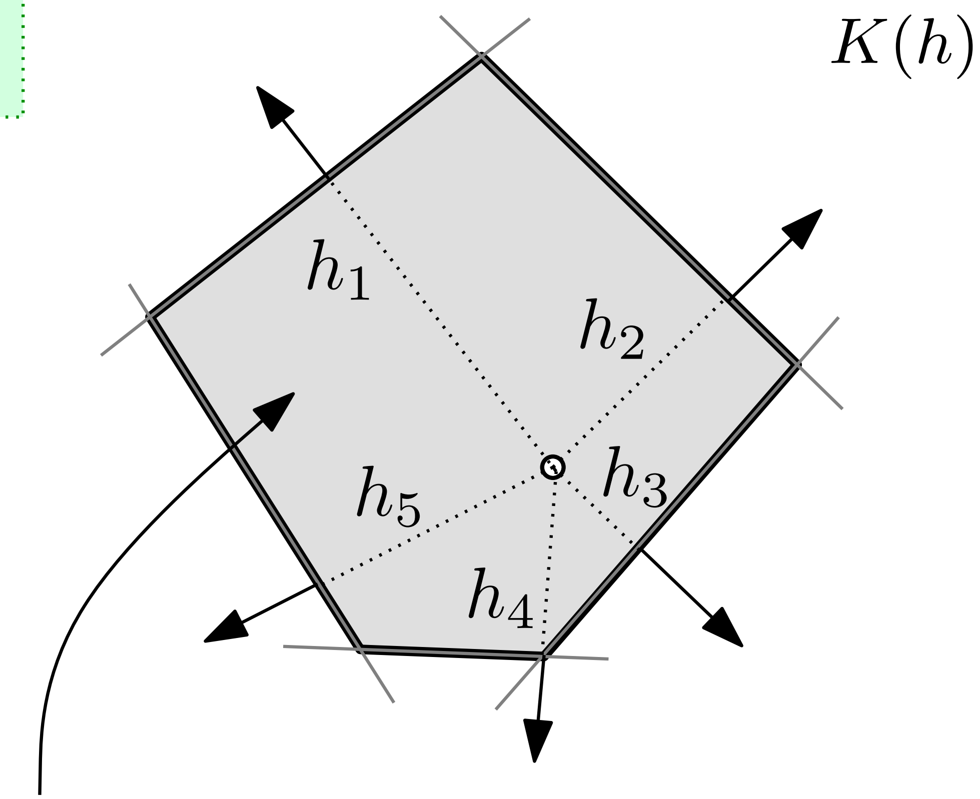

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Extension to other spaces, such as c-convex functions in optimal transport.

Application: convex bodies with constant-width

Definition: The **width** of K in direction u is

$$w_K(u) = h_K(u) + h_K(-u).$$

K has constant width α if $w_K(u) = \alpha$.

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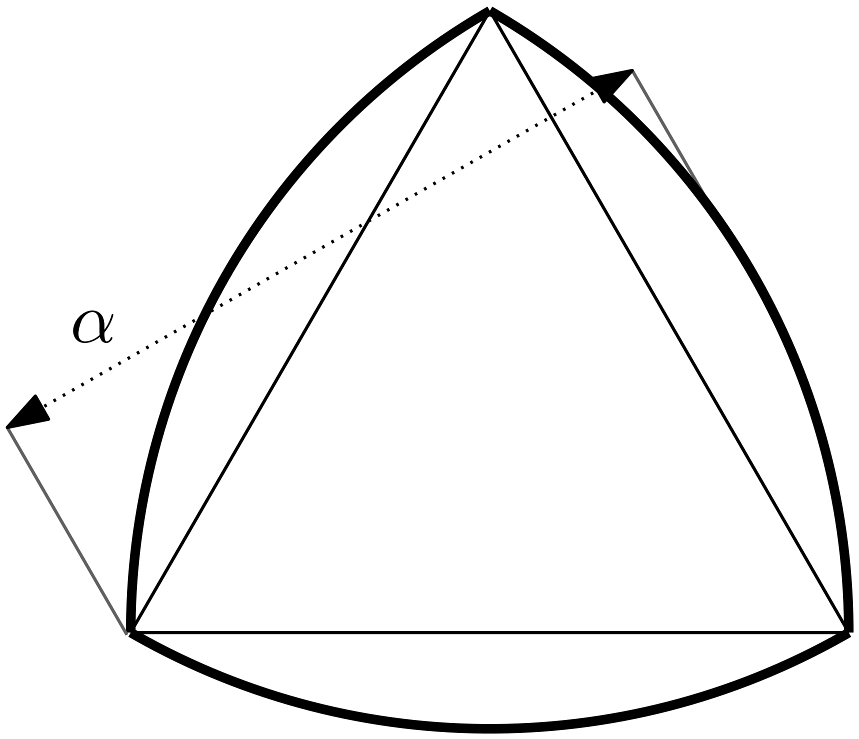

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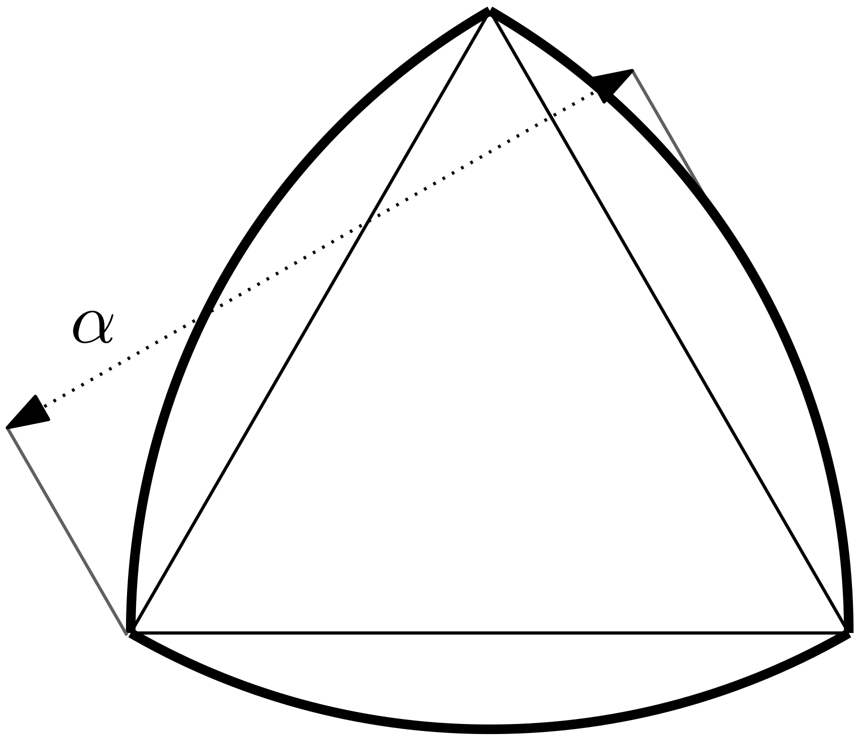

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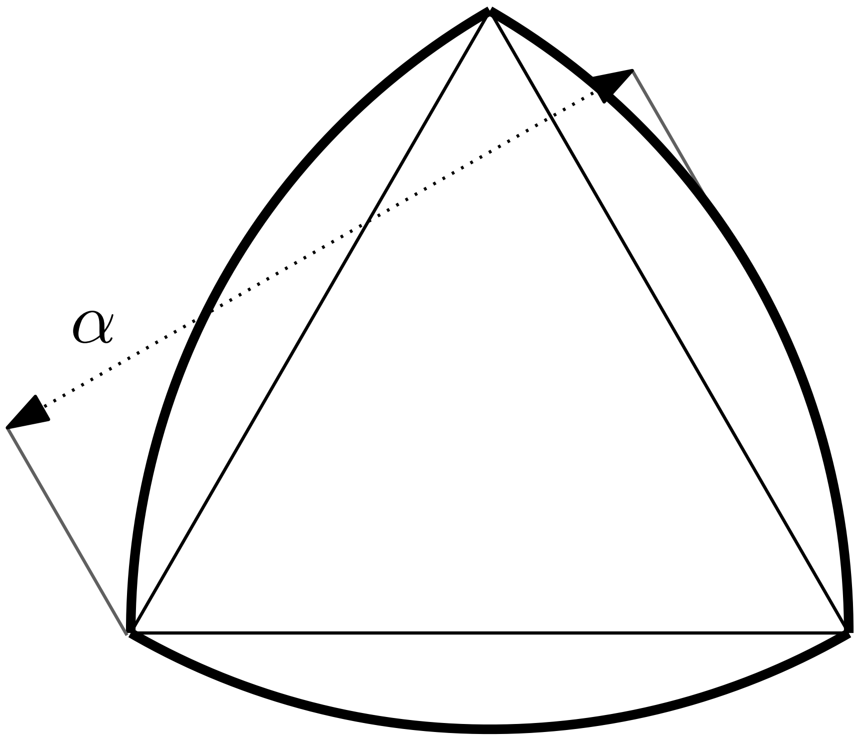

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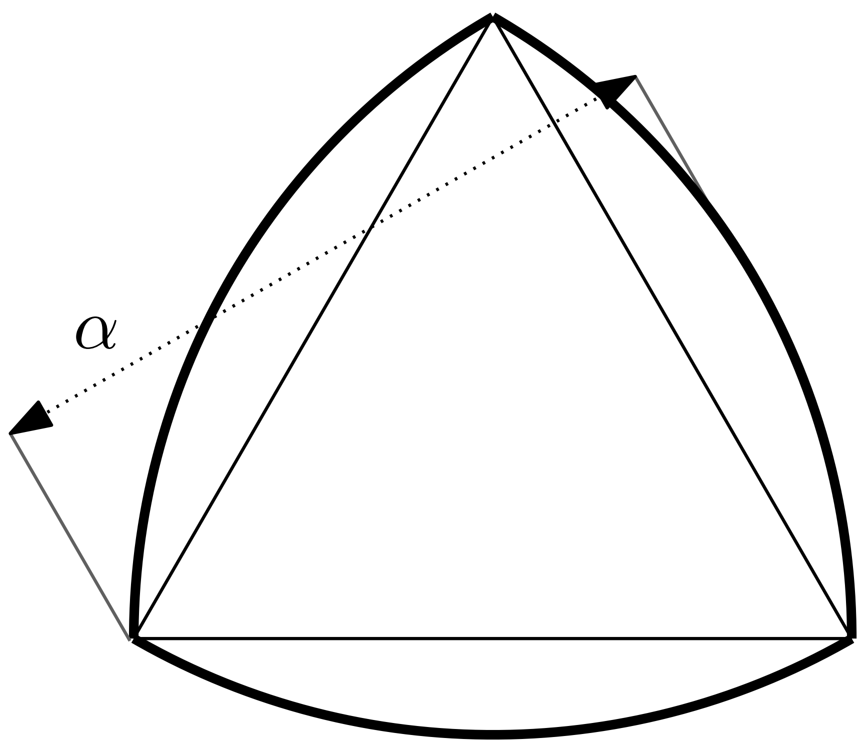

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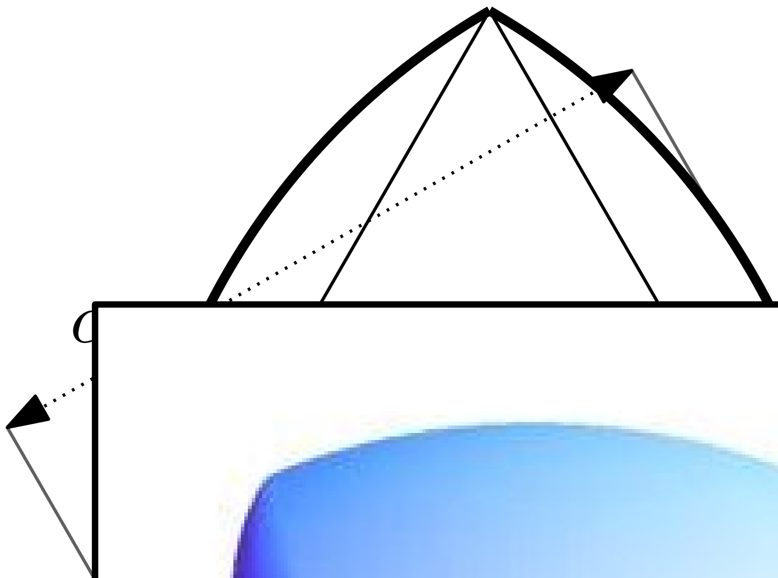

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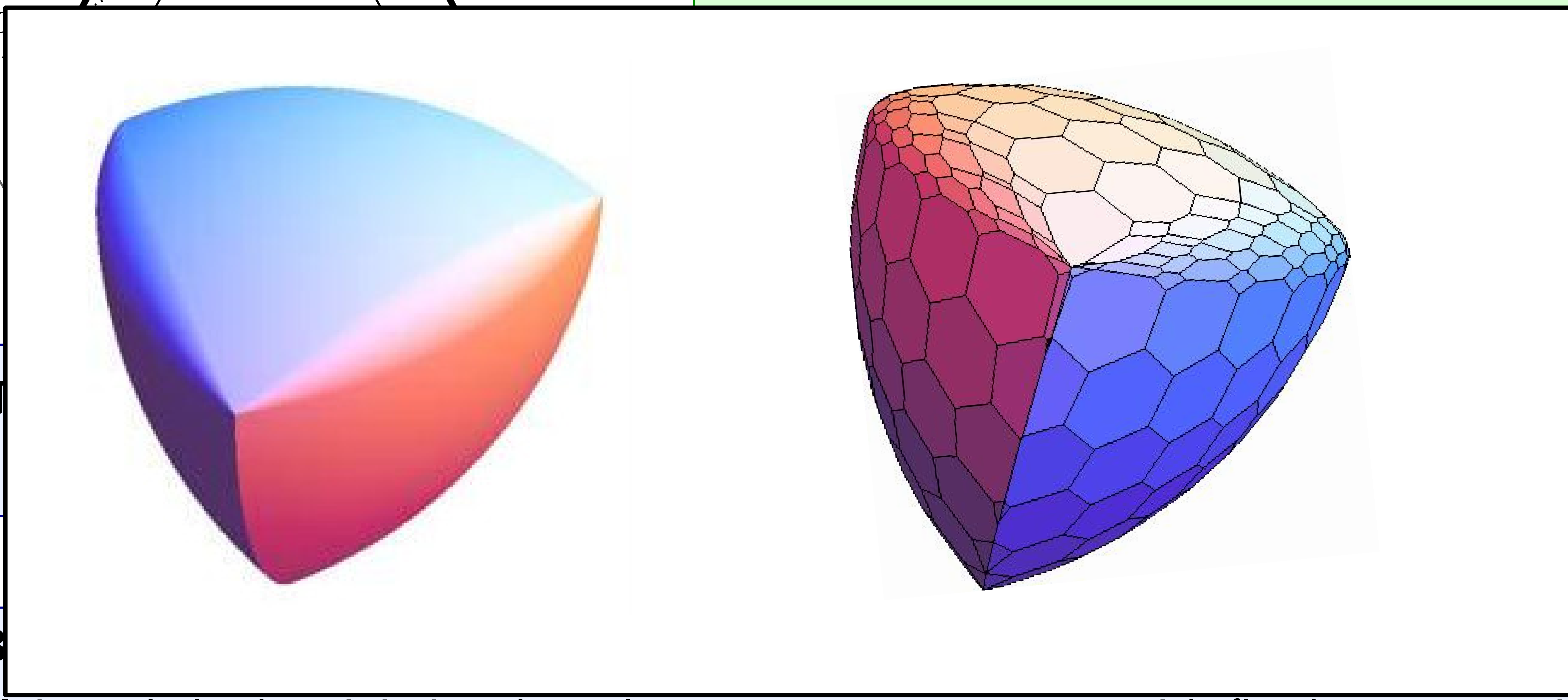

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Appendix

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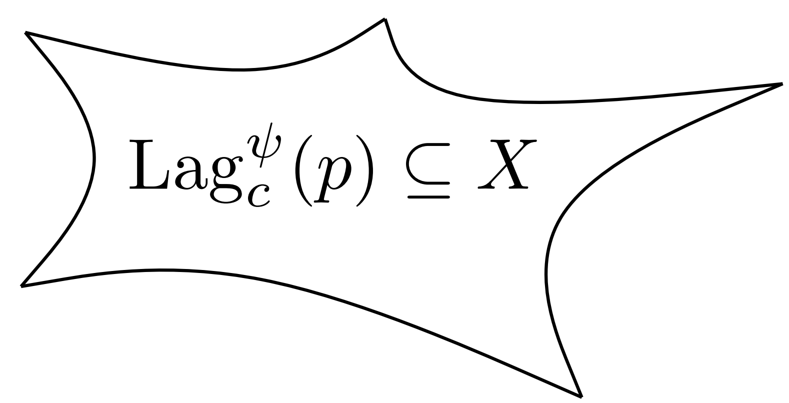

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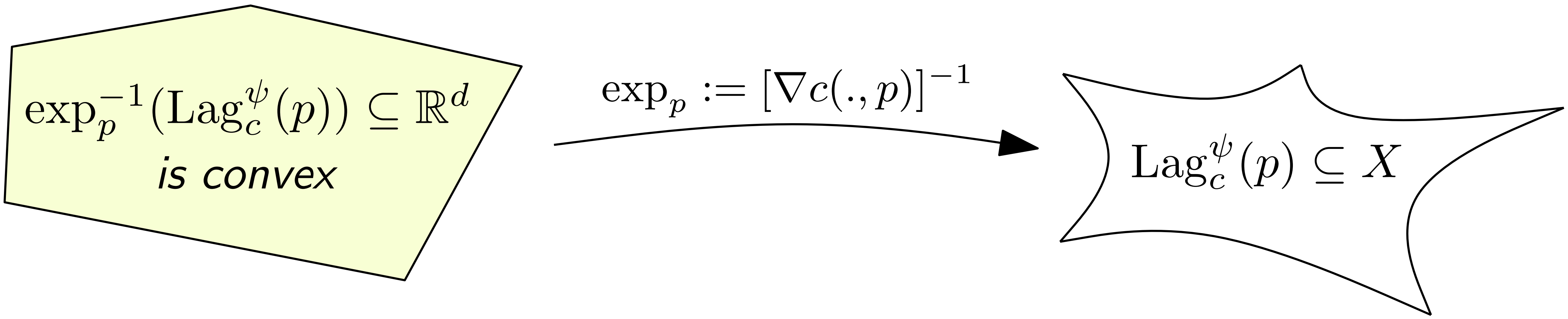

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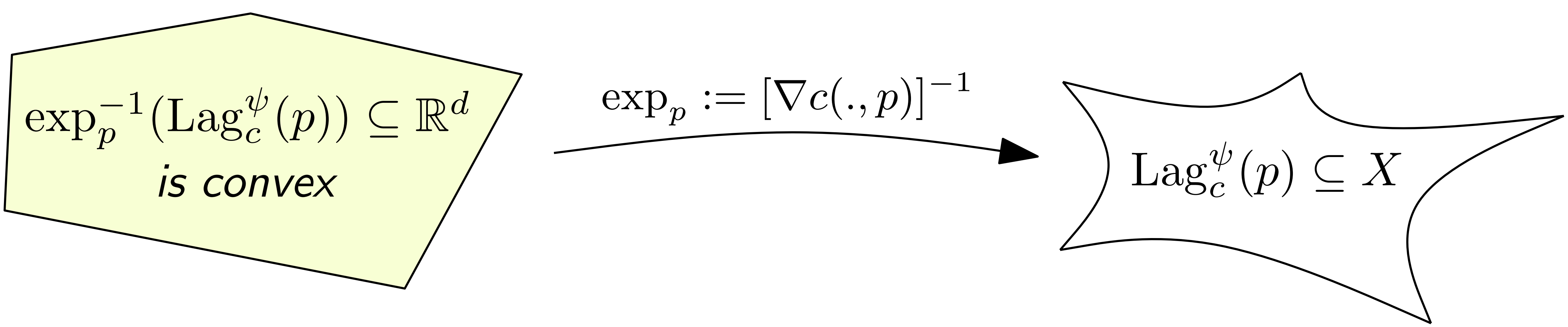

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