On the minima of the functional Mahler product

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However,asymptotic result exsits there exists a constant $c > 0$, independant of n, such that $P(K) \geq c^n P(B_2^n)$. They wherre given by Bourgain and Milman, Kuperberg, Nazarov), using of normed space theory, differential geometry and PDE.

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I mention a few results about the known exact lower bounds for $P(K)$ (K convex body or ϕ convex function). There are two cases.

a) K is centrally symmetric (ϕ is even)

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This was proved by Mahler (1939) for $n = 2$, but is still unsolved for $n \geq 3$. In some special cases, it was proved in any dimension : eg. when K is symmetric with respect to n pairwise orthogonal hyperplanes (Saint-Raymond) or when K is a zonoid (or K^* the unit ball of a subspace of L^1 (Reisner, and simpler proof by Gordon - Meyer - Reisner) or when the body has a speciel set of symmetries (Barthe-Fradelizi). The problem is made difficult by the fact that if the conjecture is true, there are many non linearly isomorphic cases of inequality. Acctually if K, L iare convex bodies in $\mathbb{R}^n$ and $\mathbb{R}^m$, then

$$P(K \times L) = P(K)P(L) / \binom{m+n}{n}$$

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The corresponding conjecture for functions $P(\phi) \geq 4^n$. These two conjectures are equivalent but with a gap of dimension. For dimension 1, there is of course no difficulty for K, but the answer for functions need at least two pages (Fradelizi-Meyer). In higher dimensions, the result is known for function with enough symmetries (Fradelizi-M).

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b) The general case : conjecture $P(K) \geq P(\Delta_n)$, where Δ_n is a simplex, and the corresponding conjecture to functions is $P(\phi) \geq e^n$. For bodies, this is known respectively for $n = 2$ (Mahler 1938 and Meyer for the case of equality, for functions for $n = 1$ (Fradelizi - Meyer).

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b) The general case : conjecture $P(K) \geq P(\Delta_n)$, where Δ_n is a simplex, and the corresponding conjecture to functions is $P(\phi) \geq e^n$. For bodies, this is known respectively for $n = 2$ (Mahler 1938 and Meyer for the case of equality, for functions for $n = 1$ (Fradelizi - Meyer). Observe that the general case is simpler, in the sense that the conjectured case of equality is affinely unique, but also is some what more difficult than the symmetric case in the sense that the point z where $\min |L^z|$ is reached, known as the Santaló point of L has only an implicit characterization : it is the unique point z which is the center of mass of L^z.

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- Adding a segment was used to treat the case of zonoids (Gordon-Meyer-Reisner).

- Moving the body in some direction (by the so-called shadow movements) was used by Meyer-Reisner to prove that polytopes in $\mathbb{R}^n$ with not more than $n+3$ vertices satisfy the conjecture (and also is generally used, through Steiner symmetrizations, to give the upper-bound (Blaschke-Santaló inequality).

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- Intersecting with halfspaces and taking the convex hull with some point was done by Reisner-Schütt-Werner to prove that bodies with minimal volume product cannot be too regular, which confirms the conjecture that they are polytopes.

Theorem 1

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$$\langle x - a, u \rangle = f_a(x_2 - a_2, \dots, x_n - a_n),$$

for some convex function f_a on a neighborhood of 0 in $\mathbb{R}^{n-1}$, where $(x_k - a_k)_{2 \leq k \leq n}$ are the coordinates of $x - a - \langle x - a, u \rangle u$ in an orthonormal basis of $u^{\perp} = \{z \in \mathbb{R}^n; \langle x, u \rangle = 0\}$.

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Theorem 1 ([RSW]) Suppose that $K_0 \subset \mathbb{R}^n$, $n \geq 2$, has positive Gauss curvature at some $a \in \partial K$, then K_0 is not a local minimum of $K \to P(K)$ among all convex bodies, and if K is centrally symmetric, among centrally symmetric convex bodies.

Theorems 2 and 3

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Definition Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We say that *$\phi$ is strictly C_2 at $x_0 \in \mathbb{R}^n$* if $\phi(x_0) < +\infty$, ϕ is differentiable at x_0 and for some *positive definite* symmetric $[n \times n]$ matrix A, one has :

$$\phi(x) = \phi(x_0) + \langle \nabla\phi(x_0), x - x_0 \rangle + \frac{1}{2}\langle x - x_0, A(x - x_0)\rangle + o(|x - x_0|^2).$$

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If $y_0 = \nabla\phi(x_0)$, it is easily shown that then $\mathcal{L}\phi$ satisfies

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Thus $\mathcal{L}\phi$ is strictly C_2 at $y_0 = \nabla\phi(x_0)$ and $\nabla(\mathcal{L}\phi)(y_0) = x_0$.

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Theorem 2 Let $\mathcal{F}$ be the cone all convex functions $\theta : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\int e^{-\theta(x)} dx < +\infty$ and $\int e^{-\mathcal{L}\theta(y)} dy < +\infty$. If $\phi \in \mathcal{F}$ is strictly C_2 at some x_0, then

$$\int_{\mathbb{R}^n} e^{-\phi(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\phi(y)} dy > \min_{\theta \in \mathcal{F}} \Big(\int_{\mathbb{R}^n} e^{-\theta(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\theta(y)} dy \Big).$$

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The case of even convex functions is treated by the following :

Theorem 3 Let $\mathcal{F}_e$ be the cone all convex even functions $\theta : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\int e^{-\theta(x)} dx < +\infty$ and $\int e^{-\mathcal{L}\theta(y)} dy < +\infty$. If $\phi \in \mathcal{F}_e$ is strictly C_2 at some x_0, then

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1) K has a unique tangent hyperplane at a and if Q_a is the positive definite quadratic form associated to K at a, there is a unique ellipsoid $\mathcal{E}$ centered at 0 tangent at a to K, such that the quadratic form associated to $\mathcal{E}$ at a is also Q_a. This is the *osculating ellipsoid $\mathcal{E}$ to K at a centered at 0* .

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2) There exists a unique $a^* \in \partial K^*$ such that $\langle a^*, a\rangle = 1$, K^* has positive Gauss curvature at a^* and $\mathcal{E}^*$ is the osculating ellipsoid to K^* at a^* centered at 0.

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A) Suppose first that 0 is the center of symmetry of K.Then $P(K) = |K||K^*|$. Define for $0 < c < 1$,

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$$\frac{|K| - |K_c|}{|\mathcal{E}| - |\mathcal{E}_c|} \to 1 \quad \text{and} \quad \frac{|(K_c)^*| - |K^*|}{|(\mathcal{E}_c)^*| - |\mathcal{E}^*|} \to 1.$$

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$$\frac{|K| - |K_c|}{|\mathcal{E}| - |\mathcal{E}_c|} \to 1 \quad \text{and} \quad \frac{|(K_c)^*| - |K^*|}{|(\mathcal{E}_c)^*| - |\mathcal{E}^*|} \to 1.$$

To compute $|\mathcal{E}| - |\mathcal{E}_c|$ and $|(\mathcal{E}_c)^*| - |\mathcal{E}^*|$, we may suppose that $\mathcal{E} = rB_2^n$, $r > 0$ (and $\mathcal{E}^* = \frac{B_2^n}{r}$). Setting $1 - c = \cos t$, $t > 0$, $t \to 0$, an easy computation gives, with $v_n = |B_2^n|$,

$$|\mathcal{E}| - |\mathcal{E}_c| \sim 2r^n v_{n-1} \frac{t^{n+1}}{n+1}\ .$$

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Thus

$$|K_c|\ |(K_c)^*| - |K||K^*| = 2v_{n-1} \frac{t^{n+1}}{n+1} \big(\frac{|K|}{nr^n} - r^n |K^*| \big) + o(t^{n+1}).$$

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$$|(\mathcal{E}_c)^*| - |\mathcal{E}^*| \sim \frac{2v_{n-1}}{r^n}\frac{t^{n+1}}{n(n+1)}.$$

Thus

$$|K_c|\ |(K_c)^*| - |K||K^*| = 2v_{n-1}\frac{t^{n+1}}{n+1}\Big(\frac{|K|}{nr^n} - r^n|K^*|\Big) + o(t^{n+1}).$$

Similarly, beginning with K^* instead of $K = (K^*)^*$, we get

$$|(K^*)_c|\ |\big((K^*)_c\big)^*| - |K^*||K| = 2v_{n-1}\frac{t^{n+1}}{n+1}\Big(\frac{r^n|K^*|}{n} - \frac{|K|}{r^n}\Big) + o(t^{n+1}).$$

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Suppose that K, and thus K^*, are local minima for $L \to P(L)$ among centrally symmetric L. Then $P(K_c) - P(K) \geq 0$ and $P\big((K^*)_c\big) - P(K^*) = P\big((K^*)_c\big) - P(K) \geq 0.$

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Remark. For general K, it is not more complicated. Define $K_c = \{x \in K; \langle x, a^* \rangle \leq 1 - c\}$ and similarly $(K^*)_c$. If $P(K)$ was minimal, one would have $P(K) \leq \min\Big(P(K_c), P\big((K^*)_c\big)\Big) \leq \min\Big(|K_c||(K_c)^*|, |(K^*)_c||\big((K^*)_c\big)^*|\Big)$. With the same computation as above, we prove that it is absurd. $\square$

Proof of theorems 2 and 3

1) A variational argument

For $t \geq 0$ and $x \in \mathbb{R}^n$, let $h_t(x) = \phi(x_0) + t + \langle \nabla\phi(x_0), x - x_0 \rangle$,

$$\phi_t = \max(\phi, h_t),\ \psi_t = \mathcal{L}\phi_t \text{ and } W_t = \{(y,s) \in \mathbb{R}^n \times \mathbb{R}; s \geq \psi_t(y)\}.$$

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Then W_t is the convex hull in $\mathbb{R}^n \times \mathbb{R}$ of W_0 and the point $\big(y_0, \psi(y_0) - t\big)$. For $t \geq 0$, let

$$C_t = \int_{\mathbb{R}^n} e^{-\phi_t(x)} dx \leq C_0 \text{ and } D_t = \int_{\mathbb{R}^n} e^{-\psi_t(y)} dy \geq D_0.$$

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(we increase ϕ so we decrease $\mathcal{L}\phi$)

2) An estimate for $C_0 - C_t$.

By approximation, we may restrict to the case where for all $x \in \mathbb{R}^n$

$$\phi(x) = \phi(x_0) + \langle \nabla\phi(x_0), x - x_0\rangle + \frac{1}{2}\langle x - x_0, A(x - x_0)\rangle$$

One has

$$C_0 - C_t = \int_{\{\phi_t > \phi\}} e^{-\phi}\left(1 - e^{\phi - \phi_t}\right)dx$$

$$= \int_{\{\phi_t > \phi\}} e^{-\phi(x)}\left(1 - e^{-t + \frac{1}{2}\langle x - x_0, A(x - x_0)\rangle}\right)dx,$$

and

$$\{\phi_t > \phi\} = \{x \in \mathbb{R}^n; \frac{1}{2}\langle x - x_0, A(x - x_0)\rangle < t\}.$$

Let $x = x_0 + \sqrt{2t}A^{-1/2}z$ and

$$g_t(z) = \phi(x_0) + \langle \nabla\phi(x_0), \sqrt{2t}A^{-\frac{1}{2}}z\rangle + t^2\langle z, A^{-1}z\rangle,$$

Proof of theorems 2 and 3

We get

$$C_0 - C_t = \frac{(2t)^{\frac{n}{2}}}{\sqrt{\det(A)}} \int_{\{|z| \leq 1\}} e^{-g_t(z)} \left(1 - e^{-t(1-|z|^2)}\right) \, dz$$

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$$C_0 - C_t \sim t^{\frac{n}{2}+1} \frac{2^{\frac{n}{2}}}{\sqrt{\det(A)}} e^{-\phi(x_0)} \int_{|z|\leq 1} (1 - |z|^2) dz$$

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and finally $C_0 - C_t \sim \dfrac{v_n e^{-\phi(x_0)}}{n+2} \dfrac{(2t)^{\frac{n}{2}+1}}{\sqrt{\det(A)}}$.

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Thus $\{\psi_t < \psi\} = \{y \in \mathbb{R}^n; \langle y - y_0, B(y - y_0)\rangle < 2t\}$.

Proof of theorems 2 and 3

Setting $y = y_0 + \sqrt{2t}\sqrt{B^{-1}}w = y_0 + \sqrt{2t}\sqrt{A}w$ and $h_t(w) = (\psi - \psi_t)(y_0 + \sqrt{2t}\sqrt{A}w)$, we get

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Since ψ_t is affine on the segment $[y_0, y_0\sqrt{2t}\sqrt{A}\frac{w}{|w|}]$, one has

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We get $(\psi - \psi_t)(y_0 + \sqrt{2t}\sqrt{A}w) = t(1-|w|)^2$. and thus

$$\frac{D_t - D_0}{\sqrt{\det(A)}(2t)^{n/2}} = \int_{|w|\leq 1} e^{-(\psi(y_0)+\langle x_0, \sqrt{2t}\sqrt{A}w\rangle + t|w|^2)}\left(e^{(t(1-|w|)^2}-1\right)\, dw$$

$$\text{or }, \frac{D_t - D_0}{\sqrt{\det(A)}(2t)^{n/2}} \sim t\frac{v_n e^{-\psi(y_0)}}{(n+1)(n+2)} \text{ when } t \to 0 \, .$$

4) The product $C_0 D_0$ cannot be minimal in $\mathcal{F}$.

From the last inequalities, we get

$$C_t D_t - C_0 D_0 \sim \frac{v_n (2t)^{\frac{n}{2}+1}}{n+2} \Big(\frac{C_0 \sqrt{\det(A)} e^{-\psi(y_0)}}{n+1} - \frac{D_0 e^{-\phi(x_0)}}{\sqrt{\det(A)}} \Big).$$

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$$D_0 \det(A^{-1}) e^{-\phi(x_0)} \geq (n+1) C_0 e^{-\psi(y_0)}.$$

Multiplying these inequalities, we get $1 \geq (n+1)^2$. ABSURD. $\square$
A simple modification allows to treat the case of even ϕ.

Two remarks

1). In some orthonormal basis of $\mathbb{R}^{n-1}$, the quadradic form Q_a attached to a body K which has positive Gauss curvature at a can be writen

$$Q_a(y_2, \dots, y_n) = \frac{1}{2}\sum_{i=2}^{n} \frac{y_i^2}{R_i} \text{ with } R_i > 0..$$

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Then $\kappa_K(a) := \left(\prod_{i=2}^{n} R_i\right)^{-\frac{1}{n-1}}$ is the *generalized Gauss curvature of K at a* .. Define $r(a) > 0$ by $|\mathcal{E}_a| = r(a)^n v_n$. A simple computation gives

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Similarly

$$r(a)^{-n} = |a|^{-\frac{n+1}{2}} \kappa_{K^*}(a^*)^{-\frac{n-1}{2}} \ .$$

We get thus the formula

$$\left(\kappa_K(a)\kappa_{K^*}(a^*)\right)^{\frac{n-1}{2}} = (|a|\ |a^*|)^{-\frac{n+1}{2}}.$$

Two remarks

2) It is tempting to try to prove that minimal bodies for volume product are actually polytopes aned minimal functions for $P(\phi)$ are piecewise affine, but such "local" methods do not seem to work.

Congratulations to ???

This will be my conclusion

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THANK YOU FOR YOUR ATTENTION

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THE END