Transference Principle for Log-Sobolev and Spectral-Gap Inequalities with Application to Conservative Spin Systems

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Technion

Phenomena in high dimensions in geometric analysis, random matrices, and computational geometry

Roscoff

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joint with Franck Barthe, Toulouse.

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Motivation - Uniqueness of Phase

$G = (V, E)$ = finite graph, vertex = particle, edges = interactions.
Hamiltonian $H : \mathbb{R}^V \to \mathbb{R}$ describes the system energy:
$H(x) = \sum_i \Psi_i(x_i) + \sum_{i \sim j} \Psi_{i,j}(x_i, x_j)$.
Gibbs probability measure $\mu = \frac{1}{Z_\beta} \exp(-\beta H(x)) dx$.

Major Problem 1: uniqueness of Gibbs measure on $\mathbb{R}^V$ in thermodynamic limit.

Connected to uniform estimates on spectral-gap / log-Sobolev inequalities associated to μ_n (uniform in n and in bc).

Major Problem 2: what happens during perturbation of H (changing β, adding interaction)? Is uniqueness violated?

Leads to question of stability of spectral-gap / log-Sobolev inequalities under perturbation of μ.

Main Idea: study stability of related inequalities:

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

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Isoperimetric Inequalities

(Ω, d, μ) - measure metric space; d - metric, μ - prob. measure.
Assume: $\Omega \subset (M^n, g)$ Riemannian manifold, d induced geodesic distance on M, $\mu = h \, vol_M|_\Omega$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski's exterior boundary measure):

$$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon},$$

$$A^d_\varepsilon := \{x \in \Omega; d(x, A) < \varepsilon\}.$$

(Ω, d, μ) satisfies $\mathcal{I}$-Isoperimetric Inequality if:

$$\forall A \subset \Omega \quad \mu^+(A) \geq \mathcal{I}(\mu(A)) \ , \ \mathcal{I} : [0, 1] \to \mathbb{R}_+ \ .$$

Since typically $\mu^+(A) = \mu^+(\Omega \setminus A)$:

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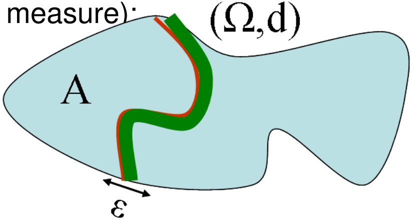

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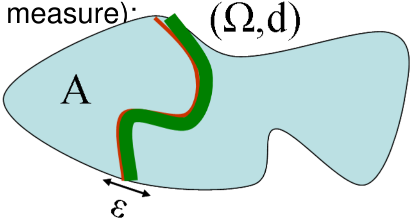

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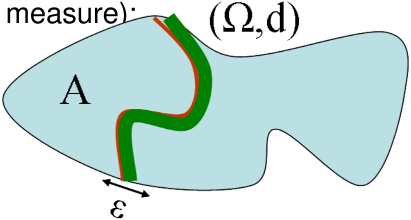

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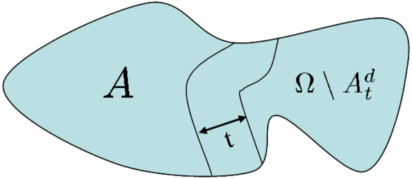
Concentration Inequalities

$\mathcal{K}$-Concentration (large deviation) Inequality:

$$\forall r > 0 \quad \forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A^d_r) \leq \mathcal{K}(r) \; .$$



Quantify how tightly μ is concentrated around $1/2$-sets ("concentration of measure").

Important: only meaningful in the large ($r \gg 0$), may not provide infinitesimal information, contrary to Isoperimetric Inqs.

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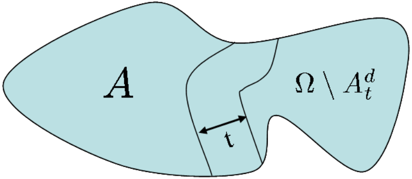

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The Hierarchy

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

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Examples: (Cheeger, Maz'ya; Gromov–V. Milman; Ledoux, Beckner; Herbst)

$\mathcal{I}(v) \geq Dv$ (Expanders, log-concave) $\Rightarrow$ $\| |\nabla f| \|_2 \geq \frac{D}{2}\sqrt{\mathrm{Var}(f)}$ (Poincaré, Spectral-Gap) $\Rightarrow$ $\mathcal{K}(r) \leq \exp(-cDr)$ (Exponential Conc)

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Remark: reverse implications are in general false
due to bottlenecks in space (geometry of (Ω, d) or measure μ).

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Hierarchy Reversal in Presence of Convexity

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

Setting: (M^n, g), d, $\Omega \subset M$ convex, $\mu = \exp(-\psi) vol_M|_\Omega$.
Bakry–Émery Curvature-Dimension Condition:

$$Ric_g + Hess_g \psi \geq -\kappa g \ , \ \kappa \geq 0.$$

- $\kappa = 0$ - "convex case" (e.g. $(\mathbb{R}^n, |\cdot|, \mu)$ log-concave)
- $\kappa > 0$ - "semi-convex case" (e.g. double-well potentials).

$\Rightarrow$ Obtain dimension independent hierarchy reversal :

- Sobolev inqs (Buser, Ledoux, Bakry–Ledoux, M.).
- Concentration inqs (M. 08-10):
  If $\kappa > 0$, require $\mathcal{K}(r) < \exp(-\frac{1}{2}\kappa r^2)$, otherwise false.

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Hierarchy Reversal Examples

In convex case ($Ric_g + Hess_g \psi \geq 0$):

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Hierarchy Reversal Examples

In semi-convex case ($Ric_g + Hess_g \psi \geq -\kappa g$, $\kappa > 0$) :

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If in addition $\mathcal{K}(r) < \exp(-\frac{1}{2}\kappa r^2)$
(needed to compensate for $-\kappa$ curvature)

Summarizing

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

$$\begin{cases} \text{Under } Ric_g + Hess_g \psi \geq -\kappa g \; , \; \kappa \geq 0 \\ \text{strong concentration condition if } \kappa > 0 \end{cases}$$

Intuition: convexity bridges between small and large extensions.

Recover all previous dim-dependent results:

- Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov: $\int_\Omega \exp(\beta(d(x,x_0)))d\mu(x) < \infty$ $\Rightarrow \mathcal{K}(r) \leq$ Markov
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This talk: *stability* of Sobolev inqs under measure perturbation:

- $\kappa = 0$ - stability of λ_{SG} w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1)$.
- General inqs - stability w.r.t. $\|\frac{d\mu_2}{d\mu_1}\|_{L^p(\mu_1)}$.

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- General inqs - stability w.r.t. $\|\frac{d\mu_2}{d\mu_1}\|_{L^p(\mu_1)}$.

Applications Overview

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

$$\begin{cases} \text{Under } Ric_g + Hess_g \psi \geq -\kappa g \ , \ \kappa \geq 0 \\ \text{strong concentration condition if } \kappa > 0 \end{cases}$$

Intuition: convexity bridges between small and large extensions.

Recover all previous dim-dependent results:

- Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov: $\int_\Omega \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ $\Rightarrow \mathcal{K}(r) \leq$ Markov
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When $\kappa = 0$, we actually have $\rho_{LS}(\mu_2) \geq c \, \frac{p-1}{p} \frac{1}{1+\log(L)} \, \rho_{LS}(\mu_1)$.

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- False without assuming $Hess V_2 \geq -\kappa Id$.
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Idea of Proof:

$$\| |\nabla f| \|_2^2 \geq \frac{\rho_{LS}}{2} \mathrm{Ent}_{\mu_1}(f^2) \;\; \Rightarrow \;\; \mathcal{K}(r) \leq \exp(-\frac{\rho_{LS}}{2} r^2)$$

(log-Sobolev for μ_1) (Gaussian Conc for μ_1)

$$\Downarrow \; \left\| \frac{d\mu_2}{d\mu_1} \right\|_{L^p(\mu_1)} \leq L$$

$$\mathcal{I}(v) \geq D' v \sqrt{\log 1/v} \Rightarrow \| |\nabla f| \|_2^2 \geq D \, \mathrm{Ent}_{\mu_2}(f^2) \;\; \Rightarrow \;\; \mathcal{K}(r) \leq L \exp(-\frac{\rho_2}{2} r^2)$$

(Gaussian isop. for μ_2) (log-Sobolev for μ_2) (Gaussian Conc for μ_2)

$$Hess V_2 \geq -\kappa Id \text{ and } \rho_2 := \frac{p-1}{4p} \rho_{LS}(\mu_1) > \kappa$$

Application: Conservative Spin Systems

$V : \mathbb{R} \to \mathbb{R}$ self-interaction potential. Work on $(\mathbb{R}^n, |\cdot|)$.

$(\mathbb{R}^n, |\cdot|, \mu_n)$ Grand Canonical Ensemble = non-interacting spins:

$$\mu_n := \frac{1}{Z_n} \exp(-H(x))dx \; , \; H(x) = \sum_{i=1}^{n} V(x_i) \; .$$

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Conservative Spin Systems: Our Contribution

By transferring Sobolev inequalities from GCE to CE, we obtain:

- Spectral-Gap: new results when V is (non-strictly) convex: $V'' \geq 0$.
- Log-Sobolev: partially recover and partially extend previous results for $V = V_{sc} + P$.
  - Soft - avoids difficult ad-hoc local Cramér Theorems.
  - Flexible - e.g. can easily incorporate weak-interactions; more generally, can handle superposition of several perturbations:

$$\mu_2 = f\, g\, h\, \mu_1 \;\Rightarrow\; \left\| \frac{d\mu_2}{d\mu_1} \right\|_{L^p(\mu_1)} \leq \|f\|_{L^{3p}(\mu_1)} \|g\|_{L^{3p}(\mu_1)} \|h\|_{L^{3p}(\mu_1)} \;.$$

Conservative Spin Systems: Our Analysis

Assume that $(\mathbb{R}, |\cdot|, \mu = \exp(-V(x))dx)$ satisfies $\rho_{LS}(\mu) > 0$ (Bobkov–Götze), and that $V'' \geq -\rho_{LS}/8$.

- Cramér's trick: by modifying $V(x) \mapsto V(x) + ax$, reduce to $s = 0$ and $\mathbb{E}_\mu(x) = 0$. Denote $E := E_0$, $\Delta := E^\perp$ the diagonal.
- Given parameter $w > 0$, introduce the thickened version of μ_E:

$$\begin{aligned} \mu_{E,w} &= \mu_E \otimes \frac{1}{2w} 1_{[-w,w]}(t) d\mathrm{vol}_\Delta(t) \\ &= \frac{1}{Z_{E,w}} \exp\left(-H(\pi_E(x))\right) 1_{|\langle x,\Delta\rangle| \leq w} dx \;,\; Z_{E,w} = 2wZ_E \;. \end{aligned}$$

- Main Step: show that $\left\| \frac{d\mu_{E,w}}{d\mu_n} \right\|_{L^4(\mu_n)} \leq D(\mu, w)$.
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- Cramér's trick: by modifying $V(x) \mapsto V(x) + ax$, reduce to $s = 0$ and $\mathbb{E}_\mu(x) = 0$. Denote $E := E_0$, $\Delta := E^\perp$ the diagonal.
- Given parameter $w > 0$, introduce the thickened version of μ_E:

$$\begin{aligned} \mu_{E,w} &= \mu_E \otimes \frac{1}{2w} 1_{[-w,w]}(t) d\mathrm{vol}_\Delta(t) \\ &= \frac{1}{Z_{E,w}} \exp\left(-H(\pi_E(x))\right) 1_{|\langle x,\Delta \rangle| \leq w} dx \; , \; Z_{E,w} = 2w Z_E \; . \end{aligned}$$

- Main Step: show that $\left\| \frac{d\mu_{E,w}}{d\mu_n} \right\|_{L^4(\mu_n)} \leq D(\mu, w)$.
- Hess $H(\pi_E(x)) \geq -\rho_{LS}/8$, and hence by Transference Principle $\rho_{LS}(\mu_{E,w}) \geq D(\rho_{LS}(\mu_n), w) = D(\rho_{LS}(\mu), w)$.
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We consequently require exponentially decaying (in r) bounds on:

$$\begin{aligned} & \mu_n \left\{ x \in \mathbb{R}^n \,;\, |\langle x, \Delta\rangle| \leq w \,,\; H(x) - H(\pi_E(x)) \geq r \right\} \\ = \; & \mu_n \left\{ x \in \mathbb{R}^n \,;\, |\langle x, \Delta\rangle| \leq w \,,\, \sum_{i=1}^n V(x_i) - \sum_{i=1}^n V(\pi_E(x)_i) \geq r \right\} \\ \leq \; & \mu_n \left\{ x \in \mathbb{R}^n \,;\, \exists t \in [-w, w] \,,\, \sum_{i=1}^n V(x_i) - \sum_{i=1}^n V(x_i + t/\sqrt{n}) \geq r \right\} . \end{aligned}$$

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Comparison with Previous Results for LSI

Compare with previous results for LSI when $V = V_{sc} + P$.

Main Drawback: smallness of negative perturbation $V'' \geq -\rho_{LS}/8$.

Advantages:

- Soft - avoids difficult ad-hoc local Cramér Theorems.
- Flexible - e.g. can easily incorporate weak-interactions.
- For $s = 0$, recover results $V''_{sc} \geq \alpha > 0$, $\|P\|_{C^2} \ll 1$.
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