Spectral Estimates, Contractions and (maybe) Hypercontractivity

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Setup - Laplacian on Weighted Riemannian Manifolds

(M^n, g, μ) - weighted Riemannian manifold:

- (M^n, g) smooth, complete, connected, oriented (no boundary).
- $\mu = e^{-V}\mathrm{Vol}_g$.

$\Delta_g f := \nabla_g \cdot \nabla_g f$, $\Delta_{g,\mu} f := e^V \nabla_g \cdot (e^{-V} \nabla_g f) = \Delta_g f - \langle \nabla_g f, \nabla_g V \rangle$.

$$\forall f, h \in C_c^\infty(M) \quad -\int (\Delta_{g,\mu} f) h \, d\mu = \int \langle \nabla f, \nabla h \rangle \, d\mu = -\int f(\Delta_{g,\mu} h) d\mu,$$

so $-\Delta_{g,\mu}$ is symmetric pos-def on $L^2(\mu)$ with domain $C_c^\infty(M)$; by completeness of M, uniquely extends to self-adjoint operator.

Spectrum $\sigma(-\Delta_{g,\mu}) \subset [0, \infty)$. If discrete $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \nearrow \infty$.
If discrete and $\mu(M) < \infty$ then $0 = \lambda_1 < \lambda_2$ (spectral-gap).

Goal: Study $\lambda_k = \lambda_k(-\Delta_{g,\mu}) = \lambda_k(M^n, g, \mu)$.

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Curvature-Dimension Condition

Def (Bakry–Émery Curvature Dimension Condition)

$(M^n, g, \mu = \exp(-V)\mathrm{Vol}_g) \in CD(\rho, \infty)$ ($\rho \in \mathbb{R}$), if:

$$\mathrm{Ric}_{g,\mu} := \mathrm{Ric}_g + \mathrm{Hess}_g V \geq \rho g \quad \text{on} \quad M.$$

$CD(\rho, \infty)$ for $\rho > 0$ is nice class: $\mu(M) < \infty$ and spectrum discrete.

Examples:

1. Gaussian Measure $(\mathbb{R}^n, |\cdot|, \gamma^n_\rho)$, Cov $= \frac{1}{\rho} Id$, $\rho > 0$.
   - $\Delta_{\gamma^n} = \Delta - \langle \nabla, x \rangle$ Ornstein-Uhlenbeck operator.
   - Eigenfunctions are Hermite polynomials; $\sigma(-\Delta_{\gamma^1}) = \{0, 1, 2, \ldots\}$, $\sigma(-\Delta_{\gamma^n}) = \oplus_{i=1}^n \sigma(-\Delta_{\gamma^1})$.
   - Satisfies $CD(\rho, \infty)$.
2. Canonical Sphere $(S^n, g_{can}, \mathrm{Vol}_{g_{can}})$.
   - $\Delta_{S^n} = \Delta_{g_{can}}$.
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Model Space for $CD(\rho,\infty)$

$(\mathbb{R}^1, |\cdot|, \gamma_\rho^1)$ is model space for numerous functional & geometric inqs for the class $(M^n, g, \mu) \in CD(\rho,\infty)$, $\rho > 0$:

- Log-Sobolev (Bakry–Émery): $LS(M^n, g, \mu) \geq LS(\mathbb{R}^1, |\cdot|, \gamma_\rho^1)$.
- Spectral-Gap (Lichnerowicz): $\lambda_2(M^n, g, \mu) \geq \lambda_2(\mathbb{R}^1, |\cdot|, \gamma_\rho^1)$.
- Isoperimetric Inq (Bakry-Ledoux):
  $\tilde{\mu}(A) = \gamma_\rho^1(-\infty, a] \Rightarrow \tilde{\mu}^+(A) \geq (\gamma_\rho^1)^+(-\infty, a]$.

These inqs are invariant under tensorization, $1 \leftrightarrow n$.

Main Question. Is it true that:
$(M^n, g, \mu) \in CD(\rho,\infty), \rho > 0 \quad \Rightarrow \quad \forall k \;\; \lambda_k(M^n, g, \mu) \geq \lambda_k(\mathbb{R}^n, |\cdot|, \gamma_\rho^n)$?

- Classical? Difficult? (λ_2, $\lambda_3 - \lambda_2$, Polya's conjecture).
- Kato's inq (Besson, Bérard, ...) - domination of $\exp(-t\Delta_{g,\mu})$.
- Ledoux, Bakry–Bentaleb: use Γ_k to control λ_k.

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Thm (M. '15)

- True for Euclidean space $(\mathbb{R}^n, |\cdot|, \mu) \in CD(\rho, \infty)$, $\rho > 0$.
- False for $(S^n, g_{can}, \mathrm{Vol}_{g_{can}}) \in CD(n-1, \infty)$, $n \geq 3$.

Why False:
$(S^n, g_{can}, \mathrm{Vol}_{g_{can}})$: $0 = \lambda_1 < \lambda_2 = \ldots = \lambda_{n+2} = n$.
$(\mathbb{R}^n, |\cdot|, \gamma^n_{n-1})$: $0 = \lambda_1 < \lambda_2 = \ldots = \lambda_{n+1} = n-1 < \lambda_{n+2} = 2(n-1)$.

There may be some topological obstructions (first eigenvalues), so:

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More Theorems

1. $(\mathbb{R}^n, |\cdot|, \mu = e^{-V}dx)$ $\text{Hess}\, V \geq \rho Id \Rightarrow \lambda_k(\mathbb{R}^n, |\cdot|, \mu) \geq \lambda_k(\gamma_\rho^n)$.
2. $(\mathbb{R}^n, |\cdot|, \mu = e^{-V}dx)$ $\text{Hess}\, V \leq \rho Id \Rightarrow \lambda_k(\mathbb{R}^n, |\cdot|, \mu) \leq \lambda_k(\gamma_\rho^n)$.
3. Let $p \in [1, 2]$, $\nu_p^n = c_p^n e^{-\sum_{i=1}^n |x_i|^p} dx$,
   $\mu = \nu_p^n \exp(-U)$, U convex, $U(\pm x_1, \ldots, \pm x_n) = U(x_1, \ldots, x_n)$.
   Then: $\lambda_k(\mathbb{R}^n, |\cdot|, \mu) \geq \lambda_k(\mathbb{R}^n, |\cdot|, \nu_p^n)$.
4. 1-D criterion: given $(\mathbb{R}, |\cdot|, \mu = f_\mu dx)$ probability measure,
   set $F_\mu(x) := \mu(-\infty, x]$, $I_\mu^\flat := f_\mu \circ F_\mu^{-1} : [0, 1] \to \mathbb{R}_+$.
   If $\mathcal{I}_{\mu_2}^\flat \geq \frac{1}{L} \mathcal{I}_{\mu_1}^\flat$ on $[0, 1]$ $\Rightarrow \lambda_k(\mathbb{R}, |\cdot|, \mu_2) \geq \frac{1}{L^2} \lambda_k(\mathbb{R}, |\cdot|, \mu_1)$.
5. Let $p \in [2, \infty]$, $\tilde{B}_p^n := \left\{ x \in \mathbb{R}^n ; \|x\|_{\ell_p^n} \leq c_p^n \right\}$ volume 1.
   Then: $\lambda_k^{Neu}(\tilde{B}_p^n, |\cdot|, Leb|_{\tilde{B}_p^n}) \geq \frac{1}{392} \lambda_k(\mathbb{R}^n, |\cdot|, \gamma^n)$.
6. Other: Riemannian submersions, minimal fibers, etc..

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5. Let $p \in [2, \infty]$, $\tilde{B}_p^n := \left\{ x \in \mathbb{R}^n ; \|x\|_{\ell_p^n} \leq c_p^n \right\}$ volume 1.
   Then: $\lambda_k^{Neu}(\tilde{B}_p^n, |\cdot|, Leb|_{\tilde{B}_p^n}) \geq \frac{1}{392}\lambda_k(\mathbb{R}^n, |\cdot|, \gamma^n)$.
6. Other: Riemannian submersions, minimal fibers, etc..

More Theorems

1. $(\mathbb{R}^n, |\cdot|, \mu = e^{-V}dx)$ Hess $V \geq \rho Id \;\Rightarrow\; \lambda_k(\mathbb{R}^n, |\cdot|, \mu) \geq \lambda_k(\gamma_\rho^n)$.
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Anti-Climax

All these results are immediate consequences of:

"Thm" (M. 15, Contraction Principle)

Let $T : (M_1, g_1, \mu_1) \to (M_2, g_2, \mu_2)$, so that:
$T_* \mu_1 = \mu_1 \circ T^{-1} = c\mu_2$, $c \in (0, \infty)$, and $d_2(Tx, Ty) \leq L\, d_1(x, y)$.
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- On general mm-space (Ω, d, μ):

$$\lambda_k^{Lip} := \inf_{\substack{F \subset W^{1,2}_{Lip} \\ \dim F = k}} \; \sup_{\substack{f \in F \\ f \neq 0}} \frac{\int |\nabla f|_d^2 \, d\mu}{\int |f|^2 \, d\mu}.$$

  On (M, g, μ), by min-max and density of $W^{1,2}_{Lip} \subset W^{1,2}$, $\lambda_k^{Lip} = \lambda_k$.
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Known Lipschitz maps and Contractions

Caffarelli's Contraction Thm

$(\mathbb{R}^n, |\cdot|, \mu) \in CD(\rho, \infty)$, $\rho > 0 \quad \Rightarrow$
$\exists T : (\mathbb{R}^n, |\cdot|, \gamma_\rho^n) \to (\mathbb{R}^n, |\cdot|, \mu)$ s.t. $T_*(\gamma_\rho^n) = c\mu$ and T is contraction.

Other contractions / Lipschitz maps:

- Kolesnikov - reverse Caffarelli.
- Kim - M. '11 - generalized Caffarelli.
- Adaptation of Bobkov-Houdré.
- Latała–Wojtaszczyk.
- Riemannian submersions.

Contraction Conjecture

Caffarelli's Thm is very power tool for transferring functional & geometric inqs (Sobolev, Isoperimetry, Concentration, Gaussian Correlation for ellipsoids), but many other methods do the same job.

Spectrum comparison goes well beyond previous capabilities.

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(Conjecture 2^* implies 1^*, consistent w/ Caffarelli in Euclidean case and log-Sobolev, Spectral-Gap, Gaussian isoperimetry).

False on S^n; formulate only for $\mathbb{R}^n$ due to topological obstruction:
If $T : \mathbb{R}^n \to M^n$ continuous, but also $1-1$ and onto $\Rightarrow$
$M^n \simeq \mathbb{R}^n$ by Brouwer's Invariance of Domain Thm.

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Extensions

Let (S^n, g, Vol_g) satisfy $CD(\rho, n)$: $\mathrm{Ric}_g \geq \rho g$, $\rho > 0$.

Conjecture 3: $\forall k \;\; \lambda_k(S^n, g, \mathrm{Vol}_g) \geq \lambda_k(S^n, g^{\rho}_{can}, \mathrm{Vol}_{g^{\rho}_{can}})$. Implied by:

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Finally, Conjecture 4 is consistent with predictions of Conjecture 3.

Conjecture 3 is consistent with (in fact, $S^n \leftrightarrow M^n$):

- Lichnerowicz: $\lambda_2(M^n, g, \mathrm{Vol}_g) \geq \lambda_2(S^n, g^\rho_{can}, \mathrm{Vol}_{g^\rho_{can}})$.
- Bérard–Gallot comparison of heat-kernel trace:
  $\sum_{k=1}^\infty \exp(-t\lambda_k(M^n, g)) \leq \sum_{k=1}^\infty \exp(-t\lambda_k(S^n, g^\rho_{can})) \;\; \forall t > 0$.
- Asymptotically true:
  $\lambda_k(M^n, g, \mathrm{Vol}_g) \geq (1 + o(1))\lambda_k(S^n, g^\rho_{can}, \mathrm{Vol}_{g^\rho_{can}})$ as $k \to \infty$,
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$$\lambda_k(M^n, g, \mathrm{Vol}_g) \geq \frac{1}{15e}\lambda_k(S^n, g^\rho_{can}, \mathrm{Vol}_{g^\rho_{can}}) \quad \forall k \geq 4^n.$$

  Proof: Cwikel–Lieb–Rozenblum inequality for $\#$ bound states +
  Sharp $CD(\rho, n)$ Sobolev inq.

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Extending Caffarelli's Theorem

Let (S^n, g, Vol_g) satisfy $CD(\rho, n)$: $\mathrm{Ric}_g \geq \rho g$, $\rho > 0$.

Conjecture 3: $\forall k \;\; \lambda_k(S^n, g, \mathrm{Vol}_g) \geq \lambda_k(S^n, g^\rho_{can}, \mathrm{Vol}_{g^\rho_{can}})$. Implied by:

Conjecture 4: $\exists T : (S^n, g^\rho_{can}, \mathrm{Vol}_{g^\rho_{can}}) \to (S^n, g, \mathrm{Vol}_g)$ s.t. $T_*(\mathrm{Vol}_{g^\rho_{can}}) = c\mathrm{Vol}_g$ and contraction.

If true, Conjecture 4 would give a single reason for variety of separate results (albeit only for S^n).

My main message to you:

Would be very interesting to extend the 2 known proofs of Caffarelli's Contraction Thm (Optimal-Transport, Heat-Flow) from scalar setting (densities) to 2-tensor one (metrics), leading to Conjecture 4.

When does it hold (with $P_t = \exp(t\Delta_{g,\mu})$):

$$\sum_{k=1}^{\infty} \exp(-t\lambda_k) = tr(P_t) = \int h_t(x,x) d\mu =: Z(t) < \infty?$$

Easy Case:

$\|P_t\|_{L^1(\mu)\to L^\infty(\mu)} < \infty$, i.e. ultracontractive, h_t bounded, so $Z(t) < \infty$.

Borderline Case:

$\|P_t\|_{L^2(\mu)\to L^q(\mu)} < \infty$, i.e. hypercontractive, iff log-Sobolev (Gross).

Thm (M. '15)

Assume $(M^n, g, \mu) \in CD(\rho, \infty)$, $\rho \in \mathbb{R}$ and satisfies $LSI(L)$. If:

$$s := \frac{4\rho}{\exp(\rho t) - 1} < L \;\Rightarrow\; Z(t) \leq \exp\left(\frac{s}{1 - s/L} \int_M d^2(x, x_0) d\mu\right) < \infty.$$

In particular, if $L > 4\rho_-$ then $\exists t > 0 \;\; Z(t) < \infty$.

Bounding λ_k on average

When does it hold (with $P_t = \exp(t\Delta_{g,\mu})$):

$$(\, k \exp(-t\lambda_k) \leq \,) \sum_{k=1}^{\infty} \exp(-t\lambda_k) = tr(P_t) = \int h_t(x,x) d\mu =: Z(t) < \infty?$$

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Thank you for the question

Conjecture 2*

$(\mathbb{R}^n, g, \mu) \in CD(\rho, \infty)$, $\rho > 0 \quad \Rightarrow$
$\exists T : (\mathbb{R}^n, |\cdot|, \gamma_\rho^n) \to (\mathbb{R}^n, g, \mu)$ s.t. $T_*(\gamma_\rho^n) = c\mu$ and T is contraction.

Conjecture 2* would imply the following Bishop thm ($\mu(\mathbb{R}^n) = 1$):

$$\exists x_0 \in \mathbb{R}^n \quad \mu(B(x_0, r)) \geq \gamma_\rho^n(B(0, r)) \quad \forall r > 0$$

(by setting $x_0 = T(0)$). In particular:

$$\exists x_0 \in \mathbb{R}^n \quad \int d(x, x_0)^2 d\mu \leq \frac{n}{\rho}.$$

This is **not** known to be true, and I am doubtful, since the $CD(\rho, \infty)$ condition does not see the dimension n.

However, under Graded Curvature-Dimension Condition $GCD(0, \rho)$:

$$\text{Ric}_g \geq 0 \,,\; \text{Hess}_g V \geq \rho g,$$

we can prove the above Bishop thm.
So replace $CD(\rho, \infty)$ by $GCD(0, \rho)$ in Conjectures 1* and 2*.

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