Mossel I

General Plan

I. Spectral / L_2 methods

II Hypercontraction

III Gaussian techniques

IV "General Spaces" $(f : \Omega^n \to \{-1,1\})$

V Sparsity of Spectrum / IEC / Open Problems

Goal for today :
- define Fourier expansion
- influences
- prove : half-cube minimizes surface area
  half-space minimizes noise sensitivity
  Kalai's proof of Arrow's Thm
- " Testing Linearity "

Basic setup

$$L^2(\{-1,1\}) = L^2(\{-1,1\}, \tfrac{1}{2}(\delta_{-1}+\delta_{+1}))$$

Claim $1, x$ is an orthonormal basis of $L^2(\{-1,1\})$

$\Rightarrow$ we can write any $f \in L_2(\{\pm 1\})$ as $f(x) = \hat{f}(\emptyset)\cdot 1 + \hat{f}(\{1\})\cdot x$

Claim $\langle f, g\rangle = \mathbb{E} fg = \hat{f}(\emptyset)\hat{g}(\emptyset) + \hat{f}(\{1\})\hat{g}(\{1\})$

$\mathbb{E} f = \hat{f}(\emptyset)$, $\quad \mathrm{Var} f = \hat{f}(\{1\})^2$.

Less basic setup

$$L_2(\{-1,1\}^n) := L_2(\{-1,1\}^n, (\tfrac{1}{2}(\delta_{-1}+\delta_{+1}))^{\otimes n})$$

Claim The functions $\{x_S\}_{S\subset[n]}$, $x_S = \prod_{i\in S} x_i$ constitute an orthonormal basis $\quad [n] = \{1,\dots,n\}$

Proof. $x_S x_T = x_{S\Delta T} \Rightarrow \mathbb{E} x_S x_T = \delta_{S,T}$

Def. The Fourier transform of $f \in L_2(\{\pm 1\}^n)$ is given by

$$f(x) = \sum_{S\subset[n]} \hat{f}(S) x_S .$$

Claim $\mathbb{E} f = \hat{f}(\emptyset)$, $\mathrm{Var} f = \sum_{S\neq\emptyset} \hat{f}(S)$

$$\langle f, g\rangle = \mathbb{E} fg = \sum_{S\subset[n]} \hat{f}(S)\hat{g}(S) .$$

Alternative interpretation

- x_S are characters of $\mathbb{Z}_2^n$
- x_S are eigenvectors of Laplacian on $\mathbb{Z}_2^n$

Examples of expansions

1) $x_{[n]}$ = parity of the number of "-1" in x

$$= \prod_{i=1}^n x_i = (-1)^{\#\{i:\ x_i=-1\}}$$

2) $\mathrm{Maj}(x_1,x_2,x_3) = \mathrm{sgn}\left(\sum_{i=1}^n x_i\right)$

$$\mathrm{Maj}(-x) = -\mathrm{Maj}(x) \Rightarrow \mathbb{E}\,\mathrm{Maj} = 0 \Rightarrow \widehat{\mathrm{Maj}}(\emptyset) = 0$$

$$\widehat{\mathrm{Maj}}(1,2) = \widehat{\mathrm{Maj}}(1,3) = \widehat{\mathrm{Maj}}(2,3) = 0$$

$$\widehat{\mathrm{Maj}}(1) = \mathbb{E}\,\mathrm{Maj}(x_1,x_2,x_3)x_1 = \mathbb{E}\left(\mathrm{Maj}(x_1,x_2,x_3) \mid x_1=+\right)$$

$$= \tfrac{3}{4}\cdot 1 + \tfrac{1}{4}(-1) = \tfrac{1}{2}$$

So

$$\mathrm{Maj} = \tfrac{1}{2}(x_1+x_2+x_3 - x_1x_2x_3).$$

Def Let $f \in L_2(\{\pm1\}^n)$ and $1 \le i \le n$

$$I_i(f) = \mathbb{E}\left[\mathrm{Var}(f(x) \mid x_{-i})\right], \qquad x_{-i} = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

i-th is missing

$$\mathrm{Var}(f(x) \mid x_{-i}) = \mathbb{E}\left(f(x)^2 \mid x_{-i}\right) - \left(\mathbb{E}(f(x) \mid x_{-i})\right)^2.$$

Example

- $f(x) = x_1$

$$I_1(f) = \mathbb{E}\,\mathrm{Var}(f \mid x_{-1}) = \mathbb{E}\,1 = 1$$

$$I_2(f) = \mathbb{E}\,\mathrm{Var}(f \mid x_{-2}) = \mathbb{E}\,0 = 0$$

- $\mathrm{Maj}(x_1,\dots,x_n) = \mathrm{sgn}\left(\sum_{i=1}^n x_i\right)$, n-odd

$$\mathrm{Var}(f \mid x_{-i}) = \begin{cases} 1 & \sum_{j\ne i} x_j = 0 \\ 0 & \end{cases} \Rightarrow \mathbb{E}\,\mathrm{Var} f = \mathbb{P}\left(\sum_{j\ne i} x_j = 0\right) \underset{\text{CLT}}{=} O\left(\tfrac{1}{\sqrt{n}}\right)$$

Claim $I_i(f) = \sum_{S:\, i\in S} \hat{f}(S)^2$.

Proof

$$\mathrm{Var}(f \mid x_{-i}) = \mathrm{Var}\left(\sum_S \hat{f}(S)x_S \mid x_{-i}\right)$$

$$= \mathrm{Var}\Big(\underbrace{\cancel{\sum_{S:\, i\notin S}}}_{\text{not random}} + \sum_{S:\, i\in S} \mid x_{-i}\Big) = \mathrm{Var}\Big(\underbrace{x_i}_{\text{gives } 1} \sum \hat{f}(S)x_{S\setminus\{i\}} \mid x_{-i}\Big)$$

$$= \left(\sum \hat{f}(S)x_{S\setminus\{i\}}\right)^2,$$

so $I_i(f) = \mathbb{E}\left[\sum_{S: i\in S} \hat{f}(S)\, x_{S\setminus\{i\}}\right]^2 = \sum_{S: i\in S} \hat{f}(S)^2$

<u>Corrolary</u> $\mathrm{Var} f \leq \sum_i I_i(f)$, $I_i(f) \leq \mathrm{Var} f$.

<u>Proof.</u> $\sum I_i(f) = \sum |S| \hat{f}(S)^2 \geq \sum_{S\neq\emptyset} \hat{f}(S)^2 \geq \sum_{S: i\in S} \hat{f}(S)^2 = I_i(f)$. $\square$

<u>Claim</u> (Harper) Among all subsets of $\{0,1\}^n$ with 2^{n-1} vertices the minimizer of edge boundary of the set is given by $A = \{x : x_1 = 0\}$.

Equivalent formulation

Among all functions $f: \{-1,1\}^n \longrightarrow \{-1,1\}$ with $\mathbb{E} f = 0$ a minimizer of $\sum I_i(f)$ is given by $f(x) = x_1$.

<u>Proof.</u> If $f = x_1$ then $\sum_{i=1}^n I_i(f) = I_1(f) = \mathrm{Var} f = \mathbb{E} f^2 - (\mathbb{E} f)^2 = 1$

For all other functions with $\mathbb{E} f = 0$

$$\sum_{i=1}^n I_i(f) \geq \mathrm{Var} f = 1. \quad \square$$

<u>Def.</u> (Bochner operator) Let $-1 \leq \eta \leq 1$, $I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, $J = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, let μ_η^n be $\left(\eta I + (1-\eta) J\right)^{\otimes n}$. For $f \in L_2(\{\pm 1\}^n)$ define

$(T_\eta f)(x) = \mathbb{E}\left(f(Y) \mid Y \sim \mu_\eta^n(x \cdot)\right)$

<u>Claim</u> $T_\eta f = \sum \hat{f}(S)\, \eta^{|S|} x_S$ (Riesz Product)

<u>Def.</u> (BKS) Noise stability of f is given by

$\langle f, T_\eta f\rangle = \mathbb{E} f\, T_\eta f = \langle \sum \hat{f}(S) x_S , \sum \hat{f}(S) \eta^{|S|} x_S \rangle = \sum \hat{f}(S) \eta^{|S|}$

<u>Interpretation</u> (in terms of voting)

$\langle f, T_\eta f\rangle$ = correlation between f-vote and η correlated f-vote.

$\{\pm1\}^n \to \{\pm 1\}$

<u>Claim</u> ① If $0 < \eta < 1$ then among all functions f with $\mathbb{E} f = 0$ the maximizers of $\langle f, T_\eta f\rangle$ are given by $f = \pm x_i$

② If $-1 < \eta < 0$ then among <u>all</u> functions ~~f with E~~ $f: \{\pm1\}^n \to \{\pm 1\}$ the ~~maximizers~~ minimizers are given by $f = \pm x_i$.

Proof ① $\langle f, T_\eta f\rangle = \sum_{S \neq \emptyset} \eta^{|S|} \hat{f}(S)^2 \leq \eta \sum \hat{f}(S) = \eta$

If $f = \pm x_i$: $\langle f, T_\eta f\rangle = \eta$

For uniqueness: only maximizers are linear functions

$$f(x) = \sum \hat{f}(i) x_i$$

② Exercise. □

Let's reformulate part II: Let $(x_i, y_i) \in \{-1,1\}^2$ be such that

① different pairs are independent

② $\mathbb{E} x_i = \mathbb{E} y_i = 0$

③ $\mathbb{E} x_i y_i = \eta$

Then the minimum of $\mathbb{E} f(x) f(y) = \langle f, T_\eta f\rangle$ is obtained for $f = \pm x_i$.

Easy claim Let (x, y, z) be uniformly distributed in $\{-1,1\}^3 \setminus \{\pm(1,1,1)\}$

~~Then $\mathbb{E} xy = \mathbb{E} xz = \mathbb{E} yz = -\frac{1}{3}$~~.

Motivation: condorcet voting

n voters ranking 3 Alternatives a, b, c using $(x_i, y_i, z_i) \in \{-1,1\}^3 \setminus \{\pm(1,1,1)\}$

The function f is "rational" if for all voting profiles

$$(f(x), f(y), f(z)) \neq \pm(1,1,1).$$

Theorem (special case of Arrow's thm) If f is rational then $f = \pm x_i$.

Proof. (Kalai) Let $H(x_1, x_2, x_3) = \mathbb{1}\{(x_1, x_2, x_3) \neq \pm(1,1,1)\}$

$$= \frac{1}{4}(3 - x_1 x_2 - x_2 x_3 - x_3 x_1).$$

f is rational iff $H(f(x), f(y), f(z)) = 1$, for all profiles x, y, z.

iff $\mathbb{E} H(f(x), f(y), f(z)) = 1$

iff $\mathbb{E} f(x) f(y) + \mathbb{E} f(y) f(z) + \mathbb{E} f(z) f(x) = -1$

Yet, for the dictator $\mathbb{E} f(x) f(y) = -\frac{1}{3}$

for another function $\mathbb{E} f(x) f(y) > -\frac{1}{3}$. □

"Testing linearity"

Claim $f = x_S$ for some S iff $\forall x, y$ $f(x) f(y) f(xy) = 1$,

$f : \{-1,1\}^n \to \{-1,1\}$.

Proof. Exercise on linear algebra.

Claim $\mathbb{P}(f = x_S) \geq 1-\eta \Rightarrow \mathbb{P}(f(x)f(y)f(xy) = 1) \geq 1-3\eta$.

Proof. Union bound.

Theorem (BLR) If $\mathbb{E}[f(x)f(y)f(xy)] \geq \eta > 0$, then

$\exists S$ s.t. $\mathbb{P}(f = x_S) \geq \frac{1+\eta}{2}$.

Proof.
$$\mathbb{E}\, f(x)f(y)f(xy) = \mathbb{E}\Big(\sum_S \hat{f}(S)x_S \sum_T \hat{f}(T)\cancel{y}_T \sum_U \hat{f}(U)x_U y_U\Big)$$
$$= \sum_{S,T,U} \hat{f}(S)\hat{f}(T)\hat{f}(U)\, \underbrace{\mathbb{E}\, x_S y_T x_U y_U}_{=\ \mathbb{E}\, x_S x_U\, \mathbb{E}\, y_T y_U \,=\, \delta_{S,U}\delta_{T,U}} = \sum_S \hat{f}(S)^3$$

We have
$$\eta < \sum_S \hat{f}(S)^3 \leq \cancel{\max} \sum_{S:\, \hat{f}(S)>0} \hat{f}(S)^3 \leq \max_S \hat{f}(S) \sum \hat{f}(S)^2 = \max_S \hat{f}(S)$$

$$\Rightarrow \exists S \quad \hat{f}(S) > \eta \Rightarrow \mathbb{E}\, f x_S \geq \eta \Rightarrow \mathbb{P}(f = x_S) \geq \frac{1+\eta}{2}. \quad \square$$

0.01.2012 Hyper-contraction

1. KKL, Talagrand (g4)
2. Hypercont. $\Rightarrow$ concentration of low-deg functions
3. Quantitative Barbara thm
4. Qualitative Theory Applications

Recall T_η is the Markov Operator corresponding to $(\eta I + (1-\eta)J)^{\otimes n}$.

Claim (~~hyper~~ contractivity) (1) $\forall -1 \leq \eta \leq 1\ \forall p \geq 1\ \forall f \in L_2(\{\pm 1\}^n)$
$$\|T_\eta f\|_p \leq \|f\|_p$$

(2) $\forall p \leq 1 \quad \|T_\eta f\|_p \geq \|f\|_p, \quad f \in L_2^+(\{\pm 1\}^n)$, $f \geq 0$.

Proof. Minkowski / Reverse Minkowski inequality gives this is true for any Markov operator. $\square$

Theorem (hypercontractivity)
(Bonami - Beckner - Gross - Nellson - ...
C. Borell

(1) If $\eta^2(q-1) \leq p-1$ then
$$\|T_\eta f\|_q \leq \|f\|_p, \quad q > p > 1$$

(2) If $\eta^2(1-q) \leq 1-p$ then
$$\|T_\eta f\|_q \geq \|f\|_p, \quad f \geq 0, \quad q < p < 1.$$

· <u>Proof idea</u> ① Tensorize (follows from convexity)
② Prove for $n=1$. $\square$

<u>Application</u> (KKL) Let $f : \{-1,1\}^n \longrightarrow \{-1,0,1\}$.

① For all $0 \leq \eta \leq 1$ $\quad \sum \eta^{2|S|} \hat{f}(S)^2 \leq \|f\|_2^{4/(1+\eta^2)}$

② $\exists C \quad \sum \dfrac{\hat{f}(S)^2}{|S|} \leq C \dfrac{\|f\|_2^2}{\log(1/\|f\|_2^2)}$.

<u>Proof.</u> ① $\sum \eta^{2|S|} \hat{f}(S)^2 = \|T_\eta f\|_2^2 \leq \|f\|_{1+\eta^2}^2 = \left(\|f\|_{1+\eta^2}^{1+\eta^2}\right)^{2/(1+\eta^2)}$

$$\underset{f \in \{\pm 1, 0\}}{=} \left(\|f\|_2^2\right)^{2/(1+\eta^2)}.$$

② Let $\eta = \frac{1}{2}$, then $\sum \left(\frac{1}{4}\right)^{|S|} \hat{f}(S)^2 \leq \|f\|_2^{8/5}$

Denote $\mathbb{P}(f \neq 0) = \|f\|_2^2 = x$ and let $k = C \log \frac{1}{x}$, $\left(\frac{1}{4}\right)^k = x^{1/5}$

$$\cancel{x^{1/5} \sum_{|S| \leq k} \hat{f}(S)^2 \leq \sum 4^{-|S|} \hat{f}(S)^2 \leq x^{8/5}}$$

$$\sum_{|S| \leq k} \hat{f}(S)^2 \leq x^{7/5}$$

Now, let's compute

$$\sum_{\emptyset \neq S} \frac{\hat{f}(S)^2}{|S|} = \sum_{|S| \leq k} + \sum_{|S| > k} \leq x^{7/5} + \frac{1}{C \log \frac{1}{x}} \sum_S \hat{f}(S)^2$$

$$\leq x^{7/5} + \frac{Cx}{\log \frac{1}{x}} \leq \frac{Cx}{\log \frac{1}{x}}. \quad \square$$

<u>Theorem</u> (KKL) Let $f : \{-1,1\}^n \longrightarrow \{-1,1\}$, $\mathbb{E} f = 0$, then

$\exists i \quad I_i(f) \geq C \dfrac{\log n}{n}$.

<u>Proof</u> $\underset{\substack{\uparrow \\ i\text{-th derivative}}}{f_i} = \frac{1}{2}\left[f(\underset{\substack{\uparrow \\ i}}{1}) - f(-1) \right] \underset{\substack{\uparrow \\ \text{need to be checked} \\ \text{for example} \\ \text{on the basis}}}{=} \sum_{S : i \in S} \hat{f}(S) x_S$,

so

$$I_i(f) = \sum_{S : i \in S} \hat{f}(S)^2 = \|f_i\|_2^2. \quad \square$$

We want to prove that for some small D $\exists i$ s.t. $\|f_i\|_2^2 \geq D \frac{\log n}{n}$.

Assume a contradiction

$\forall i \quad \|f_i\|_2^2 \leq D \frac{\log n}{n}$.

Then by claim we get that $\forall i$:

$$\sum_{S: i\in S} \frac{\hat f(S)^2}{|S|} = \sum_{S\neq\emptyset} \frac{\hat f_i(S)^2}{|S|} \le \frac{\tilde C \|f_i\|_2^2}{\log n}$$

$\log\left(\frac{1}{\|f_i\|_2^2}\right)$

Summing these inequalities over all i

$$1 = \mathrm{Var} f = \sum_{S\neq\emptyset} \hat f(S)^2 \le \frac{\tilde C}{\log n} \sum \|f_i\|_2^2 \le \frac{\tilde C}{\log n}\, n\, \frac{C\log n}{n}$$

we get a contradiction if $D \sim \frac{1}{\tilde C}$. $\square$

<u>Talagrand '94</u> $\exists C \quad \forall f \in L_2(\{\pm1\}^n)$

$$\mathrm{Var} f \le \sum_{i=1}^n \frac{\|f_i\|_2^2}{\log(\|f_i\|_2)\cdot\|f_i\|_1}.$$

How to get the concentration out of hypercontractivity?

<u>Theorem</u> Let $f \in L_2(\{\pm1\}^n)$ s.t. $f = \sum_{|S|\le d} \hat f(S) x_S$. Then

$$\|f\|_q \le (q-1)^{d/2}\|f\|_2, \quad q \ge 2.$$

<u>Proof</u> Let $f_i = \sum_{|S|=i} \hat f(S) x_S$, $\eta = \frac{1}{\sqrt{q-1}}$. Notice

$$T_\eta f_i = \eta^i f_i,$$

and by hypercontractivity

$$\|T_\eta f\|_q \le \|f\|_2$$

$$\|f\|_q \le \|T_\eta^{-1} f\|_2$$

$$\|f\|_q^2 \le \|T_\eta^{-1} f\|_2^2 = \Big\|\sum_{i=0}^d \eta^{-i} f_i\Big\|_2^2 = \sum_{i=0}^d \eta^{-2i}\|f_i\|_2^2$$

$$\le \eta^{-2d}\|f\|_2^2. \ \square$$

To get the concentration from that we prove

<u>Theorem</u> Let $f = \sum_{|S|\le d} \hat f(S) x_S$. Then, if $t \ge e^{d/2}$,

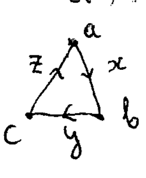
$$\mathbb{P}(|f| \ge t\|f\|_2) \le \exp\left(-\frac{d}{2e} t^{2/d}\right).$$

<u>Proof</u> Exercise (moments $\rightsquigarrow$ tails). $\square$

<u>Condorcet voting</u>

n voters vote uniformly and independently

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \text{ uniformly } \sim \{-1,1\}^3 \setminus \{\pm(1,1,1)\}$$



$$f(x_1, \dots, x_n) = \begin{cases} 1 & a>b \\ -1 & b>a \end{cases}$$

$$g(y_1, \dots, y_n) = \begin{cases} 1 & b>c \\ -1 & c>b \end{cases}$$

$$h(z_1, \dots, z_n) = \begin{cases} 1 & c>a \\ -1 & a>c \end{cases}$$

Def We say that f, g, h are <u>rational</u> if for all voting profiles $(f(x), g(y), h(z)) \neq \pm(1,1,1)$.

Barbera's theorem If $\exists\, i \neq j$ s.t. $I_i(f) > 0$, $I_j(g) > 0$ then (f, g, h) is not rational.

Quantitative version (Mosel 09) If $I_i(f) > \varepsilon$, $I_j(g) > \varepsilon$

$$\mathbb{P}(\text{non-rational outcome}) \gtrsim \frac{\varepsilon^3}{36}.$$

Proof of Barbera's theorem WLog $i = 1$, $j = 2$.

$$\exists\, x_2, \dots, x_n \quad f(+, x_2, \dots, x_n) \neq f(-, x_2, \dots, x_n),$$

$$\exists\, y_1, y_3, \dots, y_n \quad f(y_1, +, y_3, \dots, y_n) \neq f(y_1, -, y_3, \dots, y_n),$$

Let $z_1 = -y_1$, $z_i = -x_i$, $i \geq 2$. Note that $(x_i, y_i, z_i) \neq \pm(1,1,1)$. Also note that $\exists\, x_1, y_2$ s.t.

$$h(z_1, \dots, z_n) = g(y_1, \dots, y_n) = f(x_1, \dots, x_n)$$

$\Rightarrow$ this is <u>not</u> a rational outcome. $\square$

~~Lemma (Cor of Rev-Hyp-Contractivity M-O'Donell) If A is measurable~~ Regev Sudakov with respect to $(x_1, \dots, x_n)$ and B w.r.t. $(y_1, \dots, y_n)$, and $\mathbb{P}(A) \geq \varepsilon$, $\mathbb{P}(B) \geq \varepsilon$. Then

$$\mathbb{P}(A \cap B) \geq \varepsilon^3.$$

Proof of quantitative version

$$A = \{(x_3, \dots, x_n) : \exists\, x_2 \;\; f(+, x_2, \dots, x_n) \neq f(-, x_2, \dots, x_n)\}$$

$$B = \{(y_3, \dots, y_n) : \exists\, y_1 \;\; g(y_1, +, y_3, \dots, y_n) \neq g(y_1, -, y_3, \dots, y_n)\}$$

$$\left.\begin{array}{l} \mathbb{P}(A) \geq I_1(f) \geq \varepsilon \\ \mathbb{P}(B) \geq I_2(g) \geq \varepsilon \end{array}\right\} \Longrightarrow \mathbb{P}(A \cap B) \geq \varepsilon^3$$

Condition on profiles for voters $3, \dots, n$ in $A \cap B$ $\Rightarrow$ get a function on two voters where

$$\begin{array}{l} I_1(f \mid \text{cond}) > 0 \\ I_2(g \mid \text{cond}) > 0 \end{array} \Rightarrow \mathbb{P}(\text{non-rational outcome}) \geq \frac{1}{36}. \;\square$$

Proof of the lemma $\quad \mathbb{E}(x_i y_i) = -\frac{1}{3}$, $\mathbb{E}\, x_i(-y_i) = \frac{1}{3}$

$$\mathbb{P}(x \in A, y \in B) = \mathbb{P}(x \in A, -y \in -B\} = \mathbb{E}\, 1_A T_{1/3} 1_B.$$

Claim If f, g are positive functions then

$$\mathbb{E} f T_\eta g \geq \|f\|_p \|g\|_q,$$

providing that $\eta^2 \leq (1-p)(1-q)$.

The lemma follows, for

$$p = \frac{2}{3},\ q = \frac{2}{3},\quad \mathbb{P}(A \cap B) \geq \|1_A\|_{2/3} \|1_B\|_{2/3} \geq \varepsilon^{3/2} \varepsilon^{3/2} = \varepsilon^3.$$

Proof of the Claim

$\frac{1}{p'} + \frac{1}{p} = 1$. (remember that $p < 1$, but stay cool)

$$\mathbb{E} f T_\eta g \geq \|f\|_p \|T_\eta g\|_{p'} \underset{\text{hypercontractivity}}{\geq} \|f\|_p \|g\|_q. \quad \square$$

Example from coding theory (Bogdanov - Mossell)

Goal: Find a map $f : \{-1,1\}^n \to \{-1,1\}^k$, $k \ll n$ s.t.

- $\mathbb{P}(f(x) = z) = 2^{-k}$ for all $z \in \{-1,1\}^k$
- $\mathbb{P}(f(x) = f(y))$ is maximized when (x_i, y_i) are i.i.d. and

$$\mathbb{E} x_i = \mathbb{E} y_i = 0$$
$$\mathbb{E} x_i y_i = \eta > 0.$$

Yesterday we saw one case: $k = 1$, the maximizer of $\mathbb{E} f T_\eta f$ is dictator $= \pm x_k$.

Let $f_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$,

$$\mathbb{P}(f(x) = f(y)) = \left(\frac{1+\eta}{2}\right)^k = (1-\varepsilon)^k \approx e^{-\varepsilon k}.$$

Question Is this optimal?

Theorem 1 If $k \geq 10 + \frac{2}{\varepsilon}$ and n is large enough there exists $f : \{-1,1\}^n \to \{-1,1\}^k$ s.t.

- $\forall z \quad \mathbb{P}(f(x) = z) = 2^{-k}$
- $\forall z \quad \mathbb{P}(f(x) = f(y) = z \mid f(x) = f(y)) = z^{-k}$
- $\mathbb{P}(f(x) = f(y)) \geq \frac{c}{\sqrt{2k}} 2^{-k\varepsilon/(1-\varepsilon)}$.

Proof uses Hamming balls.

Question: Is this optimal? **Answer** Yes, up to $\frac{c}{\sqrt{\varepsilon k}}$.

Theorem Let $f: \{-1,1\}^n \longrightarrow \{-1,1\}^k$ satisfy $\mathbb{P}(f(x)=z) = 2^{-k}$ $\forall z$.
Then
$$\mathbb{P}(f(x)=f(y)) \leq 2^{-\frac{k\varepsilon}{1-\varepsilon}}.$$

Proof Enough to show that for all z
$$\mathbb{P}(f(x)=f(y)=z) \leq 2^{-k}\, 2^{-\frac{k\varepsilon}{1-\varepsilon}}.$$
Let $h(x) = 1_{\{f(x)=z\}}$
$$\mathbb{P}(f(x)=f(y)=z) = \mathbb{E}(h T_\eta h) = \langle T_{\sqrt{\eta}} h, T_{\sqrt{\eta}} h \rangle = \| T_{\sqrt{\eta}} h \|_2^2$$
$$\underset{\text{hyp-con}}{\leq} \|h\|_{1+\eta}^2 = \left(\|h\|_2^2\right)^{2/(1+\eta)} = 2^{-k\frac{1}{1-\varepsilon}}. \quad \square$$

1.01.2012

Gaussian connections

① sketch the proof of Gaussian hypercontraction & reverse

② Hermite expansion $\longleftrightarrow$ discrete Fourier expansion

③ Non-linear invariance : - Majority is stablest
- social choice
- Theoretical Computer Science

① **Def** For $f: \mathbb{R}^n \longrightarrow \mathbb{R}$
$$(U_\eta f)(x) = \mathbb{E}\, f(\eta x + \sqrt{1-\eta^2}\, y), \quad y \sim \mathcal{N}(0, I).$$

Theorem (Bonami - Bechner - Gross - Nelson - ...) Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be somewhat nice
If $q > p > 1$ and $\eta^2(q-1) \leq p-1$ then
$$\|U_\eta f\|_q \leq \|f\|_p.$$
Underlining measure = the standard Gaussian measure

Reverse, if $f: \mathbb{R}^n \longrightarrow \mathbb{R}_+$, $\eta^2(1-q) \leq 1-p$,
then
$$\|U_\eta f\|_q \geq \|f\|_p.$$

Proof (via CLT)
$$f: \mathbb{R}^n \longrightarrow \mathbb{R} \quad \rightsquigarrow \quad f_m : \{-1,1\}^{n\times m} \longrightarrow \mathbb{R}$$
$$f_m(x_1^1, \dots, x_1^m, \dots, x_n^1, \dots, x_n^m)$$
$$= f\left(\frac{\sum_{i=1}^m x_1^i}{\sqrt{m}}, \dots, \frac{\sum_{i=1}^m x_n^i}{\sqrt{m}}\right).$$

Let y_i^j be iid with $\mathbb{E}\, x_j^i y_j^i = \eta$, $\mathbb{E}\, x_b^a y_c^d = 0$ unless $\begin{matrix} b=c \\ a=d \end{matrix}$

$$T_\eta f_m(x) = \mathbb{E} f_m(y) = \mathbb{E} f\left(\frac{\sum_{i=1}^m y_1^i}{\sqrt{m}}, \dots \right) \approx (\mathcal{U}_y f)_m$$

$$\underset{\mathcal{N}\left(\eta \frac{\sum x_1^i}{\sqrt{m}}, \sqrt{1-\eta^2}\right)}{\mathcal{S}}$$

For each m we've seen that

$$\begin{matrix} \| T_\eta f_m \|_q & \leq & \| f_m \|_p \\ \wr\wr & & \big\downarrow{\scriptstyle m\to\infty} \\ \| \mathcal{U}_\eta f_m \|_q & & \| f \|_p \\ \big\downarrow & & \\ \| \mathcal{U}_\eta f \|_q & & \end{matrix}$$

<u>Theorem</u> Let $f \in L_2(\mathbb{R}^1, \gamma)$, $\gamma(x :\ f \text{ is not continuous at } x) = 0$,
Let $f_m : \{-1,1\}^m \to \mathbb{R}$ be as before. Let expand

$$f_m = \sum_{k=0}^m \hat{f}_m(k) \left(\frac{\sum_{|S|=k} x_S}{\binom{n}{k}^{1/2}} \right),$$

$$f = \sum \hat{f}(k) h_k, \qquad h_k \text{ - probabilistic Hermite polynomials}$$

Then for all k

$$\hat{f}_m(k) \xrightarrow[m\to\infty]{} \hat{f}(k).$$

<u>Proof</u> Exercise.

<u>Example</u> $f(x) = \operatorname{sgn} x$. Then f_m is Majority,

$$\hat{f}(k) = \begin{cases} 0 & k \text{ even} \\ \sqrt{\frac{2}{\pi k}} \sqrt{2^{-k+1} \binom{k-1}{\frac{k-1}{2}}} & \end{cases} \approx O(k^{-3/4})$$

Note

$$\sum_{l \geq k} \hat{f}(l)^2 = O\left(\frac{1}{\sqrt{k}}\right).$$

<u>Theorem</u> (Bourgain, ≈2000) If $f : \{-1,1\}^n \to \{-1,1\}$ has no influences
then
$$\sum_{|S| \geq k} \hat{f}(S)^2 \geq k^{-1/2-\delta}$$
In other words, if $I_i(f) \leq \varepsilon(\delta, k)$, then (assume $\underline{\mathbb{E} f = 0}$)

③ - Slides.