Discrete Fourier Analysis - Thinking Outside the (Hamming) Cube

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January 7, 2012

Goals and Challenges

Goal

Extend the theory of DFA to functions $f : \Omega^n \to \{-1, 1\}$

Ω some (probability / metric / metric-measure) space

Challenges

- No canonical definition of influences or Fourier expansion
- A variety of interesting extensions

Today's Plan

Today: Some Examples

- Some extensions we won't talk about.
- Reverse hyper-contraction and hitting bounds on Discrete Spaces.
- Hyper-contraction and the Variance Influence on Discrete Spaces.
- Efron Stein decomposition, query probabilities and influences.
- Gaussian Influences.

Some extensions we won't talk about

BKKKL-92 extension for Influence

Main result: There exists a variable i with

$$I_i(A) \geq \frac{\log}{n} Var(A)$$

O'Donnell-Wimmer $\sim$ 05

Influences in finite group setups.

Cordero-Erausquin and Ledoux

Abstract generalizations of O'Donnell-Wimmer and Gaussian Influences (below) based on nice Dirchlet form decompositions.

Beckner operator on general spaces

The (Bonami-Beckner) operator on general spaces

On a probability space (Ω, μ) let

$$M_\rho(x, y) = \rho\delta_{x,y} + (1 - \rho)\mu_y.$$

For $f : \Omega^n \to \mathbb{R}$, let

$$T_\rho f(x) = E[f(y)|y \sim M^{\otimes n}(x, \cdot)].$$

In words: $T_\rho f(x)$ is the average of $f(y)$ where each coordinate y_i is chosen to be x_i with probability ρ and otherwise μ-re-randomized.

Reverse-Hyper-Contractive inequalities on the Hamming cube

Borell's reverse bound

Reverse-Hyper-Contractivity (Borell 85):

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_\rho f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } \rho^2(1-q) \leq (1-p)$$

Question

What about other spaces?

Reverse Hyper-contraction on general spaces

M-Oleszkiewicz-Sen-11

Let Ω be an arbitrary space and let T_t be the corresponding Bonami-Backner semi-group. Then for all $f : \Omega^n \to \mathbb{R}_+$:

$$\|T_\rho f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } \rho(1-q) \leq 1-p$$

Comments

- Borell get ρ^2 vs. ρ.
- No dependence on the structure of the space.

Product Space Applications (M-Oleszkiewicz-Sen-11)

Correlated Pairs (generalizing MORSS-05)

Let $x, y \in (\Omega, \mu^n)$ correlated as follows:

$x \sim \mu^n$ and y is T_ρ correlated version is the Bonami-Beckner operator.

Let $A, B \subset \Omega^n$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$. Then:

$$\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-\sqrt{\rho}}}$$

Quantitative Arrow's Theorem (Generalizing M-10)

Obtain a quantitative Arrow theorem for voting distribution μ^n where μ is any non-degenerate distribution on $\{-1, 1\}^3 \setminus \{(1, 1, 1), (-1, -1, -1)\}$.

Hyper-contractivity on the Hamming cube

Bomani, Beckner, Gross:

For $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}$:

$$\|T_\rho f\|_q \leq \|f\|_p \text{ for } q > p > 1 \text{ and } (q-1)\rho^2 \leq (p-1)^2$$

And ...

What are the hyper-contractive constants on other spaces?

Hyper-contractive constants on general discrete spaces

Results for other discrete spaces

- Talagrand (94) for $\Omega_\alpha = (\{-1,1\}, p\alpha_{-1} + (1-\alpha)\delta_1)$ lose $\log(1/\alpha)$ compared to unbiased.
- Oleszkiewicz (03) - exact formula for Ω_α and two norms.
- Wolff (07) general discrete space where α smallest atom measure.
- Related derivations via the Log-Sob constant (Diaconis, Saloff-Coste 96).

Rough Summary

Lose $\log(1/\alpha)$ compared to unbiased Hamming cube.

Example of an application to Discrete Fourier Analysis

The Variance Influence

We will chose to work with the variance Influence:

$$I_i(f) = E[Var[f(X_1, \ldots, X_n)|X_{-i}]]$$

Talagrand Thm for general discrete spaces

There exists a universal constant C such that for any function $f : (\Omega, \mu^n) \to \{0, 1\}$ it holds that

$$Var(f) \leq C \log(\frac{1}{\alpha}) \sum_{i=1}^{n} \frac{I_i(f)}{\log(1/2) - \log(I_i)/2}$$

Example of an application to Discrete Fourier Analysis

Talagrand Thm for general discrete spaces

There exists a universal constant C such that for any function $f : (\Omega, \mu^n) \to \{0,1\}$ it holds that

$$Var(f) \leq C \log(\frac{1}{\alpha}) \sum_{i=1}^{n} \frac{I_i(f)}{\log(1/2) - \log(I_i)/2}$$

Comments

- This is tight: $n^{-1}\delta_1 + (1 - 1/n)\delta_0$ and $f(x) = OR$.
- Proof is the "same".
- One useful tool is Efron-Stein decomposition.

Related reading

- M'05 course on DFE
- Work by N. Keller.

The Efron Stein Decomposition

Efron-Stein / Chaos decomposition

Let $(\Omega, \mu) = \prod_{i=1}^{n} (\Omega_i, \mu_i)$. The Efron-Stein decomposition of $f : \Omega \to \mathbb{R}$ is given by

$$f(x) = \sum_{S \subseteq [n]} f_S(x_S), \quad \text{where}$$

- f_S depends only on x_S.
- For all $S \not\subseteq S'$ and all $x_{S'}$ it holds that:

$$E[f_S | X_{S'} = x_{S'}] = 0, \quad E[f_S f_{S'}] = 0 \text{ for } S \neq S'$$

Existence and Uniqueness

The Efron-Stein decomposition exists and is unique.

Existence and Uniqueness of the Efron-Stein Decomposition

Existence Pf Sketch

$$\textbf{Def:} \quad f_S(x) := \sum_{S' \subseteq S} (-1)^{|S \setminus S'|} E[f(X) | X_{S'} = x_{S'}]$$

$$\textbf{Claim 1:} \quad \sum_S f_S(x) = \sum_{S'} E[f | X_{S'} = x_{S'}] \sum_{S : S' \subseteq S} (-1)^{|S \setminus S'|} = f(x).$$

$$\textbf{Claim 2:} \quad \mathbb{E}[f_S | X_{S'} = x_{S'}] = \mathbb{E}[f_S | X_{S' \cap S} = x_{S' \cap S}]$$

For $S' \subsetneq S$:

$$\mathbb{E}[f_S | X_{S'} = x_{S'}] = \sum_{S'' \subset S} (-1)^{|S \setminus S''|} \mathbb{E}[f(X) | X_{S'' \cap S'} = x_{S'' \cap S'}]$$

$$= \sum_{S'' \subset S'} \mathbb{E}[f(X) | X_{S''} = x_{S''}] \sum_{S'' \subset \tilde{S} \subset S'' \cup (S \setminus S')} (-1)^{|S \setminus \tilde{S}|} = 0.$$

Other nice properties of Efron Stein Decompositions

Additional Properties

$$Var(f) = \sum_{S \neq \emptyset} \|f_S\|_2^2$$

$$I_i(f) = \sum_{S : i \in S} \|f_S\|_2^2.$$

$$T_\rho f_S = \rho^{|S|} f_S$$

More generally if $T = \otimes_{i=1}^n T_i$ where T_i are Markov operators then:

$$T(f_S) = (Tf)_S$$

Proof

Exercise!

A lemma by Schramm and Steif

As an example we will generalized a beautiful lemma from the Boolean to the general case

What else todo today

Plan

Talk a bit about geometric influences

More details on Reverse Hyper-contraction

A case where variance influence is "Boring"

The Gaussian Measure

Let $(\Omega, \mu) = (\mathbb{R}, \gamma)$, where γ is the standard Gaussian measure

Let t be chose so that $P(N(0,1) > t) = n^{-1}$

Let $f : \mathbb{R}^n \to \{0,1\}$ be defined by $f(x_1, \ldots, x_n) = 1$ iff $\max x_i > t$.

Clearly $Var(f) = \Theta(1)$

Clearly $I_i(f) = \Theta(1/n)$ for all i

Conclusion

No KKL, Sharp Thresholds etc. for the Variance Influence and Gaussian Measure!

: (

A new definition: Geometric influence

The *geometric influence* of the j-th coordinate on $A \subseteq (\mathbb{R}^n, \nu^{\otimes n})$ is

$$I_j^{\mathcal{G}}(A) := \mathbb{E}_x[m_\nu(A_j^x)].$$

where

$$A_j^x := \{y \in \mathbb{R} : (x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) \in A\}.$$

and

$$m_\nu(A_j^x) := \liminf_{r \downarrow 0} \frac{\nu(A_j^x + [-r, r]) - \nu(A_j^x)}{r}.$$

is lower Minkowski content of A_j^x.

- We always assume ν has a density.
- Defined in Keller-M-Sen and by Cordero-Erausquin and Ledoux.

Lemma

Let ν be a probability measure on $\mathbb{R}$ with a 'nice' density. Let $A \subset \mathbb{R}^n$ be a monotone set. Then

$$\lim_{r \downarrow 0} \frac{\nu^{\otimes n}(A + [-r, r]^n) - \nu^{\otimes n}(A)}{r} = \sum_{i=1}^{n} I_i^{\mathcal{G}}(A).$$

- In literature, $\liminf_{r \downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n)-\nu^{\otimes n}(A)}{r}$ is sometimes called 'boundary under uniform enlargement.'
- Also true for convex sets. But not for general sets, e.g. $\mathbb{Q}^n$.

KKL-type bound (Keller-M-Sen)

Theorem

Consider the product spaces $\mathbb{R}^n$ endowed with the product Gaussian measure $\mu^{\otimes n}$. Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$

$$\max_{1 \le i \le n} I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{\sqrt{\log n}}{n},$$

where $c > 0$ is a universal constant.

KKL-type bound (Keller-M-Sen)

Theorem

Consider the product spaces $\mathbb{R}^n$ endowed with the product Boltzmann measure $\mu_\rho^{\otimes n}$. Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$

$$\max_{1 \le i \le n} I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{(\log n)^{1-1/\rho}}{n},$$

where $c > 0$ is a universal constant.

- Similar statements for measures whose isoperimetric function $\mathcal{I}(t)$ satisfy

$$\mathcal{I}(t) \ge K \min\{t, 1-t\} \left(\log \frac{1}{\min\{t, 1-t\}}\right)^{\delta}$$

for all $t \in [0,1]$ and for some $\delta > 0$.

Tightness

- The dependence on n in the lower bound is tight up to constant factor.

- Example: One-sided box.

  Let $B_n := (-\infty, a_n]^n$ where a_n is chosen such that $\mu_\rho^{\otimes n}(B_n) = 1/2$. Then

  $$I_j^{\mathcal{G}}(B_n) \leq c \cdot \frac{(\log n)^{1-1/\rho}}{n},$$

  for all $1 \leq j \leq n$.

Some random words about the proof

- Hypercontractivity (Nelson-Gross-...)
  Let $f : \{0,1\}^n \to \mathbb{R}$. For $\delta > 0$, define $T_\delta f(x) = \mathbb{E} f(y)$ where $y_1, y_2, \ldots, y_n$ are i.i.d. with

$$y_i = \begin{cases} x_i & \text{w.p. } \delta \\ \text{indep } \mathrm{Ber}(1/2) & \text{w.p. } (1-\delta) \end{cases}$$

  Then

$$\|T_\delta f\|_p \le \|f\|_q, \quad 1 < q < p < \infty, \ \delta \le \sqrt{\frac{q-1}{p-1}}.$$

- Isoperimetric inequality for log-concave densities (e.g. Bobkov '96, Barthe '04).
- Uses the notion of h-influences introduced recently by Keller ('09).

Russo's formula and sharp threshold

Let $\{\nu_\theta : \theta \in \mathbb{R}\}$ denote a family of "nice" probability measures with ν_θ has a density $\lambda_\theta(x) = \lambda(x - \theta)$.

Lemma

Let $A \subseteq \mathbb{R}^n$ be increasing. Then the function $\theta \to {\nu_\theta}^{\otimes n}(A)$ is differentiable and its derivative is given by

$$\frac{d{\nu_\theta}^{\otimes n}(A)}{d\theta} = \sum_{j=1}^{n} I_j^{\mathcal{G}}(A)$$

Corollary

Let $\Phi_\theta = N(\theta, 1)$. Let $A \subset \mathbb{R}^n$ be an increasing and transitive

$$\Phi_{\theta_0}^{\otimes n}(A) > \epsilon \quad \Rightarrow \quad \Phi_{\theta_1}^{\otimes n}(A) > 1 - \epsilon$$

where $\theta_1 - \theta_0 = c \log(1/2\epsilon)(\log n)^{-1/2}$.

Gaussian isoperimetric inequality under uniform enlargement

Theorem (Sudakov & Tsirelson '74, Borell '75)

For any $n \geq 1$ and any $A \subseteq \mathbb{R}^n$

$$\liminf_{r \downarrow 0} \frac{\Phi^{\otimes n}(A + [-r, r]^n) - \Phi^{\otimes n}(A)}{r} \geq \underbrace{\phi(\Phi^{-1}(t))}_{\text{Gaussian isoperimetric fn}},$$

where $t = \Phi^{\otimes n}(A)$.

- The equality is attained by half-spaces, e.g. $A = \{x_1 \leq a\}$.

An isoperimetric result for symmetric bodies

- Question: Find a lower bound on the boundary measure (under uniform enlargement) of sets in $\mathbb{R}^n$ that are 'transitive'.
- Guess: The boundary measure should be 'large'. Symmetry restriction rules out the candidates like half spaces which have small boundary.

Theorem

Then for any transitive set $A \subset \mathbb{R}^n$ we have

$$\liminf_{r \downarrow 0} \frac{\mu^{\otimes n}(A + [-r, r]^n) - \mu^{\otimes n}(A)}{r} \geq ct(1-t)\sqrt{\log n},$$

where $t = \mu^{\otimes n}(A)$ and $c > 0$ is a universal constant .

Remark: This result also extends to all Boltzmann measures.

More results : Extension of Talagrand's result

Theorem (Talagrand '94)

For any set $A \subset \{0,1\}^n$ with uniform product bernoulli measure. If $\mu^{\otimes n}(A) = t$, then

$$\sum_{j=1}^{n} \frac{I_j(A)}{-\log I_j(A)} \geq c_1 t(1-t).$$

Theorem (Gaussian analogue)

Consider $\mathbb{R}^n$ with the product Gaussian measure $\Phi^{\otimes n}$. Let $A \subset \mathbb{R}^n$. If $\Phi^{\otimes n}(A) = t$, then

$$\sum_{j=1}^{n} \frac{I_j^{\mathcal{G}}(A)}{\sqrt{-\log I_j^{\mathcal{G}}(A)}} \geq c_2 t(1-t).$$

More results: Extension of Friedgut's result

Theorem (Friedgut)

Let $A \subset \{0,1\}^n$ with uniform product bernoulli measure. If $\sum_{j=1}^n I_j(A) \leq s$, then for any $\epsilon > 0$, $\exists B \subset \{0,1\}^n$ such that 1_B is determined by at most $\exp(c_1 s/\epsilon)$ coordinates and $\mathbb{P}\{1_A \neq 1_B\} \leq \epsilon$.

Theorem (Gaussian analogue)

Consider $\mathbb{R}^n$ with the product Gaussian measure $\Phi^{\otimes n}$. Let $A \subset \mathbb{R}^n$. Assume A is monotone and $\sum_{j=1}^n I_j^{\mathcal{G}}(A)\sqrt{-\log I_j^{\mathcal{G}}(A)} \leq s$.

Then for any $\epsilon > 0$ $\exists B \subset \mathbb{R}^n$ such that 1_B is determined by at most $\exp(c_2 s/\epsilon)$ coordinates and $\mathbb{P}\{1_A \neq 1_B\} \leq \epsilon$.

More on reverse Hypercontractivity

Notation

We write T_t for T_ρ where $\rho = e^{-t}$.

- The classical "Semi-group" notation.
- A bit confusing with respect to what we've seen so far.

Hypercontractive Inequalities

Hyper-contractivity (Bomani, Beckner, Gross):

For $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}$:

$$\|T_t f\|_q \leq \|f\|_p \text{ for } p > q > 1 \text{ and } t \geq \tfrac{1}{2} \ln \tfrac{p-1}{q-1}$$

Borell's reverse bound

Reverse-Hyper-Contractivity (Borell 85):

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \tfrac{1}{2} \ln \tfrac{1-q}{1-p}$$

Reverse-Hyper-Contractive inequalities; What?

Borell's reverse bound

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Does this make sense?

"Norms" when $p, q < 1$

Reverse Minkowski inequality

If f is non-negative $p < 1$ and T is a Markov operator then:

$$\|Tf\|_p \geq \|f\|_p$$

Reverse Hölder inequality

If f, g are non-negative and p, p' are dual norms < 1 then:

$$\mathbb{E}[fg] = \|fg\|_1 \geq \|f\|_p \|g\|_{p'}.$$

Reverse-Hyper-contractivity (Borell 85)

For all $f : \{-1, 1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Why? What? When?

Reverse-Hyper-contractivity (Borell 85)

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Why is it true?

What is it good for?

Is it true for other discrete spaces?

Why is it true? Borell's argument

I. Tensor

From Reverse Minkowski suffices to prove for $n = 1$

II. Duality

From Revese Hölder suffices to prove for $0 < q < p < 1$

III. Core of Proof

Write $f(x) = 1 + ax$ where $-1 < a < 1$

Taylor expand $\|f\|_p^p$ and $\|T_t f\|_q^p$ and compare terms

Comment

Steps I. and II. are standard and general. Step III. is at the core of the proof

Borell's argument continued

$n = 1, \quad f(x) = 1 + ax, \quad a \in (-1, 1), \quad 0 < q < p < 1$

$$\|f\|_p^p = 1 + \sum_{n=1}^{\infty} \binom{p}{2n} a^{2n}$$

$$\|T_t f\|_q^q = 1 + \sum_{n=1}^{\infty} \binom{q}{2n} e^{-2nt} a^{2n}$$

By Convexity: $\|T_t f\|_q^p \geq 1 + \frac{p}{q} \sum_{n=1}^{\infty} \binom{q}{2n} e^{-2nt} a^{2n}$

Reduces to: $\frac{p}{q} \binom{q}{2n} e^{-2nt} \geq \binom{p}{2n}$

Or: $\frac{p}{q} \binom{q}{2n} \left(\frac{p-1}{q-1}\right)^n \geq \binom{p}{2n}$

Further reduces to: $(i-q)(1-p)^{1/2} \leq (i-p)(1-q)^{1/2}$ for $i \geq 2$

What is it good for?

Correlated pairs (M-O'Donnell-Regev-Steif-Sudakov-05):

Let $x, y \in \{-1,1\}^n_{1/2}$ correlated as follows:

x is chosen uniformly and y is T_t correlated version.

i.e. $\mathbb{E}[x_i y_i] = e^{-t}$ for all i independently

Let $A, B \subset \{-1,1\}^n_{1/2}$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$

Then: $\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-e^{-t}}}$

Pf Sketch:

From Reverse Hölder and Borell's result get for any $f, g > 0$:

$$\mathbb{E}[g T_t f] \geq \|f\|_p \|g\|_q \qquad \forall t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

For $f = 1_A, g = 1_B$ optimize norms p and q

What is it good for? Cosmic coin model

Coin-Tossing Model

k players want to toss the same coin

Each player gets a y^i that is ρ correlated with $x \in \{-1,1\}^n$.

If $f(y) \in \{0,1\}$ is the coin toss then prob. of agreement is $\|T_t f\|_k^k + \|T_t(1-f)\|_k^k$.

Proposition (M-O'Donnell-Regev-Steif-Sudakov-05):

If $f \in \{0,1\}$ and $\mathbb{E}[f] \le 1/2$ then $\|T_t f\|_k^k \le k^{1-e^{2t}+o(1)}$.

Comments

This is tight!

Proof uses reverse-hyper-contraction. Standard Hyper-contractivity gives a bound of $\|T_t f\|_k^k \le 0.5^{\frac{1}{\rho^2}}$

Arrow Theorem

Setup

3 alternatives a, b, c that are ranked by n voters.

Voter i preference of a vs. b, b vs. c and c vs. a are x_i, y_i and z_i

$(x_i, y_i, z_i) \in R := \{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$ for all i.

$f, g, h : \{-1,1\}^n \to \{-1,1\}$ aggregate the pairwise preference.

Arrow's Theorem (51)

Let $f, g, h : \{-1,1\}^n \to \{-1,1\}$ satisfying for $b = \pm 1$:
$f(b,\ldots,b) = g(b,\ldots,b) = h(b,\ldots,b) = b$ (Unanimity).

Then either $\exists i$ s.t $\forall x$, $f(x) = g(x) = h(x) = x_i$ or

$\exists x, y, z \in R^n$ s.t. $f(x) = g(y) = h(z)$.

Arrow Theorem and Hyper-Contractivity

Reverse-hyper-contractivity is essential in recent quantitative proofs of Arrow's theorem by M-2011 and Keller-2011 following Kalai's paper (02) in the balanced case. For example:

Barbera's Lemma (82);

Lef $f, g, h : \{-1, 1\}^n \to \{-1, 1\}$ with $I_1(f) > 0, I_2(g) > 0$

Then $\exists x, y, z \in R^n$ s.t. $f(x) = g(y) = h(z)$

Quantitative Barbera's Lemma (M-11)

Lef $f, g, h : \{-1, 1\}^n \to \{-1, 1\}$ with $I_1(f) > \epsilon, I_2(g) > \epsilon$

Then $\mathbb{P}[f(x) = g(y) = h(z)|(x, y, z) \in R^n\}] \geq \frac{\epsilon^3}{36}$

Interest in Rev. Hyp contraction on other spaces?

Motivation 1:

Lower bounds for $\mathbb{P}[X \in A, Y \in B]$ for correlated X, Y in general product spcaes

Motivation 2:

Obtain bounds for "Cosmic Die Problem"

Motivation 3

Prove Quantitative Arrow Theorem for non-uniform distributions over voters profile

Hyper and Reverse-Hyper Contractive inequalities

Reverse-Hyper-contractivity (Borell 85)

For all $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Hyper-contractivity (Bomani, Beckner, Gross):

For $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}$:

$$\|T_t f\|_p \leq \|f\|_q \text{ for } p > q > 1 \text{ and } t \geq \frac{1}{2} \ln \frac{p-1}{q-1}$$

Question

Is Hyper-Contraction equivalent to Rev.-Hyper-Contraction?

Our Results (M-Oleszkiewicz-Sen-11)

Our Result:

Let Ω be an arbitrary space and let T_t be the corresponding Bonami-Backner semi-group. Then for all $f : \Omega^n \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \ln \frac{1-q}{1-p}$$

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For all $f : \Omega^n \to \mathbb{R}_+$:

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Compare to Borell 82, Oleszkiewicz 03, Wolff 07

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

For all $f : \{-1,1\}^n_\alpha \to \mathbb{R}_+$ need $t \gtrsim (\frac{1}{q} - \frac{1}{p}) \ln \alpha$.

Comments:

Note: inequality does not depend on underlying space!.

Sharper (but not tight) bounds are obtained in the paper

Further Results (M-Oleszkiewicz-Sen-11)

Log-Sobolev and Rev. Hyper-Contraction

Let $T_t = e^{-tL}$ be a general Markov semi-group. Suppose L satisfies 2-Logsob or 1-Logsob inequality with constant C. Then for all $q < p < 1$, all positive f and all $t \geq \frac{C}{4} \log \frac{1-q}{1-p}$ it holds that $\|T_t f\|_q \geq \|f\|_p$.

Product Space Applications (M-Oleszkiewicz-Sen-11)

Correlated Pairs

Let $x, y \in (\Omega, \mu^n)$ correlated as follows:

$x \sim \mu^n$ and y is T_t correlated version where $T_t = e^{-t(I-\mathbb{E})}$ is the Bonami-Beckner operator.

Let $A, B \subset \Omega^n$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$. Then:

$$\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-e^{-t/2}}}$$

Quantitative Arrow's Theorem

Obtain a quantitative Arrow theorem for voting distribution μ^n where μ is any non-degenerate distribution on $\{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$.

Markov Chain Applications (M-Oleszkiewicz-Sen-11)

Correlated Pairs

Let $x, y \in (\Omega, \mu)$ correlated as follows:

$x \sim \mu$ and y is T_t correlated version where $T_t = e^{-tL}$, where L satisfies 1 or 2-LogSob inequality with constant C

Let $A, B \subset \Omega^n$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$. Then:

$$\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-e^{-2t/C}}}$$

Glauber Dynamics for Ising model in High Temperatures in $[n]^d$.

$A = \{x : Maj(x) = +\}, \quad , B = \{x : Maj(x) = -\}, \quad C = \Theta(1), \quad t_{mix} = O(\log n)$

Random-Transposition Card Shuffle

General A, B. We have $C = \Theta(n), t_{mix} = \Theta(n \log n)$

A Queueing Theory Application (M-Oleszkiewicz-Sen-11)

A queueing process

Take $\{0,1\}^n_{\frac{\lambda}{n}}$ with the Bonami-Beckner operator T_t and

$$X = \sum_{i=1}^n X_i.$$

As $n \to \infty, X \sim Poisson(\lambda)$ and $T_t X$ is the following queueing process:

At $[t, t+dt]$: 1) Each customer is serviced with probability dt.

2) The probability of a new customer arriving is λdt

From our results if $\mathbb{P}[A] > \epsilon, \mathbb{P}[B] > \epsilon$ then

$$\mathbb{P}[X \in A, T_t X \in B] \geq \epsilon^{\frac{2}{1-e^{-t/2}}}$$

Process has infinite mixing time and 2-logSob. 1-logSob is known to be finite (Liming Wu 97)

Proof Sketch

Equivalence with Log-Sobolev inequalities

Using Equivalence of Log-Sob-p inequality and Hyper/Rev-Hyper inequalities work with Log-Sob-p

Main step: Monotonicity of Log-Sob

Log-Sob-$p \implies$ Log-Sob-q for all $2 \geq p > q \geq 0$

Log-Sob-1 for simple operators

Show that Log-Sob-1 holds with $C = 4$ for the semi-group $e^{-t(I - \mathbb{E})}$

Add refs: Wu, Bobkov-Ledoux, Diaconis-Saloff-Coste

Definition of logSob

Standard Definitions

$$Ent(f) = \mathbb{E}(f \log f) - \mathbb{E}f \cdot \log \mathbb{E}f$$

$$\mathcal{E}(f,g) = \mathbb{E}(fLg) = \mathbb{E}(gLf) = \mathcal{E}(g,f) = -\frac{d}{dt}\mathbb{E}fT_t g\Big|_{t=0}.$$

Definition of Log-Sob

$$p\text{-logSob(C)} \iff \forall f, Ent(f^p) \leq \frac{Cp^2}{4(p-1)}\mathcal{E}(f^{p-1}, f) \ (p \neq 0, 1)$$

$$\text{1-logSob(C)} \iff \forall f, Ent(f) \leq \frac{C}{4}\mathcal{E}(f, \log f)$$

$$\text{0-logSob(C)} \iff \forall f, Var(\log f) \leq -\frac{C}{2}\mathcal{E}(f, 1/f)$$

Notes

All functions are positive. Non-Standard normalization, 1-logSob $\sim$ modified-logSob (Defined by Bakry, Wu mid 90s)

LogSob - Easy facts

Self-Dual Definition

For $p \neq 0, 1$: $\mathcal{E}(f^{p-1}, f) = \mathcal{E}(g^{1/p}, g^{p'}), \quad g = f^p$

$$p\text{-logSob(C)} \iff \forall g, Ent(g) \leq \frac{Cpp'}{4}\mathcal{E}(g^{1/p}, g^{1/p'})$$

$$\implies (p\text{-logSob(C)} \iff p'\text{-logSob(C)}).$$

1-logSob

Claim: If $L = I - \mathbb{E}$ then L is 1-logSob(4).

$$Ent(f) = \mathbb{E}f \log f - \mathbb{E}f \cdot \log \mathbb{E}f \leq \mathbb{E}f \log f - \mathbb{E}f \cdot \mathbb{E} \log f =$$
$$\mathbb{E}f(\log f - \mathbb{E} \log f) = \mathbb{E}fL \log f = \mathcal{E}(f, \log f).$$

Log-Sob Monotonicity

Main Thm: Monotonicity

p-logSob(C) $\implies$ q-logSob(C) for $0 \leq q \leq p \leq 2$

Log-Sob Monotonicity

Thm: Monotonicity

p-logSob(C) $\implies$ q-logSob(C) for $0 \leq q \leq p \leq 2$

Main applications

$p = 1, q < 1$ Gives $I - \mathbb{E}$ satisfies Rev. Hyp. Contraction

$p = 2, q < 1$ Gives Hyp. Contraction $\implies$ Rev. Hyp. Contraction

Comments

$1 = q \leq p = 2$ is due to Gross ($q > 1$), Bakry ($q = 1$) etc.

Recall p-logSob(C) $\iff \forall g, Ent(g) \leq \frac{Cpp'}{4}\mathcal{E}(g^{1/p}, g^{1/p'})$

Using continuity at 0 and 1 suffices to show the following

Comparison of Dirichlet forms

Thm: Generalized Stroock-Varopoulos

For all $p > q$ with $p, q \in (0, 2] \setminus \{1\}$ and all $g > 0$:

$$qq'\mathcal{E}(g^{1/q}, g^{1/q'}) \geq pp'\mathcal{E}(g^{1/p}, g^{1/p'})$$

Comparison of Dirichlet forms

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For all $p > q$ with $p, q \in (0,2] \setminus \{1\}$ and all $g > 0$:

$$qq'\mathcal{E}(g^{1/q}, g^{1/q'}) \geq pp'\mathcal{E}(g^{1/p}, g^{1/p'})$$

Proof based on the following Lemma (Exercise):

$$\forall a, b > 0:$$

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'})$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
$$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply} T_t}$$
$$qq' T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq$$
$$pp' T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange}$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
$$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}\,T_t}$$
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$$pp'T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange}$$
$$qq'\left(T_t[g] - b^{1/q}T_t[g^{1/q'}] - b^{1/q'}T_t[g^{1/q}] + b\right) \geq$$
$$pp'\left(T_t[g] - b^{1/p}T_t[g^{1/p'}] - b^{1/p'}T_t[g^{1/p}] + b\right) \implies_{b->g}$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
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$$qq'\left(T_t[g] - g^{1/q} T_t[g^{1/q'}] - g^{1/q'} T_t[g^{1/q}] + g\right) \geq$$
$$pp'\left(T_t[g] - g^{1/p} T_t[g^{1/p'}] - g^{1/p'} T_t[g^{1/p}] + g\right) \implies \text{Taking } \mathbb{E}$$

Pf Sketch

$$
\begin{aligned}
& qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq \\
& pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}\, T_t} \\
& qq' T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq \\
& pp' T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange} \\
& qq'\left(T_t[g] - b^{1/q} T_t[g^{1/q'}] - b^{1/q'} T_t[g^{1/q}] + b\right) \geq \\
& pp'\left(T_t[g] - b^{1/p} T_t[g^{1/p'}] - b^{1/p'} T_t[g^{1/p}] + b\right) \implies_{b->g} \\
& qq'\left(T_t[g] - g^{1/q} T_t[g^{1/q'}] - g^{1/q'} T_t[g^{1/q}] + g\right) \geq \\
& pp'\left(T_t[g] - g^{1/p} T_t[g^{1/p'}] - g^{1/p'} T_t[g^{1/p}] + g\right) \implies \text{Taking } \mathbb{E} \\
& qq'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/q} T_t[g^{1/q'}]]\right) \geq \\
& pp'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/p} T_t[g^{1/p'}]]\right) \implies
\end{aligned}
$$

Pf Sketch

$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$
$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}\, T_t}$
$qq' T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq$
$pp' T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies$ Rearrange
$qq'\left(T_t[g] - b^{1/q} T_t[g^{1/q'}] - b^{1/q'} T_t[g^{1/q}] + b\right) \geq$
$pp'\left(T_t[g] - b^{1/p} T_t[g^{1/p'}] - b^{1/p'} T_t[g^{1/p}] + b\right) \implies_{b->g}$
$qq'\left(T_t[g] - g^{1/q} T_t[g^{1/q'}] - g^{1/q'} T_t[g^{1/q}] + g\right) \geq$
$pp'\left(T_t[g] - g^{1/p} T_t[g^{1/p'}] - g^{1/p'} T_t[g^{1/p}] + g\right) \implies$ Taking $\mathbb{E}$
$qq'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/q} T_t[g^{1/q'}]]\right) \geq$
$pp'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/p} T_t[g^{1/p'}]]\right) \implies$
Note that $t = 0$ equality holds. Therefore
$\frac{d}{dt}_{|t=0} LHS \geq \frac{d}{dt}_{|t=0} RHS \implies$
$qq'\mathcal{E}(g^{1/q}, g^{1/q'}) \geq pp'\mathcal{E}(g^{1/p}, g^{1/p'})$

Log-Sob $\iff$ (Rev) Hyper-contraction

Log Sob $\implies$ Rev-Hyper-Contraction

Proposition: r-Logsob(C) $\implies \|T_t f\|_q \geq \|f\|_p$ for all $f > 0$ and $r' \leq q \leq p \leq r$ if $t \geq \frac{C}{4} \log \frac{1-q}{1-p}$.

Proof Sketch

Assume: $0 < q \leq p \leq r$

$$\text{Let } t(q) = \frac{C}{4} \log \frac{1-q}{1-p}, \quad t(p) = 0, \quad q^2 t'(q) = \frac{Cq^2}{4(q-1)}$$

Let $\psi(q) = \|T_{t(q)} f\|_q$. Note that $\psi(p) = \|f\|_p$

$$\frac{d}{dq} \log \|T_{t(q)} f\|_q = \frac{Ent(f_{t(q)}^q) - q^2 t'(q) \mathcal{E}(f_{t(q)}^{q-1}, f_{t(q)})}{q^2 \mathbb{E} f_{t(q)}^q} \leq 0,$$

since q-logSob(C) holds which follows from r-logSob(C) in turn. The case $r' \leq q \leq p < 0$ follows by duality. The remaining cases follow by taking limits, duality and composition

Log-Sob $\iff$ (Rev) Hyper-contraction

Log Sob $\implies$ Rev-Hyper-Contraction

$\exists C, \|T_{\frac{C}{4}\log\frac{1-q}{1-p}} f\|_q \geq \|f\|_p \; \forall 0 < q < p \leq r \implies \text{r}-logSob(\text{C})$

Remark

Similar results hold for hypercontractivity

Open Problems

Tighter values?

Find tight bounds for simple/general Markov operators

For simple operators we get $\|T_t f\|_q \geq \|f\|_p$ for $q < p \leq 0$ and $t \geq \log \frac{2-q}{2-p}$ and also for $0 \leq q < p < 1$ and $t \geq \log \frac{(1-q)(2-p)}{(1-p)(2-q)}$

Applications?

Applications of strong mixing properties of Markov-chains?

Examples?

Are there examples where r-logSob holds while r'-logSob does not hold for $0 < r < r' < 1$?