Ricci on graphs and functional inequalities

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I] Functional inequalities on graphs

II] Ricci in the continuous X obstructions

III] Construction with W_1 (Dobrushin – Jordan – Ollivier – Sammer)

IV] Erbar – Maas

Survey → Gozlan – Melbourne – Perkins – R. – Tetali

I] 1) Definitions: $G=(V,E)$ finite graph, connected (undirected)

$\mathcal{P}(G)=\{$prob. measures on $G\}$.

$K=\big(K(x,y)\big)_{x,y\in V}$ is a Markov kernel if: $K(x,y)\geq 0$

$\sum_y K(x,y)=1 \ \forall x$.

$K(x,y)=0$ if $x\not\sim y$

Assume for simplicity $K(x,x)>0$

Def: Fix $C>0$, $\mu\in\mathcal{P}(G)$ and a Markov kernel K, reversible w.r.t. μ (i.e. $\forall x,y \ \mu(x)K(x,y)=\mu(y)K(y,x)$)

Then,

i) (μ,K) satisfies the Poincaré inequality with constant C if for any $f: V\to\mathbb{R}$, it holds:

$$\mathrm{Var}_\mu(f)\leq C\,\mathcal{E}_{\mu,K}(f,f)$$

where $\mathrm{Var}_\mu(f)=\mu(f^2)-\mu(f)^2$

and $\mathcal{E}_{\mu,K}(f,g)=-\mu(f\,Lg)$ with $L=I-K$.

$$=\frac{1}{2}\sum_{x,y}\mu(x)K(x,y)\,(g(y)-g(x))(f(y)-f(x))$$

(ii) (μ,K) satisfies the modified log-Sobolev inequality with constant C if for all $f: V\to\mathbb{R}_+^*$, it holds:

$$\mathrm{Ent}_\mu(f)\leq C\,\mathcal{E}_{\mu,K}(f,\ln f)$$

where $\mathrm{Ent}_\mu(f)=\int f\log\frac{f}{\int f\,d\mu}\,d\mu$

(iii) (μ, K) satisfies the log-Sobolev inequality with constant C if for any $f: V \to \mathbb{R}$, it holds:

$$\mathrm{Ent}_\mu(f^2) \leq C\, \mathcal{E}_{\mu,K}(f,f)$$

The smallest C in (i) denoted by C_P

(ii) C_{MLS}

(iii) C_{LS}

Rk: In the continuous setting, for instance for a diffusion, $\mathcal{E}(f,g) = \int \nabla f \nabla g \, d\mu$.

Then (ii) and (iii) are equivalent up to some factor (universal)

2) Two basic properties

Prop I.1: Let $\mu, K, C_P, C_{MLS}, C_{LS}$ as before. Then,

$$0 \leq C_P \leq 2C_{MLS} \leq \frac{C_{LS}}{2}.$$

. Furthermore, either $C_P = \frac{C_{LS}}{2}$, or there exists $f: V \to \mathbb{R}_+^*$, with $\mu(f^2) = 1$, non constant and solution of

(i) $L f_0 = -\frac{2}{C_{LS}} f_0 \ln f_0$

. Similarly, either $C_P = 2C_{MLS}$ or there exists $f_1: V \to \mathbb{R}_+^*$ non constant, with $\mu(f_1) = 1$, which is solution of:

(ii) $L f_1 + f_1 L \log f_1 = -\frac{1}{2C_{MLS}} f_1 \log f_1$

Rk: (i) comes from Diaconis & Saloff-Coste '96

(ii) ——— Bobkov & Tetali

Proof: $(a-b)^2 \leq \frac{1}{4}(a^2-b^2)(\log a^2 - \log b^2) \quad a,b>0$

shows that $\mathcal{E}_{\mu,K}(f,f) \leq \frac{1}{4} \mathcal{E}_{\mu,K}(f^2 \log f^2)$ ~~for f>0~~
for $f>0$.

It implies $C_{MLS} \leq \frac{C_{LS}}{4}$.

Take $f = 1 + \varepsilon g$ with $\mu(g) = 0$

$$\mathrm{Ent}_\mu(f) = \frac{\varepsilon^2}{2}\mathrm{Var}_\mu(g) + o(\varepsilon^2)$$

$$\mathcal{E}_\mu(f, f) = \varepsilon^2 \mathcal{E}_\mu(g, g)$$

It implies $C_P \leq 2 C_{MLS}$ $\boxed{}$.

Prop. I.? (tensorisation property)

· Consider n graphs $(G_i)_{i=1}^n = (V_i, E_i)_{i=1}^n$, n probability measures $\mu_i \in \mathcal{P}(G_i)$, n Markov kernels $(K_i)_{i=1}^n$ on G_i, with K_i reversible w.r.t. μ_i.

Let $\alpha_1, \dots, \alpha_n$ be s.t. $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \in [0,1]$ $\forall i$.

Then the product $\bigotimes_{i=1}^n \mu_i$ and the Markov kernel

$$K(x,y) = \sum_{i=1}^n \alpha_i \, \delta_{(x_1,y_1)} \cdots \delta_{(x_{i-1},y_{i-1})} K_i(x_i,y_i) \, \delta_{(x_{i+1},y_{i+1})} \cdots \delta_{(x_n,y_n)}$$

satisfies:

$$\min_i \{\alpha_i^{-1} C_P^i\} \leq C_P \leq \max_i \{\alpha_i^{-1} C_P^i\}$$

$$\min_i \{\alpha_i^{-1} C_{MLS}^i\} \leq C_{MLS} \leq \max_i \{\alpha_i^{-1} C_{MLS}^i\}$$

$$\min_i \{\alpha_i^{-1} C_{LS}^i\} \leq C_{LS} \leq \max_i \{\alpha_i^{-1} C_{LS}^i\}.$$

Proof: Easy consequence of:

$$\mathcal{E}_{\mu,K}(f,g) = \sum_{i=1}^n \alpha_i \sum_{\hat{x}^i} \mathcal{E}_i(f,g)(\hat{x}^i)\,\hat{\mu}^i(\hat{x}^i)$$

where $\hat{x}^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ $\quad \hat{\mu}^i = \bigotimes_{j \neq i} \mu_j$

and $\mathcal{E}_i(f,g)(\hat{x}^i) = \sum_{x_i, y_i} \mu_i(x_i) K_i(x_i, y_i) \big(f(\underbrace{\hat{x}^i x_i}_{(x_1, \dots, x_n)}) - g(\underbrace{\hat{x}^i y_i}_{(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots}) \dots$

$$\mathrm{Var}_\mu(f) \le \sum_{i=1}^n \mu\left[\mathrm{Var}_{\mu_i}(f)\right]$$

$$\mathrm{Ent}_\mu(f) \le \mu\left(\sum_i \mathrm{Ent}_{\mu_i}(f)\right)$$

<u>Examples</u>: a) Complete graph K_n:

$K(x,y) = \frac{1}{n} \quad \forall x,y$ reversible w.r.t $\mu \equiv \frac{1}{n}$.

$$\mathcal{E}_{\mu,K}(f,g) = \frac{1}{2}\sum_{x,y} \mu(x)\mu(y)\,(f(y)-f(x))(g(y)-g(x))$$

<u>Prop I.3)</u>: $C_P = 1$, $\frac{1}{2} \le C_{MLS} \le 1$, $C_{LS} = \frac{n\log(n-1)}{n-2}$.

$\left(\text{with } C_{LS} = \lim_n \frac{n\log(n-1)}{n-2} = 2 \text{ for } n=2\right)$

<u>Proof</u>: $\mathrm{Var}_\mu(f) = \mathcal{E}_\mu(f,f)$! $\to C_P = 1$

$$\mathrm{Ent}_\mu(f) = \int f\log f\,d\mu - \int f d\mu \ln \int f d\mu$$

$$\underset{\text{Jensen}}{\le} \int f\left(\log f - \int \log f\,d\mu\right)d\mu$$

$$\underset{\text{reversibility}}{=} \frac{1}{2}\sum_{x,y}(f(x)-f(y))(\log f(x) - \log f(y))\mu(x)\mu(y)$$

$$= \mathcal{E}_{\mu,K}(f,\log f) \quad \rightsquigarrow C_{MLS} \le 1.$$

<u>Rk</u>: in fact, the inequality is strict if f is not constant, thus, is $C_{MLS} < 1$?

For C_{LS}, see Diaconis & Saloff-Coste 96

<u>idea</u>: (*) $C_{LS} = \left(\frac{p-q}{\log p - \log q}\right)^{-1}$ on the two points $(\mu = \mathcal{B}(p))$

. show that $1 = C_P < \frac{C_{LS}}{2}$ (take a test function)

This implies that $Lf_0 = -\frac{2}{C_{LS}} f_0 \log f_0$ for some nontrivial f_0

- f_0 takes only 2 values
- use ⊛ with $p = \min_{x \in K_n} \mu(x) = \frac{1}{n}$

b) Hypercube $Q_n = \{0,1\}^n$

~~...~~ $K(x,y) = \frac{1}{2n}$ if $x \sim y$
$= \frac{1}{2}$ if $x = y$

K reversible w.r.t $\mu \equiv \frac{1}{2^n}$

K corresponds to the product of n Markov kernels on $\{0,1\}$ defined by $K_i(x_i, y_i) = \frac{1}{2}$ $\forall x,y \in \{0,1\}$ (reversible w.r.t $\mathcal{B}(\cdot$
and $\alpha_i = \frac{1}{n}$.

Prop°: $c_P = n = 2C_{MLS} = \frac{C_{LS}}{2}$

4) Convergence to stationarity.

$$P_t = e^{-t(I-K)} = e^{-t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} K^n$$

Thm: Given K reversible w.r.t $\mu \in \mathcal{P}(G)$. The following are equivalent:

(i) $\mathrm{Var}_\mu(f) \leq C\, \mathcal{E}_{\mu,K}(f,f)$ $\forall f$

(ii) $\mathrm{Var}_\mu(P_t f) \leq e^{-\frac{2t}{C}} \mathrm{Var}_\mu(f)$ $\forall f$ $\forall t \geq 0$.

The following are equivalent:

(i) $\mathrm{Ent}_\mu(f) \leq C\, \mathcal{E}_{\mu,K}(f, \ln f)$ $\forall f > 0$.

(ii) $\mathrm{Ent}_\mu(P_t f) \leq e^{-\frac{t}{C}} \mathrm{Ent}_\mu(f)$

Proof: $\mu(f)=0$

$$\frac{d}{dt}\mathrm{Var}_\mu(P_t f) = \frac{d}{dt}\mu\left((P_t f)^2\right) = 2\mu(P_t f\, L P_t f) = -2\mathcal{E}_{\mu,L}(P_t f, P_t f)$$

$$\leq -\frac{2}{C}\mathrm{Var}_\mu(P_t f) \quad \text{then Gronwall.}$$

$\mu(f)=1$

$$\frac{d}{dt}\mathrm{Ent}_\mu(P_t f) = \int P_t f\, L \log P_t f\, d\mu + \underbrace{\int P_t f\, \frac{L P_t f}{P_t f}\, d\mu}_{=0}$$

$$= -\mathcal{E}_\mu(P_t f, \log P_t f)$$

$$\leq -\frac{1}{C}\mathrm{Ent}_\mu(f) \quad \text{then Gronwall}$$

Rk: for any $f>0$, $\mathrm{Ent}_\mu(f) \leq \dfrac{\mathrm{Var}_\mu(f)}{\mu(f)}$.

Indeed $\mathrm{Ent}_\mu(f) = \int f \log(f+1-1)\, d\mu \leq \int f(f-1)\, d\mu = \mathrm{Var}_\mu(f)$

Notably, $\mathrm{Ent}_\mu(P_t f) \leq \dfrac{\mathrm{Var}_\mu(P_t f)}{\mu(f)} \leq \dfrac{e^{-\frac{2t}{C_P}}\mathrm{Var}_\mu(f)}{\mu(f)}$.

for the exponential decay of the entropy, this is good.
But if f is ill-behaved, $\mathrm{Var}_\mu(f)$ can be much larger than $\mathrm{Ent}_\mu(f)$ ...

Prop°: Assume there exists $C>0$ such that for all f, there exists C_f such that $\mathrm{Var}_\mu(P_t f) \leq e^{-\frac{2t}{C}} C_f$ $\forall t>0$.
Then $C_P \leq C$ and $\mathrm{Var}_\mu(P_t f) \leq e^{-\frac{2t}{C}}\mathrm{Var}_\mu(f)$

This is not true for the entropy (see Caputo - Dai Pra - Posta)

Proof: Let $\phi : \begin{cases} [0,+\infty[\longrightarrow \mathbb{R} \\ t \longmapsto \log \mathrm{Var}_\mu(P_t f) \end{cases}$ with $\mu(f)=0$.

$$\phi'(t) = 2\,\frac{\mu(P_t f\, L P_t f)}{\mu((P_t f)^2)}$$

$$\phi''(t) = \frac{4}{\mu((P_t f)^2)}\Big[\mu((LP_t f)^2)\,\mu((P_t f)^2) - \mu(P_t f\, L P_t f)^2\Big] \geq 0$$

C-S.

$$\phi'(0) \leq \frac{\phi(t)-\phi(0)}{t} \quad \forall t>0.$$

$$-\frac{2\,\mathcal{E}_\mu(f,f)}{\mathrm{Var}_\mu(f)} \leq \frac{\log \mathrm{Var}_\mu(P_t f)}{t} - \frac{\phi(0)}{t} \leq -\frac{2}{C} + \frac{\log C_f}{t} - \frac{\phi(0)}{t}$$

$$\xrightarrow{t \to +\infty} -\frac{2}{C}$$

$\blacksquare$

II) Ricci in the continuous setting

1) few equivalent formulations of $CD(K,\infty)$

M is a smooth compact, finite dimensional, separable, Riemannian manifold

$\phi : M \longrightarrow \mathbb{R}$ of class C^2, $\int e^{-\phi}\,\mathrm{vol} = 1$

- $\mu = e^{-\phi}\,\mathrm{vol}$ $\qquad L = \Delta - \nabla\phi\cdot\nabla$ $\qquad (P_t)_{t\geq 0}$

$$\Gamma(f,g) = \frac{1}{2}\big[L(fg) - fLg - gLf\big]$$

$$\Gamma_2(f,g) = \frac{1}{2}\big[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big]$$

$$\Gamma(f,f) = |\nabla f|^2$$

$$\Gamma_2(f,f) = \|\mathrm{Hess}\, f\|_{HS}^2 + (\nabla f)^T\, \mathrm{Hess}\,\phi\,(\nabla f) + \mathrm{Ricci}(\nabla f, \nabla f)$$

Def°: for $K \in \mathbb{R}$, Then $CD(K,\infty)$ holds if
$$\Gamma_2(f,f) \geq K\,\Gamma(f,f) \quad \forall f \text{ smooth.}$$

Ex: $M = \mathbb{R}^n$, $\phi(x) = \dfrac{|x|^2}{2} + \dfrac{n}{2}\ln(2\pi)$

($|\cdot|$ is Euclidean norm in $\mathbb{R}^n$)

$$\Gamma_2(f,f) = \|\mathrm{Hess} f\|_{HS}^2 + (\nabla f)^T Id_{\mathbb{R}^n}(\nabla f) + 0$$
$$\geq |\nabla f|^2 = \Gamma(f,f).$$

Hence $CD(1,+\infty)$ holds.

Thm: Given $\mu_0, \mu_1 \in \mathcal{P}(M)$, then there exists a continuous curve $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}(M), W_2)$ interpolating between μ_0 and μ_1 s.t. $W_2(\mu_s,\mu_t) = |t-s|\, W_2(\mu_0,\mu_1) \quad \forall\, 0 \leq t, s \leq 1$.
Such a path is called a constant speed geodesic in $(\mathcal{P}(M), W_2)$.
Furthermore, if μ_0 and $\mu_1 \in \mathcal{P}^{ac}(M)$ then $\mu_t \in \mathcal{P}^{ac}(M)\ \forall t$.

Proof: Villani's book, Corollary 7.22 and Thm 8.7.

Def° (Displacement K-convexity of $H(\cdot|\mu)$)
For $K \in \mathbb{R}$, Then $H(\cdot|\mu)$ is said to be K-displacement convex if for any constant speed geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}(M), W_2)$ it holds:
$$H(\mu_t|\mu) \leq (1-t)\,H(\mu_0|\mu) + t\,H(\mu_1|\mu) - \frac{K}{2}t(1-t)\,W_2^2(\mu_0,\mu_1)$$
$\forall t \in [0,1]$