Thm: (equivalent formulation of $CD(K,\infty)$)

For $K \in \mathbb{R}$. The following are equivalent:

i) $CD(K,\infty)$

ii) $|\nabla P_t f| \le e^{-Kt} P_t |\nabla f| \quad \forall f$

iii) $H(\cdot|\mu)$ is K-displacement convex

iv) The following EVI holds:

$$\frac{1}{2}\frac{d^+}{dt} W_2^2(\mu_t,\nu) + \frac{K}{2} W_2^2(\mu_t,\nu) \le H(\nu|\mu) - H(\mu_t|\mu)$$

$\forall \mu_0, \nu \in \mathcal{P}^{ac}(M)$, where $\mu_t = P_t f_0 \mu$ and $\mu_0 = f_0\mu$.

(this is a rewording of the fact that μ_t is the gradient flow of the entropy w.r.t W_2)

v) $\forall \mu_0, \mu_0' \in \mathcal{P}^{ac}(M)$, it holds:

$$W_2(\mu_t, \mu_t') \le e^{-Kt} W_2(\mu_0, \mu_0')$$

where $\mu_t = P_t f_0 \mu$ and $\mu_t' = P_t f_0' \mu$, $\mu_0 = f_0\mu$, $\mu_0' = f_0'\mu$.

Von-Renesse-Sturm 05: if $\phi = 0$ in (v), W_2 can be replaced by W_1.

Thm: Assume $CD(K,\infty)$, then:

HWI inequality: $\forall f$ s.t. $\mu(f) = 1$, it holds:

$$\mathrm{Ent}_\mu(f) \le W_2(f\mu,\mu)\sqrt{\int \frac{|\nabla f|^2}{f} d\mu} - \frac{K}{2} W_2^2(f\mu,\mu).$$

HWI implies Log-Sobolev: if $K>0$, $\mathrm{Ent}_\mu(f) \le \frac{2}{K}\int \frac{|\nabla f|^2}{f} d\mu$.

Talagrand's inequality: if $K>0$, then

$$W_2^2(\nu,\mu) \le \frac{2}{K} H(\nu|\mu) \quad \forall \nu \in \mathcal{P}(M)$$

Poincaré ($C_P \le \frac{1}{K}$)

if $d\mu(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{1}{2}}} dx$, then $K=1$ is optimal in (i)-(iv).

<u>Rq</u>: **There** is a symmetric version of HWI which is equivalent to $CD(K,\infty)$.

2) Obstructions on graphs:

- Chain rule does not hold
- There is no geodesic in W_2
- Talagrand's inequality holds on graphs iff μ is a Dirac mass:

$$W_2^2(\nu,\mu) \geq W_1(\nu,\mu) \geq \frac{1}{2}\|\nu-\mu\|_{TV}.$$

Assume that ν has support containing at least two points $x, y \in V$. Consider $\nu \ll \mu$ and $\nu = f\mu$. Then:

$$\frac{1}{2}\int|f-1|d\mu \leq W_2^2(\nu,\mu) \leq C\,H(\nu|\mu) = C\int f\log(f+1-1)d\mu$$
$$\leq C\int f(f-1)\,d\mu = C\int(f-1)^2 d\mu.$$

$\rightarrow$ Take $f = 1+\varepsilon g$ with $\mu(g)=0$, ε small s.t. $f>0$ and $\int|g|d\mu$

$g: \{x,y\} \rightarrow \mathbb{R}$.

$\rightsquigarrow \varepsilon \leq 2C\varepsilon^2\int g^2 d\mu$. a contradiction.

3) <u>Changes in the litterature</u>:

i) Schmuckenschlager 98 / Lin-Yau (preprint 2012)

Bauduo - Caputo - Dai Pra - Posta 00 }
______ 09 } Prove Poincaré by showing that $\Gamma_2 \geq \rho\Gamma$

Ané-Ledoux 00

ii) Joulin 07 09 (deviation bounds) / Joulin-Ollivier **10**

iii) Maas **11** / Erbar-Maas **12-14** }
Mielke 11-13 } iv)

- Bonciocat - Sturm 09 / Bonciocat 13

  related works: Hillion 12 - Johnson-Hillion 13 - Leonard 13.

  Gozlan - R - Samson - Tetali

  Gigli - Maas 13

2) Contraction in W_1:

Ollivier 09 - Sammer 05 - Joulin 07-09

Joulin - Ollivier 10   Ollivier (survey 10-13)

Veysseire 12 - Dobrushin 70.

Related works: Ollivier - Villani 12

Lin - Lu - Yau 11

Jost - Liu

Bauer - Liu (preprint 13)

Proba: Mu Fa Chen 96. (ν) with W_1 ~~to~~ other Poincaré inequalities)

Bob Dyer 00 (path coupling).

To be investigated: Fan - Chung - Yau 96

Bauer.

III Contraction in W_1

$$W_1(\nu,\mu) = \inf_{\pi \nearrow \nu \searrow \mu} \left\{ \int_V d(x,y)\, d\pi(x,y) \right\}$$

$$= \sup_{f \text{ 1-Lipschitz}} \left\{ \int f d\nu - \int f d\mu \right\}.$$

1) Def.

Def°: Let K be a Markov kernel on $G=(V,E)$.

Given $x,y \in V$, the coarse Ricci curvature of (G,K) at (x,y) is:

$$\kappa(x,y) = 1 - \frac{W_1(K_x, K_y)}{d(x,y)}$$

where $K_x(z) = K(x,y)$ i.e. $K_x = \delta_x K$.

$$\boldsymbol{\kappa}(G,K) := \inf_{x,y \in V} \kappa(x,y).$$

Prop III 1) : Assume that $\kappa(x,y) \geq \boldsymbol{\kappa} \in \mathbb{R}$ for all $(x,y) \in E$.

Then $\kappa(x,y) \geq \kappa$ $\forall x,y$. In particular:

$$\kappa(G,K) = \inf_{(x,y) \in E} \kappa(x,y).$$

Proof: triangular inequality for W_1 along a geodesic ... $\boxed{\kappa}$

Prop° III 2) Tensorization $(G^{i})_{i=1}^{n}$ $(K^{(i)})_{i=1}^{n}$ $\sum_i \alpha_i = 1$

$$\kappa(G,K) \geq \inf_{i=1,\dots,n} \alpha_i \kappa(G^{(i)}, K^{(i)}).$$

Proof: Convexity of W_1 in both variables.

2) Examples :

a) Complete graph : $K(x,y) = \frac{1}{n} \quad \forall x,y.$

Prop: $\kappa(x,y) = 1 \quad \forall x,y \qquad \kappa(K_n, K) = 1.$

b) Hypercube : $\Omega_n = \{0,1\}^n$

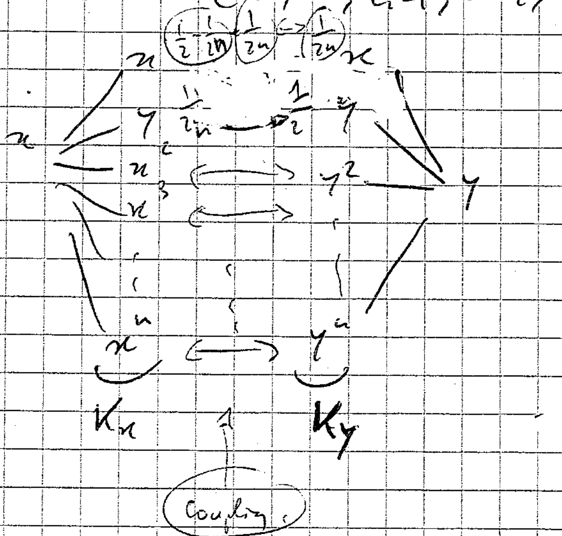
$$K(x,y) = \frac{1}{2n} \quad \text{if } x \sim y$$

$$K(x,x) = \frac{1}{2}$$

Prop° : $\forall x \sim y \quad \kappa(x,y) = \frac{1}{n}. \qquad \kappa(\Omega_n, K) = \frac{1}{n}.$

Proof : $x = (0, \dots, 0)$, $y = (1, 0, \dots, 0)$.

$$z^i = (z_1, \dots, z_{i-1}, 1-z_i, z_{i+1}, \dots, z_n)$$



$$W_1(K_x, K_y) \le (n-1) \times \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n} = 1 - \frac{1}{n}.$$

$$W_1(K_x, K_y) = \sup_{f \text{ 1-Lip}} \left\{ K_x(f) - K_y(f) \right\}$$

Take $f(x) = x_1$. $\rightarrow K_x(f) = 1 - \frac{1}{2n}$

$K_y(f) = \frac{1}{2n}.$

$$\rightarrow W_1(K_x, K_y) \geq 1 - \frac{1}{n}$$

IV Erbar-Maas' Ricci curvature

1) Definition:

$$\mathcal{D}(V) = \{\rho : V \to \mathbb{R}_+ \ / \int \rho \, d\mu = 1\}$$
$$\mathcal{D}_*(V) = \{\rho \in \mathcal{D}(V), \ \rho > 0\}$$

Given $\rho_1, \rho_2 \in \mathcal{D}(V)$

$$W^2(\rho_0, \rho_1) := \inf_{(\rho_t)_{t\in[0,1]}, (\psi_t)_{t\in[0,1]}} \frac{1}{2}\int_0^1 \sum_{x,y\in V} (\psi_t(y) - \psi_t(x))^2 \, \theta(\rho_t(x), \rho_t(y)) \, K(x,y)\mu(y) \, dt$$

where the inf. runs over all (ρ_t) of class C^∞, and all ψ_t measurable satisfying the ~~continuity~~ continuity equation:

$$\begin{cases} \dfrac{d}{dt}\rho_t(x) + \sum_{y\in V} (\psi_t(y) - \psi_t(x))\,\theta(\rho_t(x), \rho_t(y))\,K(x,y) = 0, \ \forall x \\ \rho_0 = \rho_0 \quad \text{and} \quad \rho_1 = \rho_1. \end{cases}$$

and $\theta(r,s) = \int_0^1 r^t s^{1-t} \, dt = \begin{cases} \dfrac{r-s}{\log r - \log s} & \text{if } r \neq s \\ r & \text{if } r = s. \end{cases}$

Rq: $u(t,x) := P_t f(x) \qquad P_t = e^{-tL}$

$\rho_t = u$ and $\psi_t = -\log u$

$\partial_t u = Lu$

- $\forall \rho_0, \rho_1 \in \mathcal{D}(V)$, there exists a geodesic $(\rho_t)_{t\in[0,1]}$ joining ρ_0 to ρ_1 in $(\mathcal{D}(V), W)$: $W(\rho_s, \rho_t) = |t-s|\, W(\rho_0, \rho_1)$.

Def°: Fix $\mu \in \mathcal{P}(V)$ with $\mu > 0$ and K a reversible Markov kernel.
Then (G, K) has Ricci curvature bounded from below by $\kappa \in \mathbb{R}$
if for all one speed geodesics $(\rho_t)_{t\in[0,1]}$ in $(\mathcal{D}(V), W)$,
it holds:

$$\mathrm{Ent}_\mu(\rho_t) \leq (1-t)\, \mathrm{Ent}_\mu(\rho_0) + t\, \mathrm{Ent}_\mu(\rho_1) - \frac{\kappa}{2} t(1-t) W(\rho_0, \rho_1)^2 .$$

Prop: tensorization : $(G_i)_{i=1}^n$, $(K_i)_{i=1}^n$, $\mu^{(i)}$ –

$$\mathrm{Ric}(G, K) \geq \min_i \{ \alpha_i\, \mathrm{Ric}(G^{(i)}, K^{(i)}) \}.$$

Proof: based on the mapping representation by Bourdon – Caputo – Dai Pra – Costa.

2) Examples:

a) The two-point space $\Omega_1 = \{0,1\} = K_2$

$\Omega_1 = \{0,1\} = K_2 \qquad K(x,y) = \frac{1}{2} \ \forall x,y \qquad \mu = \frac{1}{2}$.

$\nu_\beta := \frac{1}{2}\left[(1-\beta)\delta_0 + (1+\beta)\delta_1\right] \qquad$ for $\beta \in [-1,1]$.

$\nu_\beta = \rho^\beta \mu \qquad$ with $\rho^\beta(0) = 1-\beta$, $\rho^\beta(1) = 1+\beta$.

Prop°: Let $-1 \leq \alpha \leq \beta \leq 1$ and ρ^α and ρ^β the corresponding densities.
Then:

i) $$W(\rho^\alpha, \rho^\beta) = \int_\alpha^\beta \sqrt{\frac{\operatorname{argth}(r)}{r}}\, dr$$

ii) There exists a unique one speed geodesic $(\rho^{\gamma(t)})_{t\in[0,1]}$ joining ρ^α to ρ^β (in $(\mathcal{D}(\Omega_1), W)$). Moreover γ is the solution of $$\gamma'(t) = W(\rho^\alpha, \rho^\beta) \sqrt{\frac{\gamma(t)}{\operatorname{argth} \gamma(t)}} \quad \text{with } \gamma(0) = \alpha$$

iii) $\mathrm{Ric}(\Omega_1, K) \geq 1$

Corollary: on $\Omega_n = \{0,1\}^n$, $K(x,y) = \frac{1}{2n}$ if $x \sim y$ and $K(x,x) = \frac{1}{2}$, $\mu = \frac{1}{2^n}$. Then $Ric(\Omega_n, K) \geq \frac{1}{n}$.

Proof: Part i):

Let (ρ_t) and (ψ_t) [illegible] of the continuity equations.

$\rho_t(0) + \rho_t(1) = 2$

$$\theta(\rho_t(1), \rho_t(0)) = \theta(2-\rho_t(0), \rho_t(0)) = \frac{2-2\rho_t(0)}{\log(2-\rho_t(0)) - \log \rho_t(0)}$$

> 0 if $\rho_t(0) \notin \{0,2\}$

if $\rho_t(0) \in \{0,2\}$ then $\theta(\rho_t(0), \rho_t(1)) = 0$.

Hence $$(\psi_t(1) - \psi_t(0))^2 = 4 \frac{\dot{\rho_t}(0)^2}{\theta(\rho_t(0), 2-\rho_t(0))^2} \quad \text{when } \rho_t(0) \neq 0,2$$

$$W(\rho^\alpha, \rho^\beta) = \inf_{\substack{\gamma : [0,1] \to [0,2] \\ \gamma(0) = 1+\alpha \\ \gamma(1) = 1+\beta}} \sqrt{\int_0^1 \frac{\dot\gamma^2(t)}{\theta(\gamma(t), 2-\gamma(t))} \mathbb{1}_{\gamma(t) \neq 0,2} \, dt}$$

$$\frac{\dot\gamma(t)}{\theta(\gamma(t), 2-\gamma(t))} = \frac{\dot{\tilde\gamma}(t)}{-2\tilde\gamma} \log \frac{1-\tilde\gamma(t)}{1+\tilde\gamma(t)} = \frac{\dot{\tilde\gamma}(t)}{\tilde\gamma(t)} \operatorname{argth}(\tilde\gamma(t))$$

$$\operatorname{argth}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

$$W^2(\rho^\alpha, \rho^\beta) = \inf_{\substack{\gamma \\ \gamma(0)=\alpha \\ \gamma(1)=\beta \\ \gamma \nearrow}} \left\{ \int_0^1 \frac{\dot\gamma(t)^2}{\gamma(t)} \operatorname{argth}(\gamma(t)) \, dt \right\}.$$

$g(t) := \frac{\operatorname{argth} t}{t}$. The inf is reached at a unique curve ξ satisfying $2\ddot\xi(t) g(\xi(t)) + \dot\xi^2(t) g'(\xi(t)) = 0$

$t \longmapsto \xi'(t)\sqrt{g(\xi(t))}$ is constant.

$$\xi'(t)\sqrt{g(\xi(t))} = C \geq 0$$

$$\therefore \ldots \quad C = \int_\alpha^\beta \sqrt{g(r)}\,dr$$

Point (ii): let $\varphi(s) = \int_0^s \sqrt{\frac{\operatorname{argth} r}{r}}\,dr \quad s \in [-1,1]$

$$W(\rho^\delta, \rho^{\delta'}) = |\varphi(\delta) - \varphi(\delta')|$$

$$(\mathcal{D}(\Omega_1), W) \ni \rho^\delta \longmapsto \varphi(\delta) \in ([\varphi(-1), \varphi(1)], |\cdot|)$$ isometry.

~~[illegible]~~ Take γ solution of $\gamma'(t) = W(\rho^\alpha, \rho^\beta)\sqrt{\frac{\gamma(t)}{\operatorname{argth}(\gamma(t))}}$

with $\gamma(0) = 0$

$$\varphi(\gamma(t)) - \varphi(\gamma(s)) = \int_s^t \varphi'(\gamma(r))\gamma'(r)\,dr = W(\rho^\alpha, \rho^\beta)\int_s^t dr$$

$$= (t-s)\,W(\rho^\alpha, \rho^\beta)$$

$$\to W(\rho^{\gamma(t)}, \rho^{\gamma(s)}) = |t-s|\,W(\rho^\alpha, \rho^\beta)$$

This proves that $(\rho^{\gamma(t)})_{t\in[0,1]}$ is the geodesic joining ρ^α to ρ^β

(since $\gamma(1) = \beta$)

Point iii): $\quad H(t) = \mathrm{Ent}_\mu(\rho^{\gamma(t)})$

$$H'(t) = W(\rho^\alpha, \rho^\beta)\,\underbrace{\frac{1}{2}\left(1 + \frac{\gamma(t)}{(1-\gamma(t)^2)\operatorname{argth}\gamma(t)}\right)}_{\text{inf } \frac{1}{2}\left(1 + \frac{x}{(1-x^2)\operatorname{argth} x}\right) = 1}$$

$$\geq W(\rho^\alpha, \rho^\beta)$$

Let $B(\rho,\psi) = \frac{1}{4}\sum_{x,y,z} (\psi(x)-\psi(y))^2 \Big(\partial_1\theta(\rho(x),\rho(y))(\rho(z)-\rho(x))K(x,z) + \partial_2\theta(\rho(x),\rho(y))(\rho(z)-\rho(y))K(y,z)\Big) K(x,y)\pi(x)$

$$- \frac{1}{2}\sum_{x,y,z} \Big(K(x,z)(\psi(z)-\psi(x)) - K(y,z)(\psi(z)-\psi(y))\Big)(\psi(x)-\psi(y))\hat{\rho}(x,y)K(x,y)\pi(x)$$

and $A(\rho,\psi) = \frac{1}{2}\sum_{x,y,z} (\psi(y)-\psi(x))^2\hat{\rho}(x,y)K(x,y)\pi(x)$.

Thm: TFAE:

i) $\mathrm{Ric}(G,K) \geq \kappa$

ii) $\forall \rho\in\mathcal{D}_*(V)$, $\forall \psi: V\to\mathbb{R}$,

$$B(\rho,\psi) \geq \kappa A(\rho,\psi)$$

iii) $EVI(\kappa)$ holds:

$$\frac{1}{2}\frac{d^+}{dt}W^2(P_t\rho,\rho') + \frac{\kappa}{2}W(P_t\rho,\rho')^2 \leq \mathrm{Ent}_\mu(\rho') - \mathrm{Ent}_\mu(P_t\rho)$$

$\forall t\geq 0$.

Prop: $\mathrm{Ric}(K_n, K) \geq \frac{1}{2}+\frac{1}{n}$.

$K_n \to$ complete graph. $K \to$ lazy RW.