On some extentions to the FKN theorem

Jacek Jendrej, Krzysztof Oleszkiewicz,
and Jakub O. Wojtaszczyk

Institute of Mathematics
University of Warsaw & Polish Academy of Sciences

Phenomena in high dimensions in geometric analysis,
random matrices, and computational geometry
Roscoff, June 25 – June 29, 2012

The FKN result

Theorem

There exists a universal constant $L > 0$ with the following property. For $f : \{-1,1\}^n \to \{-1,1\}$ let $\rho = \left(\sum_{A \subseteq [n] : |A| \geq 2} |\hat{f}(A)|^2 \right)^{1/2}$.

Then there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A \subseteq [n] : |A| \leq 1, A \neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 \ln(2/\rho),$$

$$|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 \ln(2/\rho).$$

In the result of Friedgut, Kalai, and Naor (2002) there was $O(\rho^2)$ instead of $O\left(\rho^4 \ln(2/\rho)\right)$. An elementary proof with the $2\rho^2 + o(\rho^2)$ bound was a bit later given by Kindler and Safra.

The improved bound is of the optimal order. It was also recently (independently) proved by O'Donnell.

The FKN result

Theorem

There exists a universal constant $L > 0$ with the following property. For $f : \{-1,1\}^n \to \{-1,1\}$ let $\rho = \left(\sum_{A\subseteq[n]:|A|\geq 2} |\hat{f}(A)|^2\right)^{1/2}$.

Then there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A\subseteq[n]:|A|\leq 1, A\neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 \ln(2/\rho),$$

$$|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 \ln(2/\rho).$$

In the result of Friedgut, Kalai, and Naor (2002) there was $O(\rho^2)$ instead of $O\left(\rho^4 \ln(2/\rho)\right)$. An elementary proof with the $2\rho^2 + o(\rho^2)$ bound was a bit later given by Kindler and Safra.

The improved bound is of the optimal order. It was also recently (independently) proved by O'Donnell.

Main idea of FKN

Friedgut, Kalai and Naor have shown that if the variance of the absolute value of a sum of weighted Rademacher variables is much smaller than the variance of the sum, then one of the summands dominates the sum. We provide a simple and elementary proof of this result, and extend it to a more general setting.

Consider a family of independent random variables $(X_i)_{i=1}^n$. It is easy to prove that if the distribution of their sum is supported on a set of cardinality 2 then all summands but one are constant a.s. In our paper we investigate stability of this phenomenon. Namely, we prove that if the distribution of the sum is concentrated around a two-point set then there exists $k \in \{1, 2, \ldots, n\}$ such that $\sum_{i:i \neq k} X_i$ is concentrated around some point. We provide various strict quantitative variants of this heuristic statement.

Main idea of FKN

Friedgut, Kalai and Naor have shown that if the variance of the absolute value of a sum of weighted Rademacher variables is much smaller than the variance of the sum, then one of the summands dominates the sum. We provide a simple and elementary proof of this result, and extend it to a more general setting.

Consider a family of independent random variables $(X_i)_{i=1}^n$. It is easy to prove that if the distribution of their sum is supported on a set of cardinality 2 then all summands but one are constant a.s. In our paper we investigate stability of this phenomenon. Namely, we prove that if the distribution of the sum is concentrated around a two-point set then there exists $k \in \{1, 2, \ldots, n\}$ such that $\sum_{i:i \neq k} X_i$ is concentrated around some point. We provide various strict quantitative variants of this heuristic statement.

Sample results: symmetric case

Theorem

Let $(X_i)_{i=1}^n$ be a sequence of independent symmetric random variables. Then for some $k \in \{1, 2, \ldots, n\}$ we have

$$\operatorname{Var}\Big(\sum_{i \le n: i \neq k} X_i\Big) \le C \cdot \inf_{x \in \mathbb{R}} \operatorname{Var}\Big(\Big|x + \sum_{i \le n} X_i\Big|\Big),$$

where C is a universal constant. The result holds true with $C = (7 + \sqrt{17})/2 \approx 5.6$.

A simple example of $n = 3$ and X_1, X_2, X_3 i.i.d. symmetric ± 1 random variables indicates that the constant C cannot be less than $8/3 \approx 2.7$ even if we restrict to $x = 0$ only.

Sample results: symmetric case

Theorem

Let $(X_i)_{i=1}^n$ be a sequence of independent symmetric random variables. Then for some $k \in \{1, 2, \dots, n\}$ we have

$$\mathrm{Var}\Big(\sum_{i \le n: i \neq k} X_i \Big) \le C \cdot \inf_{x \in \mathbb{R}} \mathrm{Var}\Big(\Big| x + \sum_{i \le n} X_i \Big| \Big),$$

where C is a universal constant. The result holds true with $C = (7 + \sqrt{17})/2 \approx 5.6$.

A simple example of $n = 3$ and X_1, X_2, X_3 i.i.d. symmetric ± 1 random variables indicates that the constant C cannot be less than $8/3 \approx 2.7$ even if we restrict to $x = 0$ only.

Sample results: i.i.d. case

Theorem

Let ξ be a square-integrable random variable which is not constant a.s., and let $\xi_1, \xi_2, \ldots$ be its i.i.d. copies. Then there exists a constant K_ξ, depending only on the distribution of ξ, such that for any real numbers $a_1, a_2, \ldots, a_n$ there is some $k \leq n$ for which

$$\sum_{i \leq n : i \neq k} a_i^2 \leq K_\xi \cdot \inf_{x \in \mathbb{R}} \mathrm{Var}\left(\left| x + \sum_{i \leq n} a_i \xi_i \right| \right).$$

The proof of this theorem will be skipped in the presentation (due to lack of time). There are many other variants, some of them valid also for the Banach space valued random variables, with probability bounds replacing variance.

Key lemma

Lemma

Let X and Y be independent square-integrable random variables, at least one of them symmetric. Then

$$\min\Big(\mathrm{Var}(X), \mathrm{Var}(Y)\Big) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|X+Y|).$$

An example of X with distribution $\frac{1}{8}\delta_{-2} + \frac{3}{4}\delta_0 + \frac{1}{8}\delta_2$ and ± 1 symmetric Y indicates that the constant $(7+\sqrt{17})/4 \approx 2.78$ cannot be replaced by any number less than $16/7 \approx 2.29$.

Key lemma

Lemma

Let X and Y be independent square-integrable random variables, at least one of them symmetric. Then

$$\min\Big(\mathrm{Var}(X), \mathrm{Var}(Y)\Big) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|X+Y|).$$

An example of X with distribution $\frac{1}{8}\delta_{-2} + \frac{3}{4}\delta_0 + \frac{1}{8}\delta_2$ and ± 1 symmetric Y indicates that the constant $(7+\sqrt{17})/4 \approx 2.78$ cannot be replaced by any number less than $16/7 \approx 2.29$.

Proof of the key lemma

Obviously, $\mathbb{E}|X+Y|^2 = \mathbb{E}X^2 + \mathbb{E}Y^2$ because $\mathbb{E}XY = \mathbb{E}X \cdot \mathbb{E}Y = 0$. Since $|X+Y|$ and $|X-Y|$ have the same distribution, there is

$$\mathbb{E}|X+Y| = \mathbb{E}(|X+Y| + |X-Y|)/2 = \mathbb{E}\max(|X|,|Y|).$$

Hence

$$\mathrm{Var}(|X+Y|) = \mathbb{E}(X^2+Y^2) - (\mathbb{E}|X+Y|)^2 =$$
$$\mathbb{E}\Big((\max(|X|,|Y|))^2 + (\min(|X|,|Y|))^2\Big) - (\mathbb{E}\max(|X|,|Y|))^2 =$$
$$\mathrm{Var}(\max(|X|,|Y|)) + \mathrm{Var}(\min(|X|,|Y|)) + (\mathbb{E}\min(|X|,|Y|))^2 \geq$$
$$\frac{1}{2}\mathrm{Var}\Big(\max(|X|,|Y|) + \min(|X|,|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2 =$$
$$\frac{1}{2}\mathrm{Var}(|X|+|Y|) + (\mathbb{E}\min(|X|,|Y|))^2 =$$
$$\frac{1}{2}\Big(\mathrm{Var}(|X|) + \mathrm{Var}(|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2.$$

$\geq$:

$$2\mathrm{Var}(V) + 2\mathrm{Var}(W) = \mathrm{Var}(V+W) + \mathrm{Var}(V-W) \geq \mathrm{Var}(V+W).$$

Proof of the key lemma

Obviously, $\mathbb{E}|X+Y|^2 = \mathbb{E}X^2 + \mathbb{E}Y^2$ because $\mathbb{E}XY = \mathbb{E}X \cdot \mathbb{E}Y = 0$. Since $|X+Y|$ and $|X-Y|$ have the same distribution, there is

$$\mathbb{E}|X+Y| = \mathbb{E}(|X+Y| + |X-Y|)/2 = \mathbb{E}\max(|X|,|Y|).$$

Hence

$$\mathrm{Var}(|X+Y|) = \mathbb{E}(X^2+Y^2) - (\mathbb{E}|X+Y|)^2 =$$

$$\mathbb{E}\Big((\max(|X|,|Y|))^2 + (\min(|X|,|Y|))^2\Big) - (\mathbb{E}\max(|X|,|Y|))^2 =$$

$$\mathrm{Var}(\max(|X|,|Y|)) + \mathrm{Var}(\min(|X|,|Y|)) + (\mathbb{E}\min(|X|,|Y|))^2 \geq$$

$$\frac{1}{2}\mathrm{Var}\Big(\max(|X|,|Y|) + \min(|X|,|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2 =$$

$$\frac{1}{2}\mathrm{Var}(|X|+|Y|) + (\mathbb{E}\min(|X|,|Y|))^2 =$$

$$\frac{1}{2}\Big(\mathrm{Var}(|X|) + \mathrm{Var}(|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2.$$

$\geq$:

$$2\mathrm{Var}(V) + 2\mathrm{Var}(W) = \mathrm{Var}(V+W) + \mathrm{Var}(V-W) \geq \mathrm{Var}(V+W).$$

Proof of the key lemma

Obviously, $\mathbb{E}|X+Y|^2 = \mathbb{E}X^2 + \mathbb{E}Y^2$ because $\mathbb{E}XY = \mathbb{E}X \cdot \mathbb{E}Y = 0$. Since $|X+Y|$ and $|X-Y|$ have the same distribution, there is

$$\mathbb{E}|X+Y| = \mathbb{E}(|X+Y| + |X-Y|)/2 = \mathbb{E}\max(|X|,|Y|).$$

Hence

$$\mathrm{Var}(|X+Y|) = \mathbb{E}(X^2+Y^2) - (\mathbb{E}|X+Y|)^2 =$$
$$\mathbb{E}\Big((\max(|X|,|Y|))^2 + (\min(|X|,|Y|))^2\Big) - (\mathbb{E}\max(|X|,|Y|))^2 =$$
$$\mathrm{Var}(\max(|X|,|Y|)) + \mathrm{Var}(\min(|X|,|Y|)) + (\mathbb{E}\min(|X|,|Y|))^2 \geq$$
$$\frac{1}{2}\mathrm{Var}\Big(\max(|X|,|Y|) + \min(|X|,|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2 =$$
$$\frac{1}{2}\mathrm{Var}(|X|+|Y|) + (\mathbb{E}\min(|X|,|Y|))^2 =$$
$$\frac{1}{2}\Big(\mathrm{Var}(|X|) + \mathrm{Var}(|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2.$$

$\geq$:

$$2\mathrm{Var}(V) + 2\mathrm{Var}(W) = \mathrm{Var}(V+W) + \mathrm{Var}(V-W) \geq \mathrm{Var}(V+W).$$

Proof of the key lemma - auxiliary notation

We have proved:

$$\mathrm{Var}(|X+Y|) \geq \frac{1}{2}\Big(\mathrm{Var}(|X|) + \mathrm{Var}(|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2.$$

Thus $s \leq 2\sigma^2$ and $\mathbb{E}\min(|X|,|Y|) \leq (\sigma^2 - \frac{1}{2}s)^{1/2}$, where

$$\sigma = \Big(\mathrm{Var}(|X+Y|)\Big)^{1/2}$$

and

$$s = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|).$$

We want to prove:

$$\min\left(\mathrm{Var}(X), \mathrm{Var}(Y)\right) \leq \frac{7+\sqrt{17}}{4} \cdot \sigma^2.$$

Proof of the key lemma - auxiliary notation

We have proved:

$$\mathrm{Var}(|X+Y|) \geq \frac{1}{2}\Big(\mathrm{Var}(|X|) + \mathrm{Var}(|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2.$$

Thus $s \leq 2\sigma^2$ and $\mathbb{E}\min(|X|,|Y|) \leq (\sigma^2 - \frac{1}{2}s)^{1/2}$, where

$$\sigma = \Big(\mathrm{Var}(|X+Y|)\Big)^{1/2}$$

and

$$s = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|).$$

We want to prove:

$$\min\left(\mathrm{Var}(X), \mathrm{Var}(Y)\right) \leq \frac{7+\sqrt{17}}{4} \cdot \sigma^2.$$

Proof of the key lemma - auxiliary notation

We have proved:

$$\mathrm{Var}(|X+Y|) \geq \frac{1}{2}\Big(\mathrm{Var}(|X|) + \mathrm{Var}(|Y|)\Big) + (\mathbb{E}\min(|X|,|Y|))^2.$$

Thus $s \leq 2\sigma^2$ and $\mathbb{E}\min(|X|,|Y|) \leq (\sigma^2 - \frac{1}{2}s)^{1/2}$, where

$$\sigma = \Big(\mathrm{Var}(|X+Y|)\Big)^{1/2}$$

and

$$s = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|).$$

We want to prove:

$$\min\left(\mathrm{Var}(X), \mathrm{Var}(Y)\right) \leq \frac{7+\sqrt{17}}{4} \cdot \sigma^2.$$

Proof of key lemma - continued

The identity $a + b - 2\min(a,b) = |a - b|$ yields a pointwise bound:

$$|X| + |Y| - 2\min(|X|,|Y|) = \Big||X| - |Y|\Big| \leq$$

$$\Big|\mathbb{E}|X| - \mathbb{E}|Y|\Big| + \Big|(|X| - \mathbb{E}|X|) - (|Y| - \mathbb{E}|Y|)\Big| =$$

$$\mathbb{E}|X| + \mathbb{E}|Y| - 2\min(\mathbb{E}|X|,\mathbb{E}|Y|) + \Big|(|X| - |Y|) - \mathbb{E}(|X| - |Y|)\Big|.$$

By taking expectations, upon cancellations we arrive at

$$\min(\mathbb{E}|X|,\mathbb{E}|Y|) \leq \mathbb{E}\min(|X|,|Y|) + \frac{1}{2}\mathbb{E}\Big|(|X|-|Y|) - \mathbb{E}(|X|-|Y|)\Big| \leq$$

$$\mathbb{E}\min(|X|,|Y|) + \frac{1}{2}\Big(\mathrm{Var}(|X| - |Y|)\Big)^{1/2} \leq \Big(\sigma^2 - \frac{1}{2}s\Big)^{1/2} + \frac{1}{2}\sqrt{s},$$

where we have used the Schwarz inequality and independence of $|X|$ and $|Y|$ - so that $\mathrm{Var}(|X| - |Y|) = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|) = s$.

Proof of key lemma - continued

The identity $a + b - 2\min(a, b) = |a - b|$ yields a pointwise bound:

$$|X| + |Y| - 2\min(|X|, |Y|) = \Big||X| - |Y|\Big| \leq$$

$$\Big|\mathbb{E}|X| - \mathbb{E}|Y|\Big| + \Big|(|X| - \mathbb{E}|X|) - (|Y| - \mathbb{E}|Y|)\Big| =$$

$$\mathbb{E}|X| + \mathbb{E}|Y| - 2\min(\mathbb{E}|X|, \mathbb{E}|Y|) + \Big|(|X| - |Y|) - \mathbb{E}(|X| - |Y|)\Big|.$$

By taking expectations, upon cancellations we arrive at

$$\min(\mathbb{E}|X|, \mathbb{E}|Y|) \leq \mathbb{E}\min(|X|, |Y|) + \frac{1}{2}\mathbb{E}\Big|(|X| - |Y|) - \mathbb{E}(|X| - |Y|)\Big| \leq$$

$$\mathbb{E}\min(|X|, |Y|) + \frac{1}{2}\Big(\mathrm{Var}(|X| - |Y|)\Big)^{1/2} \leq \Big(\sigma^2 - \frac{1}{2}s\Big)^{1/2} + \frac{1}{2}\sqrt{s},$$

where we have used the Schwarz inequality and independence of $|X|$ and $|Y|$ - so that $\mathrm{Var}(|X| - |Y|) = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|) = s$.

Proof of key lemma - continued

The identity $a + b - 2\min(a, b) = |a - b|$ yields a pointwise bound:

$$|X| + |Y| - 2\min(|X|, |Y|) = \Big| |X| - |Y| \Big| \leq$$

$$\Big| \mathbb{E}|X| - \mathbb{E}|Y| \Big| + \Big| (|X| - \mathbb{E}|X|) - (|Y| - \mathbb{E}|Y|) \Big| =$$

$$\mathbb{E}|X| + \mathbb{E}|Y| - 2\min(\mathbb{E}|X|, \mathbb{E}|Y|) + \Big| (|X| - |Y|) - \mathbb{E}(|X| - |Y|) \Big|.$$

By taking expectations, upon cancellations we arrive at

$$\min(\mathbb{E}|X|, \mathbb{E}|Y|) \leq \mathbb{E}\min(|X|, |Y|) + \frac{1}{2}\mathbb{E}\Big| (|X| - |Y|) - \mathbb{E}(|X| - |Y|) \Big| \leq$$

$$\mathbb{E}\min(|X|, |Y|) + \frac{1}{2}\Big(\mathrm{Var}(|X| - |Y|)\Big)^{1/2} \leq \Big(\sigma^2 - \frac{1}{2}s\Big)^{1/2} + \frac{1}{2}\sqrt{s},$$

where we have used the Schwarz inequality and independence of $|X|$ and $|Y|$ - so that $\mathrm{Var}(|X| - |Y|) = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|) = s$.

Proof of key lemma - continued

The identity $a + b - 2\min(a, b) = |a - b|$ yields a pointwise bound:

$$|X| + |Y| - 2\min(|X|, |Y|) = \Big||X| - |Y|\Big| \le$$

$$\Big|\mathbb{E}|X| - \mathbb{E}|Y|\Big| + \Big|(|X| - \mathbb{E}|X|) - (|Y| - \mathbb{E}|Y|)\Big| =$$

$$\mathbb{E}|X| + \mathbb{E}|Y| - 2\min(\mathbb{E}|X|, \mathbb{E}|Y|) + \Big|(|X| - |Y|) - \mathbb{E}(|X| - |Y|)\Big|.$$

By taking expectations, upon cancellations we arrive at

$$\min(\mathbb{E}|X|, \mathbb{E}|Y|) \le \mathbb{E}\min(|X|, |Y|) + \frac{1}{2}\mathbb{E}\Big|(|X| - |Y|) - \mathbb{E}(|X| - |Y|)\Big| \le$$

$$\mathbb{E}\min(|X|, |Y|) + \frac{1}{2}\Big(\mathrm{Var}(|X| - |Y|)\Big)^{1/2} \le \Big(\sigma^2 - \frac{1}{2}s\Big)^{1/2} + \frac{1}{2}\sqrt{s},$$

where we have used the Schwarz inequality and independence of $|X|$ and $|Y|$ - so that $\mathrm{Var}(|X| - |Y|) = \mathrm{Var}(|X|) + \mathrm{Var}(|Y|) = s$.

Proof of the key lemma - the end

We have proved that $s \leq 2\sigma^2$ and

$$\min(\mathbb{E}|X|, \mathbb{E}|Y|) \leq \left(\sigma^2 - \frac{1}{2}s\right)^{1/2} + \frac{1}{2}\sqrt{s}.$$

Therefore

$$\min(\mathrm{Var}(X), \mathrm{Var}(Y)) \leq \min(\mathbb{E}X^2, \mathbb{E}Y^2) =$$

$$\min\left(\mathrm{Var}(|X|) + (\mathbb{E}|X|)^2, \mathrm{Var}(|Y|) + (\mathbb{E}|Y|)^2\right) \leq$$

$$s + (\min(\mathbb{E}|X|, \mathbb{E}|Y|))^2 \leq \sigma^2 + \frac{3}{4}s + \left(\sigma^2 s - \frac{1}{2}s^2\right)^{1/2} \leq$$

$$\sigma^2 + \sup_{t \in [0, 2\sigma^2]} \left(\frac{3}{4}t + (\sigma^2 t - t^2/2)^{1/2}\right) = \frac{7 + \sqrt{17}}{4} \cdot \sigma^2.$$

Proof of the key lemma - the end

We have proved that $s \leq 2\sigma^2$ and

$$\min(\mathbb{E}|X|, \mathbb{E}|Y|) \leq \left(\sigma^2 - \frac{1}{2}s\right)^{1/2} + \frac{1}{2}\sqrt{s}.$$

Therefore

$$\min(\mathrm{Var}(X), \mathrm{Var}(Y)) \leq \min(\mathbb{E}X^2, \mathbb{E}Y^2) =$$

$$\min\left(\mathrm{Var}(|X|) + (\mathbb{E}|X|)^2, \mathrm{Var}(|Y|) + (\mathbb{E}|Y|)^2\right) \leq$$

$$s + (\min(\mathbb{E}|X|, \mathbb{E}|Y|))^2 \leq \sigma^2 + \frac{3}{4}s + \left(\sigma^2 s - \frac{1}{2}s^2\right)^{1/2} \leq$$

$$\sigma^2 + \sup_{t \in [0, 2\sigma^2]} \left(\frac{3}{4}t + (\sigma^2 t - t^2/2)^{1/2}\right) = \frac{7 + \sqrt{17}}{4} \cdot \sigma^2.$$

Proof of Theorem - notation

Theorem

Let $(X_i)_{i=1}^n$ be a sequence of independent symmetric random variables. Then for some $k \in \{1, 2, \ldots, n\}$ we have

$$\mathrm{Var}\Big(\sum_{i\le n: i\ne k} X_i\Big) \le C \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\le n} X_i\Big|\Big).$$

We prove the theorem with $C = (7+\sqrt{17})/2 \approx 5.56$.

For $x \in \mathbb{R}$ let $\xi_1 = x + X_1$, and $\xi_i = X_i$ for $i \ge 2$.

Set $S = \sum_{i=1}^n \xi_i$.

For $J \subseteq \{1, 2, \ldots, n\}$ let $S_J = \sum_{i\in J} \xi_i$.

Proof of Theorem - notation

Theorem

Let $(X_i)_{i=1}^n$ be a sequence of independent symmetric random variables. Then for some $k \in \{1, 2, \ldots, n\}$ we have

$$\mathrm{Var}\Big(\sum_{i\le n: i\neq k} X_i\Big) \le C \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\le n} X_i\Big|\Big).$$

We prove the theorem with $C = (7+\sqrt{17})/2 \approx 5.56$.

For $x \in \mathbb{R}$ let $\xi_1 = x + X_1$, and $\xi_i = X_i$ for $i \ge 2$.

Set $S = \sum_{i=1}^n \xi_i$.

For $J \subseteq \{1, 2, \ldots, n\}$ let $S_J = \sum_{i\in J} \xi_i$.

Proof of Theorem - main part

Let I be a minimal, in the sense of inclusion, subset of $[n]$ such that $\mathrm{Var}(S_I) > \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$ - if no subset satisfies this condition then the assertion follows trivially. Obviously, $I \neq \emptyset$. Choose $k \in I$.

Certainly, $\mathrm{Var}(S_{I\setminus\{k\}}) \overset{*}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. Since S_I and $S_{[n]\setminus I}$ are independent and at least one of them is symmetric, Lemma yields

$$\min\Big(\mathrm{Var}(S_I), \mathrm{Var}(S_{[n]\setminus I})\Big) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|),$$

so that $\mathrm{Var}(S_{[n]\setminus I}) \overset{**}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. By * and ** there is

$$\mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) = \mathrm{Var}(S_{I\setminus\{k\}}) + \mathrm{Var}(S_{[n]\setminus I}) \leq \frac{7+\sqrt{17}}{2} \cdot \mathrm{Var}(|S|).$$

Thus we have proved that

$$\min_{k\leq n} \mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) \leq \frac{7+\sqrt{17}}{2} \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\leq n} X_i\Big|\Big).$$

Proof of Theorem - main part

Let I be a minimal, in the sense of inclusion, subset of $[n]$ such that $\mathrm{Var}(S_I) > \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$ - if no subset satisfies this condition then the assertion follows trivially. Obviously, $I \neq \emptyset$. Choose $k \in I$. Certainly, $\mathrm{Var}(S_{I\setminus\{k\}}) \overset{*}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. Since S_I and $S_{[n]\setminus I}$ are independent and at least one of them is symmetric, Lemma yields

$$\min\left(\mathrm{Var}(S_I), \mathrm{Var}(S_{[n]\setminus I})\right) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|),$$

so that $\mathrm{Var}(S_{[n]\setminus I}) \overset{**}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. By * and ** there is

$$\mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) = \mathrm{Var}(S_{I\setminus\{k\}}) + \mathrm{Var}(S_{[n]\setminus I}) \leq \frac{7+\sqrt{17}}{2} \cdot \mathrm{Var}(|S|).$$

Thus we have proved that

$$\min_{k\leq n} \mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) \leq \frac{7+\sqrt{17}}{2} \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\leq n} X_i\Big|\Big).$$

Proof of Theorem - main part

Let I be a minimal, in the sense of inclusion, subset of $[n]$ such that $\mathrm{Var}(S_I) > \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$ - if no subset satisfies this condition then the assertion follows trivially. Obviously, $I \neq \emptyset$. Choose $k \in I$. Certainly, $\mathrm{Var}(S_{I\setminus\{k\}}) \overset{*}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. Since S_I and $S_{[n]\setminus I}$ are independent and at least one of them is symmetric, Lemma yields

$$\min\Big(\mathrm{Var}(S_I), \mathrm{Var}(S_{[n]\setminus I})\Big) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|),$$

so that $\mathrm{Var}(S_{[n]\setminus I}) \overset{**}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. By * and ** there is

$$\mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) = \mathrm{Var}(S_{I\setminus\{k\}}) + \mathrm{Var}(S_{[n]\setminus I}) \leq \frac{7+\sqrt{17}}{2} \cdot \mathrm{Var}(|S|).$$

Thus we have proved that

$$\min_{k\leq n} \mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) \leq \frac{7+\sqrt{17}}{2} \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\leq n} X_i\Big|\Big).$$

Proof of Theorem - main part

Let I be a minimal, in the sense of inclusion, subset of $[n]$ such that $\mathrm{Var}(S_I) > \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$ - if no subset satisfies this condition then the assertion follows trivially. Obviously, $I \neq \emptyset$. Choose $k \in I$. Certainly, $\mathrm{Var}(S_{I\setminus\{k\}}) \overset{*}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. Since S_I and $S_{[n]\setminus I}$ are independent and at least one of them is symmetric, Lemma yields

$$\min\Big(\mathrm{Var}(S_I), \mathrm{Var}(S_{[n]\setminus I})\Big) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|),$$

so that $\mathrm{Var}(S_{[n]\setminus I}) \overset{**}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. By * and ** there is

$$\mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) = \mathrm{Var}(S_{I\setminus\{k\}}) + \mathrm{Var}(S_{[n]\setminus I}) \leq \frac{7+\sqrt{17}}{2} \cdot \mathrm{Var}(|S|).$$

Thus we have proved that

$$\min_{k\leq n} \mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) \leq \frac{7+\sqrt{17}}{2} \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\leq n} X_i\Big|\Big).$$

Proof of Theorem - main part

Let I be a minimal, in the sense of inclusion, subset of $[n]$ such that $\mathrm{Var}(S_I) > \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$ - if no subset satisfies this condition then the assertion follows trivially. Obviously, $I \neq \emptyset$. Choose $k \in I$. Certainly, $\mathrm{Var}(S_{I\setminus\{k\}}) \overset{*}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. Since S_I and $S_{[n]\setminus I}$ are independent and at least one of them is symmetric, Lemma yields

$$\min\left(\mathrm{Var}(S_I), \mathrm{Var}(S_{[n]\setminus I})\right) \leq \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|),$$

so that $\mathrm{Var}(S_{[n]\setminus I}) \overset{**}{\leq} \frac{7+\sqrt{17}}{4} \cdot \mathrm{Var}(|S|)$. By * and ** there is

$$\mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) = \mathrm{Var}(S_{I\setminus\{k\}}) + \mathrm{Var}(S_{[n]\setminus I}) \leq \frac{7+\sqrt{17}}{2} \cdot \mathrm{Var}(|S|).$$

Thus we have proved that

$$\min_{k\leq n} \mathrm{Var}\Big(\sum_{i\leq n: i\neq k} X_i\Big) \leq \frac{7+\sqrt{17}}{2} \cdot \inf_{x\in\mathbb{R}} \mathrm{Var}\Big(\Big|x + \sum_{i\leq n} X_i\Big|\Big).$$

Another slight extension of the FKN theorem

Theorem

There exists a universal constant $L > 0$ with the following property. For $f : \{-1,1\}^n \to \{-1,1\}$ let $\rho = \left(\sum_{A\subseteq[n]:|A|\geq 2} |\hat{f}(A)|^2\right)^{1/2}$.

Then there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A\subseteq[n]:|A|\leq 1, A\neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 \ln(2/\rho),$$

$$|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 \ln(2/\rho).$$

The bound $O\Big(\rho^4 \ln(2/\rho)\Big)$ is better than $O(\rho^2)$ obtained in [FKN]. Also, it is of the optimal order. For any $2 \leq m \leq n$ consider just

$$f(x) = 1 - \frac{1}{2^{m-1}} \prod_{i=1}^{m} (1 + x_i).$$

Another slight extension of the FKN theorem

Theorem

There exists a universal constant $L > 0$ with the following property. For $f : \{-1,1\}^n \to \{-1,1\}$ let $\rho = \left(\sum_{A\subseteq[n]:|A|\geq 2} |\hat{f}(A)|^2\right)^{1/2}$.

Then there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A\subseteq[n]:|A|\leq 1, A\neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 \ln(2/\rho),$$

$$|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 \ln(2/\rho).$$

The bound $O\left(\rho^4 \ln(2/\rho)\right)$ is better than $O(\rho^2)$ obtained in [FKN]. Also, it is of the optimal order. For any $2 \leq m \leq n$ consider just

$$f(x) = 1 - \frac{1}{2^{m-1}} \prod_{i=1}^{m} (1 + x_i).$$

Assumptions and Notation (A & N)

$\xi_1, \xi_2, \ldots, \xi_n$ - independent random variables, $\mathbb{E}\xi_i = 0$, $\mathbb{E}\xi_i^2 = 1$

Hilbert space $L^2 = L^2(\mathbb{R}^n, \mu)$, where $\mu = \mu_{\xi_1} \otimes \mu_{\xi_2} \otimes \ldots \otimes \mu_{\xi_n}$ is the joint distribution of the random vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

Let $\xi_0 \equiv 1$, so that $(\xi_i)_{i=0}^n$ is an orthonormal system in L^2.

Let $\mathcal{A}$ be a linear (finite dimensional and thus closed) subspace of L^2 consisting of all affine real-valued functions on $\mathbb{R}^n$.

We define coordinate projection functions $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $\pi_i(x) = x_i$ for $1 \le i \le n$, and $\pi_0 \equiv 1$.
Let $\mathcal{A}_\pi = \{\pi_0, -\pi_0, \pi_1, -\pi_1, \ldots, \pi_n, -\pi_n\}$.

For a Boolean (i.e. $\{-1, 1\}$-valued) Borel function f on $\mathbb{R}^n$ by $f_\mathcal{A} : \mathbb{R}^n \longrightarrow \mathbb{R}$ we will denote its orthogonal projection in L^2 onto the subspace $\mathcal{A}$:
$f_\mathcal{A}(x) = a_0 + a_1 x_1 + \ldots + a_n x_n$, i.e. $f_\mathcal{A} = \sum_{i=0}^n a_i \pi_i$.

Assumptions and Notation (A & N)

$\xi_1, \xi_2, \ldots, \xi_n$ - independent random variables, $\mathbb{E}\xi_i = 0$, $\mathbb{E}\xi_i^2 = 1$

Hilbert space $L^2 = L^2(\mathbb{R}^n, \mu)$, where $\mu = \mu_{\xi_1} \otimes \mu_{\xi_2} \otimes \ldots \otimes \mu_{\xi_n}$ is the joint distribution of the random vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

Let $\xi_0 \equiv 1$, so that $(\xi_i)_{i=0}^n$ is an orthonormal system in L^2.

Let $\mathcal{A}$ be a linear (finite dimensional and thus closed) subspace of L^2 consisting of all affine real-valued functions on $\mathbb{R}^n$.

We define coordinate projection functions $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $\pi_i(x) = x_i$ for $1 \leq i \leq n$, and $\pi_0 \equiv 1$.
Let $\mathcal{A}_\pi = \{\pi_0, -\pi_0, \pi_1, -\pi_1, \ldots, \pi_n, -\pi_n\}$.

For a Boolean (i.e. $\{-1, 1\}$-valued) Borel function f on $\mathbb{R}^n$ by $f_\mathcal{A} : \mathbb{R}^n \longrightarrow \mathbb{R}$ we will denote its orthogonal projection in L^2 onto the subspace $\mathcal{A}$:
$f_\mathcal{A}(x) = a_0 + a_1x_1 + \ldots + a_nx_n$, i.e. $f_\mathcal{A} = \sum_{i=0}^n a_i\pi_i$.

We may and will use the same notation for a Borel Boolean function f defined only on the support of μ, since obviously it may be extended to a Borel Boolean function F on the whole $\mathbb{R}^n$, and $F_{\mathcal{A}}$ does not depend on the choice of the extension.

Let us define the sign function in a slightly non-standard way as $\mathbf{1}_{[0,\infty)} - \mathbf{1}_{(-\infty,0)}$, to make the function Boolean (setting $\mathrm{sign}(0) = -1$ would work as well).

$$\rho = \mathrm{dist}_{L^2}(f, \mathcal{A}), \quad d = \mathrm{dist}_{L^2}(f, \mathcal{A}_\pi)$$

Easy: $\rho \leq 1$ and $d \leq \sqrt{2}$.

Finally, let us define random variables $S = f_{\mathcal{A}}(\xi)$ and $R = (f - f_{\mathcal{A}})(\xi)$.

A & N - continued

We may and will use the same notation for a Borel Boolean function f defined only on the support of μ, since obviously it may be extended to a Borel Boolean function F on the whole $\mathbb{R}^n$, and $F_{\mathcal{A}}$ does not depend on the choice of the extension.

Let us define the sign function in a slightly non-standard way as $\mathbf{1}_{[0,\infty)} - \mathbf{1}_{(-\infty,0)}$, to make the function Boolean (setting $\mathrm{sign}(0) = -1$ would work as well).

$$\rho = \mathrm{dist}_{L^2}(f, \mathcal{A}), \quad d = \mathrm{dist}_{L^2}(f, \mathcal{A}_\pi)$$

Easy: $\rho \leq 1$ and $d \leq \sqrt{2}$.

Finally, let us define random variables $S = f_{\mathcal{A}}(\xi)$ and $R = (f - f_{\mathcal{A}})(\xi)$.

A & N - continued

We may and will use the same notation for a Borel Boolean function f defined only on the support of μ, since obviously it may be extended to a Borel Boolean function F on the whole $\mathbb{R}^n$, and $F_{\mathcal{A}}$ does not depend on the choice of the extension.

Let us define the sign function in a slightly non-standard way as $\mathbf{1}_{[0,\infty)} - \mathbf{1}_{(-\infty,0)}$, to make the function Boolean (setting $\mathrm{sign}(0) = -1$ would work as well).

$$\rho = \mathrm{dist}_{L^2}(f, \mathcal{A}), \quad d = \mathrm{dist}_{L^2}(f, \mathcal{A}_\pi)$$

Easy: $\rho \leq 1$ and $d \leq \sqrt{2}$.

Finally, let us define random variables $S = f_{\mathcal{A}}(\xi)$ and $R = (f - f_{\mathcal{A}})(\xi)$.

A & N - continued

We may and will use the same notation for a Borel Boolean function f defined only on the support of μ, since obviously it may be extended to a Borel Boolean function F on the whole $\mathbb{R}^n$, and $F_{\mathcal{A}}$ does not depend on the choice of the extension.

Let us define the sign function in a slightly non-standard way as $\mathbf{1}_{[0,\infty)} - \mathbf{1}_{(-\infty,0)}$, to make the function Boolean (setting $\mathrm{sign}(0) = -1$ would work as well).

$$\rho = \mathrm{dist}_{L^2}(f, \mathcal{A}), \quad d = \mathrm{dist}_{L^2}(f, \mathcal{A}_\pi)$$

Easy: $\rho \leq 1$ and $d \leq \sqrt{2}$.

Finally, let us define random variables $S = f_{\mathcal{A}}(\xi)$ and $R = (f - f_{\mathcal{A}})(\xi)$.

Symmetric case - general bound

Theorem

Under **A & N**, *if* $\xi_1, \xi_2, \ldots, \xi_n$ *are additionally symmetric then there exists some* $k \in \{0, 1, \ldots, n\}$ *such that*

$$a_k^2 \geq 1 - \frac{9+\sqrt{17}}{2} \cdot \rho^2.$$

Also, $\rho \leq d \leq (9+\sqrt{17})^{1/2} \cdot \rho$, *and* $d \leq \left(\frac{9+\sqrt{17}}{2}\right)^{1/2} \cdot \rho + o(\rho)$ *as* $\rho \to 0^+$ *(uniformly over Boolean functions).*

Proof: $|f| \equiv 1$, thus the triangle inequality yields a pointwise bound

$$1 - |f - f_{\mathcal{A}}| \leq |f_{\mathcal{A}}| \leq 1 + |f - f_{\mathcal{A}}|, \text{ i.e. } \Big||S| - 1\Big| \leq |R|,$$

so $\mathrm{Var}(|S|) = \mathbb{E}(|S|-1)^2 - (\mathbb{E}|S|-1)^2 \leq \mathbb{E}R^2 = \|f - f_{\mathcal{A}}\|_{L^2}^2 = \rho^2$.

Theorem

Under **A & N**, *if* $\xi_1, \xi_2, \ldots, \xi_n$ *are additionally symmetric then there exists some* $k \in \{0, 1, \ldots, n\}$ *such that*

$$a_k^2 \geq 1 - \frac{9+\sqrt{17}}{2} \cdot \rho^2.$$

Also, $\rho \leq d \leq (9+\sqrt{17})^{1/2} \cdot \rho$, *and* $d \leq \left(\frac{9+\sqrt{17}}{2}\right)^{1/2} \cdot \rho + o(\rho)$ *as* $\rho \to 0^+$ *(uniformly over Boolean functions).*

Proof: $|f| \equiv 1$, thus the triangle inequality yields a pointwise bound

$$1 - |f - f_{\mathcal{A}}| \leq |f_{\mathcal{A}}| \leq 1 + |f - f_{\mathcal{A}}|, \text{ i.e. } \Big||S| - 1\Big| \leq |R|,$$

so $\mathrm{Var}(|S|) = \mathbb{E}(|S|-1)^2 - (\mathbb{E}|S|-1)^2 \leq \mathbb{E}R^2 = \|f - f_{\mathcal{A}}\|_{L^2}^2 = \rho^2$.

Symmetric case general bound - continued

Consider independent $(X_i)_{i=0}^n$ given by $X_i = a_i \xi_i$ for $1 \le i \le n$, and with X_0 being a symmetric $\pm a_0$ random variable. The sum $|\sum_{i=0}^n X_i|$ has the same distribution as $|S|$ and thus

$$\mathrm{Var}\Big(\Big|\sum_{i=0}^n X_i\Big|\Big) = \mathrm{Var}(|S|) \le \rho^2$$

so by using our result for sums (for $n+1$ instead of n summands) with $x = 0$ we get that for some $k \in \{0, 1, \ldots, n\}$ there is

$$\sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2 = \mathrm{Var}\Big(\sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} X_i\Big) \le \frac{7+\sqrt{17}}{2} \cdot \rho^2.$$

Since by orthogonality we have

$$\sum_{i=0}^n a_i^2 = \|f_{\mathcal{A}}\|_{L^2}^2 = \|f\|_{L^2}^2 - \|f - f_{\mathcal{A}}\|_{L^2}^2 = 1 - \rho^2,$$

this ends the proof of the first assertion.

Symmetric case general bound - continued

Consider independent $(X_i)_{i=0}^n$ given by $X_i = a_i\xi_i$ for $1 \le i \le n$, and with X_0 being a symmetric $\pm a_0$ random variable. The sum $|\sum_{i=0}^n X_i|$ has the same distribution as $|S|$ and thus

$$\mathrm{Var}\Big(\Big|\sum_{i=0}^n X_i\Big|\Big) = \mathrm{Var}(|S|) \le \rho^2$$

so by using our result for sums (for $n+1$ instead of n summands) with $x = 0$ we get that for some $k \in \{0, 1, \dots, n\}$ there is

$$\sum_{i \in \{0,1,\dots,n\}\setminus\{k\}} a_i^2 = \mathrm{Var}\Big(\sum_{i \in \{0,1,\dots,n\}\setminus\{k\}} X_i\Big) \le \frac{7+\sqrt{17}}{2} \cdot \rho^2.$$

Since by orthogonality we have

$$\sum_{i=0}^n a_i^2 = \|f_{\mathcal{A}}\|_{L^2}^2 = \|f\|_{L^2}^2 - \|f - f_{\mathcal{A}}\|_{L^2}^2 = 1 - \rho^2,$$

this ends the proof of the first assertion.

Symmetric case general bound - continued

Consider independent $(X_i)_{i=0}^n$ given by $X_i = a_i\xi_i$ for $1 \le i \le n$, and with X_0 being a symmetric $\pm a_0$ random variable. The sum $|\sum_{i=0}^n X_i|$ has the same distribution as $|S|$ and thus

$$\mathrm{Var}\Big(\Big|\sum_{i=0}^n X_i\Big|\Big) = \mathrm{Var}(|S|) \le \rho^2$$

so by using our result for sums (for $n+1$ instead of n summands) with $x = 0$ we get that for some $k \in \{0, 1, \ldots, n\}$ there is

$$\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 = \mathrm{Var}\Big(\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} X_i\Big) \le \frac{7+\sqrt{17}}{2}\cdot\rho^2.$$

Since by orthogonality we have

$$\sum_{i=0}^n a_i^2 = \|f_{\mathcal{A}}\|_{L^2}^2 = \|f\|_{L^2}^2 - \|f - f_{\mathcal{A}}\|_{L^2}^2 = 1 - \rho^2,$$

this ends the proof of the first assertion.

Symmetric case general bound - the end

The inequality $\rho \leq d$ follows from $\mathcal{A}_\pi \subseteq \mathcal{A}$.

$$d^2 \leq \|f - \mathrm{sign}(a_k)\pi_k\|_{L^2} = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\mathrm{sign}(a_k)\langle f, \pi_k\rangle_{L^2}$$

$$= 1 + 1 - 2\mathrm{sign}(a_k)a_k = 2(1 - |a_k|) = 2(1 - a_k^2)/(1 + |a_k|)$$

$$\leq (9 + \sqrt{17})\rho^2/(1 + |a_k|).$$

Thus $d \leq \left(9 + \sqrt{17}\right)^{1/2} \cdot \rho$.

We skip the proof of the last (asymptotic) claim.

Symmetric case general bound - the end

The inequality $\rho \leq d$ follows from $\mathcal{A}_\pi \subseteq \mathcal{A}$.

$$d^2 \leq \|f - \mathrm{sign}(a_k)\pi_k\|_{L^2} = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\mathrm{sign}(a_k)\langle f, \pi_k\rangle_{L^2}$$

$$= 1 + 1 - 2\mathrm{sign}(a_k)a_k = 2(1 - |a_k|) = 2(1 - a_k^2)/(1 + |a_k|)$$

$$\leq (9 + \sqrt{17})\rho^2/(1 + |a_k|).$$

Thus $d \leq \left(9 + \sqrt{17}\right)^{1/2} \cdot \rho$.

We skip the proof of the last (asymptotic) claim.

Symmetric case general bound - the end

The inequality $\rho \leq d$ follows from $\mathcal{A}_\pi \subseteq \mathcal{A}$.

$$d^2 \leq \|f - \mathrm{sign}(a_k)\pi_k\|_{L^2} = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\mathrm{sign}(a_k)\langle f, \pi_k\rangle_{L^2}$$

$$= 1 + 1 - 2\mathrm{sign}(a_k)a_k = 2(1 - |a_k|) = 2(1 - a_k^2)/(1 + |a_k|)$$

$$\leq (9 + \sqrt{17})\rho^2/(1 + |a_k|).$$

Thus $d \leq \left(9 + \sqrt{17}\right)^{1/2} \cdot \rho$.

We skip the proof of the last (asymptotic) claim.

Distance to a sign of a coordinate function

Corollary

*Under assumptions of Theorem (**A & N** plus symmetry) there is some $k \in \{0, 1, \ldots, n\}$ such that*

$$\min\left(\|f-(\mathrm{sign}\circ\pi_k)\|_{L^2}, \|f+(\mathrm{sign}\circ\pi_k)\|_{L^2}\right) \leq 2d \leq 2\left(9+\sqrt{17}\right)^{1/2}\cdot\rho$$

Proof: Let k be as in the proof of Theorem, i.e. such that $\|f-\pi_k\|_{L^2} = d$ (it may also happen that $\|f+\pi_k\|_{L^2} = d$). Note that for any $s \in \{-1, 1\}$ and $u \in \mathbb{R}$ there is $|s-u| \geq |\mathrm{sign}(u)-u|$ (and $|s+u| \geq |\mathrm{sign}(u)-u|$). Hence $|f-\pi_k| \geq |(\mathrm{sign}\circ\pi_k)-\pi_k|$ (and $|f+\pi_k| \geq |(\mathrm{sign}\circ\pi_k)-\pi_k|$) pointwise, so that $\|(\mathrm{sign}\circ\pi_k)-\pi_k\|_{L^2} \leq d$.

The assertion follows by the triangle inequality.

Distance to a sign of a coordinate function

Corollary

*Under assumptions of Theorem (**A & N** plus symmetry) there is some $k \in \{0, 1, \ldots, n\}$ such that*

$$\min\left(\|f-(\mathrm{sign}\circ\pi_k)\|_{L^2}, \|f+(\mathrm{sign}\circ\pi_k)\|_{L^2}\right) \leq 2d \leq 2\left(9+\sqrt{17}\right)^{1/2} \cdot \rho$$

Proof: Let k be as in the proof of Theorem, i.e. such that $\|f - \pi_k\|_{L^2} = d$ (it may also happen that $\|f + \pi_k\|_{L^2} = d$). Note that for any $s \in \{-1, 1\}$ and $u \in \mathbb{R}$ there is $|s - u| \geq |\mathrm{sign}(u) - u|$ (and $|s + u| \geq |\mathrm{sign}(u) - u|$). Hence $|f - \pi_k| \geq |(\mathrm{sign} \circ \pi_k) - \pi_k|$ (and $|f + \pi_k| \geq |(\mathrm{sign} \circ \pi_k) - \pi_k|$) pointwise, so that $\|(\mathrm{sign} \circ \pi_k) - \pi_k\|_{L^2} \leq d$.

The assertion follows by the triangle inequality.

Distance to a sign of a coordinate function

Corollary

*Under assumptions of Theorem (**A & N** plus symmetry) there is some $k \in \{0, 1, \dots, n\}$ such that*

$$\min\left(\|f-(\mathrm{sign}\circ\pi_k)\|_{L^2}, \|f+(\mathrm{sign}\circ\pi_k)\|_{L^2}\right) \leq 2d \leq 2\left(9+\sqrt{17}\right)^{1/2}\cdot\rho$$

Proof: Let k be as in the proof of Theorem, i.e. such that $\|f - \pi_k\|_{L^2} = d$ (it may also happen that $\|f + \pi_k\|_{L^2} = d$). Note that for any $s \in \{-1, 1\}$ and $u \in \mathbb{R}$ there is $|s - u| \geq |\mathrm{sign}(u) - u|$ (and $|s + u| \geq |\mathrm{sign}(u) - u|$). Hence $|f - \pi_k| \geq |(\mathrm{sign} \circ \pi_k) - \pi_k|$ (and $|f + \pi_k| \geq |(\mathrm{sign} \circ \pi_k) - \pi_k|$) pointwise, so that $\|(\mathrm{sign} \circ \pi_k) - \pi_k\|_{L^2} \leq d$.

The assertion follows by the triangle inequality.

Discrete cube

Now let us see how to strengthen the result of Friedgut, Kalai, and Naor. For a function f defined on the discrete cube $\{-1,1\}^n$ we consider its standard Walsh-Fourier expansion $\sum_A \hat{f}(A) w_A$, where $w_A(x) = \prod_{i \in A} x_i$.

Theorem

There exists a universal constant $L > 0$ with the following property. For $f : \{-1,1\}^n \to \{-1,1\}$ let $\rho = \left(\sum_{A \subseteq [n] : |A| \geq 2} |\hat{f}(A)|^2 \right)^{1/2}$.

Then there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A \subseteq [n] : |A| \leq 1, A \neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 \ln(2/\rho),$$

$$|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 \ln(2/\rho).$$

Proof of the discrete cube result - auxiliary notation

Proof: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent symmetric ± 1, so that the definition of ρ is consistent with the one from **A & N**.

We also have $a_i = \langle f, \pi_i \rangle_{L^2} = \hat{f}(\{i\})$ for $i \in [n]$, and $a_0 = \hat{f}(\emptyset)$.

Let $\theta = \left(4 \log_2(2/d) - 1\right)^{-1}$. There is $\theta \in (0, 1]$ because $d \leq \sqrt{2}$.

Let $k \in \{0, 1, \ldots, n\}$ be such that $d = \|f - \pi_k\|_{L^2}$ (if the point of $\mathcal{A}_\pi$ closest to f is of the form $-\pi_k$ then a similar reasoning works).

Hence $d^2 = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\langle f, \pi_k \rangle_{L^2} = 2(1 - a_k)$.
Remember:

$$(1 - a_k)^2 = d^4/4.$$

Proof of the discrete cube result - auxiliary notation

Proof: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent symmetric ± 1, so that the definition of ρ is consistent with the one from **A & N**.

We also have $a_i = \langle f, \pi_i \rangle_{L^2} = \hat{f}(\{i\})$ for $i \in [n]$, and $a_0 = \hat{f}(\emptyset)$.

Let $\theta = \left(4 \log_2(2/d) - 1\right)^{-1}$. There is $\theta \in (0, 1]$ because $d \leq \sqrt{2}$.

Let $k \in \{0, 1, \ldots, n\}$ be such that $d = \|f - \pi_k\|_{L^2}$ (if the point of $\mathcal{A}_\pi$ closest to f is of the form $-\pi_k$ then a similar reasoning works).

Hence $d^2 = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\langle f, \pi_k \rangle_{L^2} = 2(1 - a_k)$.
Remember:

$$(1 - a_k)^2 = d^4/4.$$

Proof of the discrete cube result - auxiliary notation

Proof: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent symmetric ± 1, so that the definition of ρ is consistent with the one from **A & N**.

We also have $a_i = \langle f, \pi_i \rangle_{L^2} = \hat{f}(\{i\})$ for $i \in [n]$, and $a_0 = \hat{f}(\emptyset)$.

Let $\theta = \Big(4\log_2(2/d) - 1\Big)^{-1}$. There is $\theta \in (0,1]$ because $d \leq \sqrt{2}$.

Let $k \in \{0, 1, \ldots, n\}$ be such that $d = \|f - \pi_k\|_{L^2}$ (if the point of $\mathcal{A}_\pi$ closest to f is of the form $-\pi_k$ then a similar reasoning works).

Hence $d^2 = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\langle f, \pi_k \rangle_{L^2} = 2(1 - a_k)$.
Remember:

$$(1 - a_k)^2 = d^4/4.$$

Proof of the discrete cube result - auxiliary notation

Proof: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent symmetric ± 1, so that the definition of ρ is consistent with the one from **A & N**.

We also have $a_i = \langle f, \pi_i \rangle_{L^2} = \hat{f}(\{i\})$ for $i \in [n]$, and $a_0 = \hat{f}(\emptyset)$.

Let $\theta = \Big(4 \log_2(2/d) - 1\Big)^{-1}$. There is $\theta \in (0, 1]$ because $d \leq \sqrt{2}$.

Let $k \in \{0, 1, \ldots, n\}$ be such that $d = \|f - \pi_k\|_{L^2}$ (if the point of $\mathcal{A}_\pi$ closest to f is of the form $-\pi_k$ then a similar reasoning works).

Hence $d^2 = \|f\|^2_{L^2} + \|\pi_k\|^2_{L^2} - 2\langle f, \pi_k \rangle_{L^2} = 2(1 - a_k)$.
Remember:

$$(1 - a_k)^2 = d^4/4.$$

Proof of the discrete cube result - auxiliary notation

Proof: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent symmetric ± 1, so that the definition of ρ is consistent with the one from **A & N**.

We also have $a_i = \langle f, \pi_i \rangle_{L^2} = \hat{f}(\{i\})$ for $i \in [n]$, and $a_0 = \hat{f}(\emptyset)$.

Let $\theta = \Big(4 \log_2(2/d) - 1\Big)^{-1}$. There is $\theta \in (0, 1]$ because $d \leq \sqrt{2}$.

Let $k \in \{0, 1, \ldots, n\}$ be such that $d = \|f - \pi_k\|_{L^2}$ (if the point of $\mathcal{A}_\pi$ closest to f is of the form $-\pi_k$ then a similar reasoning works).

Hence $d^2 = \|f\|_{L^2}^2 + \|\pi_k\|_{L^2}^2 - 2\langle f, \pi_k \rangle_{L^2} = 2(1 - a_k)$.
Remember:

$$(1 - a_k)^2 = d^4/4.$$

Hypercontractivity

Since a function $h = f - \pi_k$ is $\{-2, 0, 2\}$-valued we get

$$\mu(h \neq 0) = \mu(\{x \in \{-1,1\}^n : h(x) \neq 0\}) = \frac{1}{4}\|h\|_{L^2}^2 = (d/2)^2.$$

Therefore

$$d^4/2 = 4(d/2)^{\frac{4}{1+\theta}} = 4\Big(\mu(h \neq 0)\Big)^{\frac{2}{1+\theta}} = \|h\|_{L^{1+\theta}}^2 \overset{B-B}{\geq}$$

($B - B$ is the classical $L^2 - L^{1+\theta}$ Bonami-Beckner inequality)

$$\sum_{A \subseteq [n]} \theta^{|A|} \cdot |\hat{h}(A)|^2 \geq \theta \cdot \sum_{A \subseteq [n] : |A| \leq 1} |\hat{h}(A)|^2 =$$

$$\theta \cdot \Big((1 - a_k)^2 + \sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2\Big) = \theta \cdot \Big(\frac{d^4}{4} + \sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2\Big),$$

so that

$$\sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2 \leq (2\theta^{-1} - 1)d^4/4 \leq 2d^4 \log_2(2/d). \quad (1)$$

Hypercontractivity

Since a function $h = f - \pi_k$ is $\{-2, 0, 2\}$-valued we get

$$\mu(h \neq 0) = \mu(\{x \in \{-1,1\}^n : h(x) \neq 0\}) = \frac{1}{4}\|h\|_{L^2}^2 = (d/2)^2.$$

Therefore

$$d^4/2 = 4(d/2)^{\frac{4}{1+\theta}} = 4\Big(\mu(h \neq 0)\Big)^{\frac{2}{1+\theta}} = \|h\|_{L^{1+\theta}}^2 \overset{B-B}{\geq}$$

($B - B$ is the classical $L^2 - L^{1+\theta}$ Bonami-Beckner inequality)

$$\sum_{A \subseteq [n]} \theta^{|A|} \cdot |\hat{h}(A)|^2 \geq \theta \cdot \sum_{A \subseteq [n] : |A| \leq 1} |\hat{h}(A)|^2 =$$

$$\theta \cdot \Big((1 - a_k)^2 + \sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2\Big) = \theta \cdot \Big(\frac{d^4}{4} + \sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2\Big),$$

so that

$$\sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2 \leq (2\theta^{-1} - 1)d^4/4 \leq 2d^4 \log_2(2/d). \quad (1)$$

Hypercontractivity

Since a function $h = f - \pi_k$ is $\{-2,0,2\}$-valued we get

$$\mu(h \neq 0) = \mu(\{x \in \{-1,1\}^n : h(x) \neq 0\}) = \frac{1}{4}\|h\|_{L^2}^2 = (d/2)^2.$$

Therefore

$$d^4/2 = 4(d/2)^{\frac{4}{1+\theta}} = 4\Big(\mu(h \neq 0)\Big)^{\frac{2}{1+\theta}} = \|h\|_{L^{1+\theta}}^2 \overset{B-B}{\geq}$$

($B - B$ is the classical $L^2 - L^{1+\theta}$ Bonami-Beckner inequality)

$$\sum_{A \subseteq [n]} \theta^{|A|} \cdot |\hat{h}(A)|^2 \geq \theta \cdot \sum_{A \subseteq [n]:|A| \leq 1} |\hat{h}(A)|^2 =$$

$$\theta \cdot \Big((1-a_k)^2 + \sum_{i \in \{0,1,\dots,n\} \setminus \{k\}} a_i^2\Big) = \theta \cdot \Big(\frac{d^4}{4} + \sum_{i \in \{0,1,\dots,n\} \setminus \{k\}} a_i^2\Big),$$

so that

$$\sum_{i \in \{0,1,\dots,n\} \setminus \{k\}} a_i^2 \leq (2\theta^{-1} - 1)d^4/4 \leq 2d^4 \log_2(2/d). \quad (1)$$

Since a function $h = f - \pi_k$ is $\{-2, 0, 2\}$-valued we get

$$\mu(h \neq 0) = \mu(\{x \in \{-1,1\}^n : h(x) \neq 0\}) = \frac{1}{4}\|h\|_{L^2}^2 = (d/2)^2.$$

Therefore

$$d^4/2 = 4(d/2)^{\frac{4}{1+\theta}} = 4\Big(\mu(h \neq 0)\Big)^{\frac{2}{1+\theta}} = \|h\|_{L^{1+\theta}}^2 \overset{B-B}{\geq}$$

($B - B$ is the classical $L^2 - L^{1+\theta}$ Bonami-Beckner inequality)

$$\sum_{A \subseteq [n]} \theta^{|A|} \cdot |\hat{h}(A)|^2 \geq \theta \cdot \sum_{A \subseteq [n] : |A| \leq 1} |\hat{h}(A)|^2 =$$

$$\theta \cdot \Big((1 - a_k)^2 + \sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2\Big) = \theta \cdot \Big(\frac{d^4}{4} + \sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2\Big),$$

so that

$$\sum_{i \in \{0,1,\ldots,n\} \setminus \{k\}} a_i^2 \leq (2\theta^{-1} - 1)d^4/4 \leq 2d^4 \log_2(2/d). \quad (1)$$

Hypercontractivity

Since a function $h = f - \pi_k$ is $\{-2, 0, 2\}$-valued we get

$$\mu(h \neq 0) = \mu(\{x \in \{-1,1\}^n : h(x) \neq 0\}) = \frac{1}{4}\|h\|_{L^2}^2 = (d/2)^2.$$

Therefore

$$d^4/2 = 4(d/2)^{\frac{4}{1+\theta}} = 4\Big(\mu(h \neq 0)\Big)^{\frac{2}{1+\theta}} = \|h\|_{L^{1+\theta}}^2 \overset{B-B}{\geq}$$

($B - B$ is the classical $L^2 - L^{1+\theta}$ Bonami-Beckner inequality)

$$\sum_{A\subseteq[n]} \theta^{|A|} \cdot |\hat{h}(A)|^2 \geq \theta \cdot \sum_{A\subseteq[n]:|A|\leq 1} |\hat{h}(A)|^2 =$$

$$\theta \cdot \Big((1-a_k)^2 + \sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2\Big) = \theta \cdot \Big(\frac{d^4}{4} + \sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2\Big),$$

so that

$$\sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \leq (2\theta^{-1} - 1)d^4/4 \leq 2d^4\log_2(2/d). \quad (1)$$

Proof of the discrete cube result - the end

$$\sum_{i=0}^{n} a_i^2 = \left(1-\frac{d^2}{2}\right)^2 + \sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \overset{(1)}{\leq} \left(1-\frac{d^2}{2}\right)^2 + \frac{1}{4}(2\theta^{-1}-1)d^4$$

$$= 1 - d^2 + \frac{1}{2}\theta^{-1}d^4 \leq 1 - d^2 + 2d^4\log_2(2/d),$$

so

$$\rho^2 = \sum_{A\subseteq[n]:|A|\geq 2} |\hat{f}(A)|^2 = 1 - \sum_{i=0}^{n} a_i^2 \geq d^2 - 2d^4\log_2(2/d). \quad (2)$$

We finish the proof by observing that

$$\sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \overset{(1)}{\leq} 2d^4\log_2(2/d) \overset{(2)}{\leq} 2\left(\rho^2+2d^4\log_2(2/d)\right)^2\log_2(2/d)$$

$$\overset{d\geq\rho}{\leq} 2\left(\rho^2+2d^4\log_2(2/\rho)\right)^2\log_2(2/\rho) \overset{d=O(\rho)}{=} 2\rho^4\log_2(2/\rho)+o(\rho^5).$$

uniformly, as $\rho \to 0^+$.

Proof of the discrete cube result - the end

$$\sum_{i=0}^{n} a_i^2 = \left(1-\frac{d^2}{2}\right)^2 + \sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \overset{(1)}{\leq} \left(1-\frac{d^2}{2}\right)^2 + \frac{1}{4}(2\theta^{-1}-1)d^4$$

$$= 1 - d^2 + \frac{1}{2}\theta^{-1}d^4 \leq 1 - d^2 + 2d^4\log_2(2/d),$$

so

$$\rho^2 = \sum_{A\subseteq[n]:|A|\geq 2} |\hat{f}(A)|^2 = 1 - \sum_{i=0}^{n} a_i^2 \geq d^2 - 2d^4\log_2(2/d). \quad (2)$$

We finish the proof by observing that

$$\sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \overset{(1)}{\leq} 2d^4\log_2(2/d) \overset{(2)}{\leq} 2\left(\rho^2+2d^4\log_2(2/d)\right)^2\log_2(2/d)$$

$$\overset{d\geq\rho}{\leq} 2\left(\rho^2+2d^4\log_2(2/\rho)\right)^2\log_2(2/\rho) \overset{d=O(\rho)}{=} 2\rho^4\log_2(2/\rho)+o(\rho^5).$$

uniformly, as $\rho\to 0^+$.

Proof of the discrete cube result - the end

$$\sum_{i=0}^{n} a_i^2 = \left(1-\frac{d^2}{2}\right)^2 + \sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \stackrel{(1)}{\le} \left(1-\frac{d^2}{2}\right)^2 + \frac{1}{4}(2\theta^{-1}-1)d^4$$

$$= 1 - d^2 + \frac{1}{2}\theta^{-1}d^4 \le 1 - d^2 + 2d^4\log_2(2/d),$$

so

$$\rho^2 = \sum_{A\subseteq[n]:|A|\ge 2} |\hat{f}(A)|^2 = 1 - \sum_{i=0}^{n} a_i^2 \ge d^2 - 2d^4\log_2(2/d). \quad (2)$$

We finish the proof by observing that

$$\sum_{i\in\{0,1,\dots,n\}\setminus\{k\}} a_i^2 \stackrel{(1)}{\le} 2d^4\log_2(2/d) \stackrel{(2)}{\le} 2\left(\rho^2+2d^4\log_2(2/d)\right)^2\log_2(2/d)$$

$$\stackrel{d\ge\rho}{\le} 2\left(\rho^2+2d^4\log_2(2/\rho)\right)^2\log_2(2/\rho) \stackrel{d=O(\rho)}{=} 2\rho^4\log_2(2/\rho)+o(\rho^5).$$

uniformly, as $\rho \to 0^+$.

Proof of the discrete cube result - the end

$$\sum_{i=0}^{n} a_i^2 = \left(1-\frac{d^2}{2}\right)^2 + \sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 \overset{(1)}{\leq} \left(1-\frac{d^2}{2}\right)^2 + \frac{1}{4}(2\theta^{-1}-1)d^4$$

$$= 1 - d^2 + \frac{1}{2}\theta^{-1}d^4 \leq 1 - d^2 + 2d^4\log_2(2/d),$$

so

$$\rho^2 = \sum_{A\subseteq[n]:|A|\geq 2} |\hat{f}(A)|^2 = 1 - \sum_{i=0}^{n} a_i^2 \geq d^2 - 2d^4\log_2(2/d). \quad (2)$$

We finish the proof by observing that

$$\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 \overset{(1)}{\leq} 2d^4\log_2(2/d) \overset{(2)}{\leq} 2\left(\rho^2+2d^4\log_2(2/d)\right)^2\log_2(2/d)$$

$$\overset{d\geq\rho}{\leq} 2\left(\rho^2+2d^4\log_2(2/\rho)\right)^2\log_2(2/\rho) \overset{d=O(\rho)}{=} 2\rho^4\log_2(2/\rho)+o(\rho^5).$$

uniformly, as $\rho \to 0^+$.