Strong contraction and influences in tail spaces

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Convexity, probability and discrete structures, a geometric view point conference

Marne-la-Vallée, October 29, 2015

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- What information can be extracted from $f \in L^{>k}$?

Influences

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$$I_i(f) = \sum_{S \ni i} |\hat{f}(S)|^2$$

the influence of the i-th variable on f.

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- We have $I_i(f) = \mathbb{E} Var_i[f]$.
- For $f : \{-1,1\}^n \to \{-1,1\}$, this is equal to the probability that a randomly selected edge of the discrete cube in direction i connects values -1 and 1 of f.

The Bonami-Beckner Operator and Contraction

- P_t re-randomizes each coordinate with probability $1 - e^{-t}$:

$$(P_t f)(x) := \mathbb{E}[f(y)|y \sim_{e^{-t}} x] = \sum_S e^{-t|S|}\hat{f}(S)x_S$$

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- Mendel and Naor: What about other norms?
- Motivation: Study of "super-expanders" (with respect to all convex spaces).

Contraction in Tail Spaces

For $f \in L^{\geq k}$ and $p > 1$:

$$\|P_t f\|_p \leq e^{-c(p)k \min(t,t^2)}\|f\|_p, \quad p \geq 2, \quad \text{Meyer, Mendel-Naor}$$

$$\|P_t f\|_p \leq e^{-c(p)kt}\|f\|_p, \quad p > 1, \quad \text{Conjecture: Mendel-Naor}$$

$$\|P_t f\|_p \leq e^{-c(p)kt}\|f\|_p, \quad \text{if } f \text{ is } \{-1,0,1\}\text{-valued} \quad \text{HMO}$$

An Easy Proof

For $f \in L^{\geq k}$:

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- For $p \geq 2$:

$$\mathbb{E}[|P_t f|^p] \leq \mathbb{E}[|P_t f|^2] \leq e^{-2tk} \mathbb{E}[f^2] = e^{-2tk} \mathbb{E}[|f|^p].$$

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- For $1 < p < 2$ by $(1/(2-p), 1/(p-1))$ Hölder's inequality

$$\begin{aligned} \mathbb{E}[|P_t f|^p] &= \mathbb{E}[|P_t f|^{2-p}|P_t f|^{2p-2}|] \leq (\mathbb{E}[|P_t f|])^{2-p}(\mathbb{E}[|P_t f|^2])^{p-1} \\ &\leq (\mathbb{E}[|f|])^{2-p}e^{-2tk(p-1)}(\mathbb{E}[|f|^2])^{p-1} = e^{-2tk(p-1)}\mathbb{E}[|f|^p] \end{aligned}$$

Contraction in the first ($k = 1$) tail space

For $f \in L^{\geq 1}(\{-1,1\}^n)$, i.e. with $\mathbb{E}[f] = 0$:

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$$\|P_t f\|_p \leq M(n,p)e^{-\delta(n,p)t}\|f\|_p, \quad p > 1, \quad \text{Hino}$$

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- Proof 1: Based on a new type of the Poincaré inequality.
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- Proof 1: Based on a new type of the Poincaré inequality.
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- Proof 2: Based on interpolation.
- Inspired by Nazarov.

An interpolation proof for $p \geq 2$

- Our operator P on $\Omega = (\{-1,1\}^n, \mu)$ satisfies $\|Pf\|_2^2 \leq (1-\varepsilon)\|f\|_2^2$ when $\mathbb{E}[f] = 0$.

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$$\nu(A \times \{0\}) = \mu(A), \quad \nu(A \times \{1\}) = 4\varepsilon \cdot \mu(A)$$

- A new operator $T : L^2(\Omega, \mu) \to L^2(\tilde{\Omega}, \nu)$ by:

$$(Tf)(\omega, 0) = 2 \cdot (Pf)(\omega), \quad (Tf)(\omega, 1) = f(\omega) - \mathbb{E}f$$

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- $$\implies \|Pg\|_p \leq (1 - 2^{2-p}\varepsilon)^{1/p}\|g\|_p.$$

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- Constructions may be useful elsewhere.