Small-ball probabilities for the volume of random convex sets

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Setting:

- $\mathcal{P}_{[n]}$ - Lebesgue abs. cont. prob. measures on $\mathbb{R}^n$
- For $\mu \in \mathcal{P}_{[n]}$, let $f_\mu := \frac{d\mu}{dx}$
- $\mathcal{P}_n^b := \{\mu \in \mathcal{P}_n : \|f_\mu\|_\infty = 1\}$.
- $N \geqslant n$
- $X_1, \dots, X_N$ id random vectors $\sim \mu_1, \cdots \mu_N$

Absolute convex hull:

$$K_N := \operatorname{conv}\{\pm X_1, \dots, \pm X_N\} \tag{1}$$

Zonotope = Minkowski sum of segments $[-X_i, X_i]$:

$$\begin{aligned} Z_N &:= \sum_{i=1}^N [-X_i, X_i] \\ &= \Big\{ \sum_{i=1}^N \lambda_i X_i : \lambda_i \in [-1,1],\ i = 1, \dots, N \Big\}. \end{aligned}$$

Setting

- $n \times N$ random matrix

$$[X_1 \dots X_N] : \mathbb{R}^N \to \mathbb{R}^n$$

- If $C \subset \mathbb{R}^N$ is a convex body, then

$$[X_1 \dots X_N]C = \left\{ \sum_{i=1}^{N} c_i X_i : (c_i) \in C \right\} \subset \mathbb{R}^n.$$

- $K_N := [X_1 \dots X_N] B_1^N$.
- $Z_N := [X_1 \dots X_N] B_\infty^N$.

- D_n- Euclidean ball of $\mathrm{vol}_n(D_n) = 1$
- λ_{D_n} - $\mathrm{vol}_n(\cdot)$ restricted to D_n

Theorem

Suppose $N \geqslant n$ and $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$. Let $C \subset \mathbb{R}^N$ be a convex body and $p > 0$. Then

$$\mathbb{E}_{\otimes \mu_i} \mathrm{vol}_n([X_1 \ldots X_N]C)^p \geqslant \mathbb{E}_{\otimes \lambda_{D_n}} \mathrm{vol}_n([X_1 \ldots X_N]C)^p .$$

- For $C = B_1^N$: get K_N ([Groemer, '74])
- For $C = B_\infty^N$: get Z_N ([Bourgain, Meyer, Milman, Pajor, '88]

Proposition

If $n \leqslant N \leqslant e^n$, then

$$\left(\mathbb{E}_{\otimes\lambda_n} \operatorname{vol}_n(K_N)\right)^{\frac{1}{n}} \simeq \left(\mathbb{E}_{\otimes\gamma_n} \operatorname{vol}_n(K_N)\right)^{\frac{1}{n}} \simeq \sqrt{\frac{\log(2N/n)}{n}}.$$

Moreover,

$$\left(\mathbb{E}_{\otimes\lambda_n} \operatorname{vol}_n(Z_N)\right)^{\frac{1}{n}} \simeq \left(\mathbb{E}_{\otimes\gamma_n} \operatorname{vol}_n(Z_N)\right)^{\frac{1}{n}} \simeq \frac{N}{\sqrt{n}}.$$

Small deviations

Main goal: Find the dependence on n, N and ε in the small-ball probability:

$$\mathbb{P}_{\otimes\mu_i}\left(\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \varepsilon\sqrt{\frac{\log(2N/n)}{n}}\right)$$

and

$$\mathbb{P}_{\otimes\mu_i}\left(\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant \varepsilon\frac{N}{\sqrt{n}}\right)$$

for small ε (uniformly for $\mu_i \in \mathcal{P}_n^b$).

Theorem (Giannopoulos, Hartzoulaki, Tsolomitis '05)

Let $K \subset \mathbb{R}^n$ be a convex body, $\mathrm{vol}_n(K) = 1$ and let $\mu = \mathrm{vol}_n(\cdot)$ restricted to K. Then

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n(K_N)^{1/n} \leqslant c\sqrt{\frac{\ln(2N/n)}{n}}\right) \leqslant e^{-n},$$

Theorem (Litvak, Pajor, Rudelson, Tomczak-J., '05)

Assume

- *coordinates of each X_i are iid, symmetric sub-Gaussian;*
- *$N \geqslant (1+\zeta)n$ and $\beta \in (0, 1/2)$.*

Then

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n(K_N)^{1/n} \leqslant c(\zeta)\sqrt{\frac{\beta\ln(2N/n)}{n}}\right) \leqslant \exp(-c_1 n^{\beta} N^{1-\beta}).$$

Theorem

Let $N \geqslant n$ and $\mu_1, \dots, \mu_N \in \mathcal{P}_n^b$. Let $\varepsilon \in (0,1)$. Then

$$\mathbb{P}_{\otimes \mu_i}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant \frac{c\varepsilon N}{\sqrt{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}.$$

Asymptotically correct as $\varepsilon \to 0$.

Theorem

Let $n \leqslant N \leqslant e^n$ *and let* $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$. *Let* $\delta > 1$ *and* $\varepsilon \in (0,1)$. *Then*

$$\mathbb{P}_{\otimes \mu_i}\left(\left\{\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \frac{c_1\varepsilon}{\delta}\sqrt{\frac{\log(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{c_2 N^{1-1/\delta^2} n^{1/\delta^2}}$$

and, if $N \leqslant n e^{\delta^2}$, *then*

$$\mathbb{P}_{\otimes \mu_i}\left(\left\{\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \frac{c_3\varepsilon}{\delta}\sqrt{\frac{\log(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}.$$

Step 1: Symmetrization

Proposition

Let $\mu_1, \cdots, \mu_N \in \mathcal{P}^b_{[n]}$ and let C 1-uncoditional convex body in $\mathbb{R}^N$. Then

$$(*) \qquad \mathbb{P}_{\otimes\mu_i}\left(\mathrm{vol}_n([X_1, \cdots, X_N]C)^{1/n} \leqslant \varepsilon\right) \leqslant \mathbb{P}_{\otimes\lambda_{D_n}}\left(\mathrm{vol}_n([X_1, \cdots, X_N]C)^{1/n} \leqslant \varepsilon\right).$$

Proof of the proposition in two steps:

- 1) $(*)$ is minimized when $f_{\mu_i} \to f^*_{\mu_i}$.
- 2) Among all rotationally invariant measures the λ_n is the minimizer.

Symmetrization/Rearrangements

For an integrable $f : \mathbb{R}^n \to \mathbb{R}^+$, set

$$f^*(x) = \inf\{t > 0 : \mathrm{vol}_n\left(B(0, |x|)\right) \geqslant \mathrm{vol}_n\left(\{f > t\}\right)\}.$$

- $f^*(x) = f^*(y)$ if $|x| = |y|$.
- $f^*(x) \geqslant f^*(y)$ if $|x| \leqslant |y|$.
- $\mathrm{vol}\left(\{f > t\}\right) = \mathrm{vol}\left(\{f^* > t\}\right)$ for each t.

Theorem (Brascamp, Lieb & Luttinger, '74)

Let $f_1, \dots, f_M : \mathbb{R} \to \mathbb{R}^+$, $u_1, \dots, u_M \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{M} f_i(\langle x, u_i \rangle) dx \leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{M} f_i^*(\langle x, u_i \rangle) dx$$

Corollary

Let $L = -L \subset \mathbb{R}^n$ be convex, $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}^+$. Then

$$\int_L \prod_{i=1}^{n} f_i(x_i) dx \leqslant \int_L \prod_{i=1}^{n} f_i^*(x_i) dx.$$

Groemer

Proposition

Let $F(x_1, \cdots, x_N) := \mathrm{vol}_n([x_1, \cdots, x_N]C)$ *and* $\theta \in S^{n-1}$. *Then for every* $y_1, \cdots y_N \in \theta^\perp$ *the function*

$$F_Y(t_1, \cdots t_N) := F(y_1 + t_1\theta, \cdots, y_N + t_N\theta)$$

is even and convex.

Step 2: Passing to Gaussian measure

Proposition

Let $n < N \leqslant e^n$ and set $m = N/2 + (n-1)/2$. Then for any $p \in (0, (N-n+1)/4)$, we have

$$\left(\mathbb{E}_{\otimes_{i=1}^{m}\overline{\gamma}_n} \operatorname{vol}_n(K_m)^{-p}\right)^{-\frac{1}{pn}} \leqslant c\left(\mathbb{E}_{\otimes_{i=1}^{N}\lambda_{D_n}} \operatorname{vol}_n(K_N)^{-p}\right)^{-\frac{1}{pn}} \quad (2)$$

and

$$\left(\mathbb{E}_{\otimes_{i=1}^{m}\overline{\gamma}_n} \operatorname{vol}_n(Z_m)^{-p}\right)^{-\frac{1}{pn}} \leqslant c\left(\mathbb{E}_{\otimes_{i=1}^{N}\lambda_{D_n}} \operatorname{vol}_n(Z_N)^{-p}\right)^{-\frac{1}{pn}}, \quad (3)$$

where $\overline{\gamma}_n$ be the Gaussian measure on $\mathbb{R}^n$ with density $d\overline{\gamma}_n(x) = e^{-\pi|x|^2}dx$.

Gaussian case

Goal: Prove a reverse-Hölder inequality for standard Gaussian measure γ_n:

$$\left(\mathbb{E}_{\otimes\gamma_n} \mathrm{vol}_n\left(K_N\right)^{-p}\right)^{-\frac{1}{np}} \geqslant c\left(\mathbb{E}_{\otimes\gamma_n} \mathrm{vol}_n\left(K_N\right)\right)^{\frac{1}{n}}$$

for $p > 0$.

For the largest p, we apply Markov:

$$\mathbb{P}_{\otimes\gamma_n}\left(\mathrm{vol}_n\left(K_N\right)^{\frac{1}{n}} \leqslant c_1\varepsilon\left(\mathbb{E}_{\otimes\gamma_n} \mathrm{vol}_n\left(K_N\right)\right)^{\frac{1}{n}}\right) \leqslant \varepsilon^{pn}.$$

Step 3: Intrinsic volumes

Steiner Formula: For a convex body $C \subset \mathbb{R}^N$,

$$\mathrm{vol}_N\left(C + \lambda B_2^N\right) = \sum_{n=0}^{N} \omega_n V_{N-n}(C)\lambda^n.$$

- V_n - n-th intrinsic volume
- $V_N = \mathrm{vol}_N(\cdot)$.
- $V_{N-1} = c_n |\partial K|$ (surface area).
- $V_1 = c_n' w(K)$ (mean-width).

Kubota:

$$V_n(C) = c_{N,n} \int_{G_{N,n}} \mathrm{vol}_n\left(P_E C\right) d\nu_{N,n}(E)$$

Alternate name/normalization: quermassintegrals

Gaussian Representation of Intrinsic Volumes

Theorem (Tsirelson '85, Vitale '07)

Assume

- $C \subset \mathbb{R}^N$ *- convex body;*
- $g_1, \dots, g_N$ *independent standard Gaussian vectors in* $\mathbb{R}^n$.

Then

$$V_n(C) = c_{N,n}\mathbb{E}\left(\mathrm{vol}_n\left([g_1 \dots g_N]C\right)\right),$$

By Kubota,

$$\mathbb{E}\left(\mathrm{vol}_n\left([g_1 \dots g_N]C\right)\right) = c_{N,n}\int_{G_{N,n}} \mathrm{vol}_n\left(P_E C\right) d\nu_{N,n}(E).$$

An extension of the Gaussian rep. of intrinsic volumes

For a convex body $C \subset \mathbb{R}^N$ and $p \in [-\infty, \infty]$, set

$$W_{[n,p]}(C) := \left(\int_{G_{N,n}} \mathrm{vol}_n \left(P_F C \right)^p d\nu_{N,n}(F) \right)^{\frac{1}{np}}.$$

Proposition

Let $N \geqslant n$ and assume that

- *G is an $n \times N$ random matrix with iid N(0,1) entries;*
- *$C \subset \mathbb{R}^N$ a convex set with non-empty interior;*
- *$p > -(N - n + 1)$.*

Then

$$\left(\mathbb{E}\, \mathrm{vol}_n \left(GC \right)^p \right)^{\frac{1}{p}} = \left(\mathbb{E} \det \left(GG^* \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} W^n_{[n,p]}(C).$$

Proposition

Let $N \geqslant n$ and let G be an $n \times N$ random matrix with independent standard Gaussian entries. Then for all $p \in [-(N-n+1-e^{-n(N-n+1)}), N]$,

$$\left(\mathbb{E}\det(GG^*)^{\frac{p}{2}}\right)^{\frac{1}{pn}} \simeq \sqrt{N}.$$

Step 4 :Affine quermassintegrals

Let $C \subset \mathbb{R}^N$ be a convex body and set

$$\Phi_{[n]}(C) := W_{[n,-N]}(C) = \left(\int_{G_{N,n}} \mathrm{vol}_n \left(P_F C \right)^{-N} d\nu_{N,n}(F) \right)^{-\frac{1}{nN}}.$$

[Grinberg '91]: $\Phi_{[n]}$ is $SL(N)$-invariant.

Conjecture (Lutwak, '88)

Let $C \subset \mathbb{R}^N$ be a convex body, $\mathrm{vol}_N(C) = 1$ and $1 < n < N - 1$. Then

$$\Phi_{[n]}(C) \geqslant \Phi_{[n]}(D_N),$$

with equality iff C is an ellipsoid.

Theorem

Let $C \subset \mathbb{R}^N$ be an origin-symmetric convex body of volume one. Then for $1 \leqslant n \leqslant N-1$,

$$\Phi_{[n]}(C) = W_{[n,-N]}(C) \geqslant c\sqrt{\frac{N}{n}}.$$

Since

$$W_{[n,1]}(B_\infty^N) \simeq \sqrt{\frac{N}{n}},$$

we have

$$W_{[n,-p]}(B_\infty^N) \geqslant cW_{[n,1]}(B_\infty^N)$$

for $p = N - n + 1$.

Set $C^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \ \forall y \in K\}$.

Theorem

Let C be an origin-symmetric convex body. Then

$$c \leqslant \left(\frac{\mathrm{vol}_n\,(C)\,\mathrm{vol}_n\,(C^\circ)}{\mathrm{vol}_n\,(B_2^n)} \right)^{1/n} \leqslant 1.$$

- RHS: Blaschke-Santalo
- LHS: [Bourgain, Milman '87], Best-known c: [Kuperberg, '08]

Theorem (Grinberg, '91)

Let $K \subset \mathbb{R}^N$ be a compact set of volume 1. Then

$$\left(\int_{G_{N,n}} \mathrm{vol}_n\,(K \cap F)^N \, d\mu(F) \right)^{\frac{1}{nN}} \leqslant \left(\int_{G_{N,n}} \mathrm{vol}_n\,(D_N \cap F)^N \, d\nu_{N,n}(F) \right)^{\frac{1}{nN}}$$

Set

$$\widetilde{C^\circ} = C^\circ / \operatorname{vol}_n (C^\circ)^{1/N} .$$

By Blaschke-Santalo and Bourgain-Milman,

$$\left(\int_{G_{N,n}} \operatorname{vol}_n (P_F C)^{-N} \, d\mu(F) \right)^{\frac{1}{Nn}} \simeq \frac{n}{N} \left(\int_{G_{N,n}} \operatorname{vol}_n \left(\widetilde{C^\circ} \cap F \right)^N d\mu(F) \right)^{\frac{1}{nN}}$$

By Grinberg,

$$\begin{aligned} \left(\int_{G_{N,n}} \operatorname{vol}_n (P_F C)^{-N} \, d\mu(F) \right)^{\frac{1}{Nn}} &\leqslant c_1 \frac{n}{N} \left(\int_{G_{N,n}} \operatorname{vol}_n (D_N \cap F)^N \, d\mu(F) \right)^{\frac{1}{nN}} \\ &\leqslant c_2 \sqrt{\frac{n}{N}}. \end{aligned}$$

The case of B_∞^N (zonotopes)

Let $m = \frac{N}{2} + \frac{n-1}{2}$ and $p := \frac{N-n-1}{4}$.

$$\frac{\left(\mathbb{E}\,\mathrm{vol}_n\left(GB_\infty^m\right)^{-p}\right)^{-\frac{1}{pn}}}{\left(\mathbb{E}\,\mathrm{vol}_n\left(GB_\infty^m\right)\right)^{\frac{1}{n}}} \simeq \frac{\left(\int_{G_{N,n}} \mathrm{vol}_n\left(P_F B_\infty^m\right)^{-p} d\nu_{m,n}\right)^{-\frac{1}{pn}}}{\left(\int_{G_{m,n}} \mathrm{vol}_n\left(P_F B_\infty^m\right) d\nu_{m,n}\right)^{\frac{1}{n}}} \geq$$

$$\frac{\left(\int_{G_{m,n}} \mathrm{vol}_n\left(P_F B_\infty^m\right)^{-m} d\nu_{m,n}\right)^{-\frac{1}{mn}}}{\left(\int_{G_{m,n}} \mathrm{vol}_n\left(P_F B_\infty^N\right) d\nu_{m,n}\right)^{\frac{1}{n}}} \geq c\,\frac{\sqrt{\frac{m}{n}}}{\sqrt{\frac{m}{n}}} \simeq 1$$

$$\left(\mathbb{E}_{\otimes^N \lambda_n} \mathrm{vol}_n\left(Z_N\right)^{-p}\right)^{-\frac{1}{pn}} \geqslant c\left(\mathbb{E}\, \mathrm{vol}_n\left(GB_\infty^m\right)^{-p}\right)^{-\frac{1}{pn}} \geqslant c' \sqrt{\frac{N}{n}}$$

Markov's inequality

$$\mathbb{P}_{\otimes^N \lambda_n}\left(\mathrm{vol}_n\left(Z_N\right) \leqslant \varepsilon c \frac{N}{\sqrt{n}}\right) \leqslant \varepsilon^p = \varepsilon^{\frac{N-n-1}{4}}$$

Symmetrization

$$\mathbb{P}_{\otimes^N \mu_i}\left(\mathrm{vol}_n\left(Z_N\right) \leqslant \varepsilon c \frac{N}{\sqrt{n}}\right) \leqslant \mathbb{P}_{\otimes^N \lambda_n}\left(\mathrm{vol}_n\left(Z_N\right) \leqslant \varepsilon c \frac{N}{\sqrt{n}}\right) \leqslant \varepsilon^{\frac{N-n-1}{4}}$$

The case of B_1^N (polytopes)

For a convex body $C \subset \mathbb{R}^N$ and $p \in [-\infty, \infty]$, set

$$W_p(C) := \left(\int_{S^{n-1}} h_C(\theta)^p d\sigma(\theta) \right)^{\frac{1}{p}} = W_{[1,p]}(C).$$

Proposition

Let C be a centrally-symmetric convex body in $\mathbb{R}^N$. Then

$$W_{[n,-p]}(C) \geqslant \frac{c}{\sqrt{n}} W_{-np}(C).$$

Proposition

Let $1 \leqslant p \leqslant N$. Then

$$W_{-p}(B_1^N) \simeq \frac{\sqrt{\log \frac{2N}{p}}}{\sqrt{N}}.$$

Negative moments for norms: [Klartag, Vershynin, '07]

Volume bounds for convex hulls

Theorem

Let $N \leqslant e^n$ and let $x_1, \dots, x_N \in \mathbb{R}^n$ with $|x_i| \leqslant M$ for $i = 1, \dots, N$. Then

$$\left(\mathrm{vol}_n\left(\mathrm{conv}\left\{\pm x_1, \dots, \pm x_N\right\}\right)\right)^{1/n} \leqslant \frac{cM\sqrt{\log(2N/n)}}{n}.$$

[Gluskin, '88], [Carl-Pajor '88], [Ball-Pajor, '90], [Barany-Füredi, 87-88]

Volume bounds for zonotopes

Lemma

Let $n \leqslant N$ and let $x_1, \dots, x_N \in \mathbb{R}^n$ with $|x_i| \leqslant M$ for each $i = 1, \dots, N$. Then

$$\mathrm{vol}_n\left(\sum_{i=1}^{N}[-x_i, x_i]\right)^{1/n} \leqslant \frac{cNM}{n}.$$

Proof.

$$\mathrm{vol}_n\left(\sum_{i=1}^{N}[-x_i, x_i]\right) = 2^n \sum_{|I|=n} \left|\det[x_i]_{i\in I}\right|.$$

Hadamard: $|\det[x_i]| \leqslant \prod_i |x_i|$

$$\mathrm{vol}_n\left([x_1 \dots x_N]B_\infty^N\right) \leqslant 2^n \binom{N}{n} \max_{i\in I} \left|\det[x_i]_{i\in I}\right| \leqslant \left(\frac{2eN}{n}\right)^n M^n.$$

Lower bound for the small ball

Let $\mu_1, \cdots \mu_N \in \mathcal{P}^b_{[n]}$ and assume that f_{μ_i} are continuous at 0 and $f_{\mu_i}(0) = \|f_{\mu_i}\|_\infty = 1$. Then there exists $\varepsilon_0 = \varepsilon_0(\mu)$ such that $|f_{\mu_i}(x)| \geqslant 1/2$ whenever $|x| \leq \varepsilon_0 c\sqrt{n}$, so,

$$\begin{aligned}
\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n(K_N)^{1/n} \leqslant c\varepsilon\sqrt{\frac{\ln(2N/n)}{n}}\right) &\geqslant \mathbb{P}_{\otimes\mu}\left(|X_i| \leqslant c\varepsilon\sqrt{n}, i = 1, N\right) \\
&= \mathbb{P}_{\mu}\left(|X_1| \leqslant c\varepsilon\sqrt{n}\right)^N \\
&\geqslant \varepsilon^{nN}.
\end{aligned}$$

Same argument for Z_N.

Isotropicity

Let $\mathcal{P}_n^{\mathrm{cov}}$:= the set of measures $\mu \in \mathcal{P}_n$ with bounded densities such that the covariance matrix of μ is well-defined.

$\mu \in \mathcal{P}_n^{\mathrm{cov}}$ is isotropic if its covariance matrix is the identity. When μ is isotropic, we define its isotropic constant L_μ by $L_\mu := \|f_\mu\|_\infty^{1/n}$.

Theorem

Let $n \leqslant N \leqslant e^n$. Let $\mu \in \mathcal{P}_n^{\mathrm{cov}}$ and assume that μ is isotropic. Then, for every $\varepsilon \in (0,1)$,

$$\mathbb{P}_{\otimes\mu}\left(\left\{\mathrm{vol}_n\,(K_N)^{1/n} \leqslant \frac{c\varepsilon}{\delta L_\mu}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{c_1 N^{1-1/\delta^2} n^{1/\delta^2}} \quad (4)$$

and, if $N \leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes\mu}\left(\left\{\mathrm{vol}_n\,(K_N)^{1/n} \leqslant \frac{c\varepsilon}{\delta L_\mu}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}.$$

Similarly,

$$\mathbb{P}_{\otimes\mu}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{L_\mu}\frac{N}{\sqrt{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}.$$

log-concave case

Hyperplane conjecture $\leftrightarrow$ "For every log-concave isotropic measure $L_\mu \simeq 1$".

If μ is an isotropic log-concave measure probability measure on $\mathbb{R}^n$ with bounded isotropic constant and $n \leqslant N \leqslant e^n$, then

$$\left(\mathbb{E}_{\otimes\mu} \operatorname{vol}_n(K_N)\right)^{\frac{1}{n}} \simeq \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \simeq \left(\mathbb{E}_{\otimes\lambda_{D_n}} \operatorname{vol}_n(K_N)\right)^{\frac{1}{n}};$$

Theorem

Let $n \leqslant N \leqslant e^n$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ with bounded isotropic constant. Then, for every $\varepsilon \in (0,1)$,

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n(K_N)^{\frac{1}{n}} \leqslant \frac{c\varepsilon}{\delta}\left(\mathbb{E}_{\otimes\mu}\,\mathrm{vol}_n(K_N)\right)^{\frac{1}{n}}\right) \leqslant \varepsilon^{c_1 N^{1-1/\delta^2} n^{1/\delta^2}} \tag{5}$$

and, if $N \leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n(K_N)^{\frac{1}{n}} \leqslant \frac{c\varepsilon}{\delta}\left(\mathbb{E}_{\otimes\mu}\,\mathrm{vol}_n(K_N)\right)^{\frac{1}{n}}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}, \tag{6}$$

where c and c_1 are absolute constants.

Proposition

Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Let X be a random vector distributed according to μ. Then for every $\varepsilon \in (0,1)$,

$$\mathbb{P}_\mu\left(|X| \leqslant c_1 \varepsilon \sqrt{n}\right) \geqslant \varepsilon^n,$$

where $c_1 > 0$ is an absolute constant.

Proposition

Let $n \leqslant N \leqslant e^n$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then for every $\varepsilon \in (0,1)$,

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(K_N\right)^{\frac{1}{n}} \leqslant c_1 \varepsilon \sqrt{\frac{\ln(2N/n)}{n}}\right) \geqslant \varepsilon^{Nn},$$

and

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant \frac{c_2 \varepsilon N}{\sqrt{n}}\right) \geqslant \varepsilon^{Nn},$$

where c_1, c_2 are absolute constants.