Random Matrices and Subexponential Operator Spaces

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Our initial goal was to study a generalization of the notion of exact operator space for which the operator space version of Grothendieck's theorem obtained in joint work with Shlyakhtenko is still valid.

Motivation

Let E be a d-dimensional Banach space
In * **Remarques sur un résultat non publié de B. Maurey (1980/1981)**
I introduced the number

$$k_E(C) = \inf\{k \mid E \overset{C}{\subset} \ell_\infty^k\}$$

Gaussian random variables can be used to give a quick proof of the fact that if either $E = \ell_2^d$ or $E = \ell_1^d$ then $\exists \delta = \delta_C > 0$ such that

$$k_E(C) \geq \exp(\delta d).$$

In * it is shown same holds (with $\delta = \delta(C, C') > 0$) whenever E^* has type $p > 1$ with constant at most C'.
Open Problem (dichotomy conjecture): Same holds whenever E has finite cotype $q < \infty$

Geometrically

Recall: Essentially **any (symmetric convex) body** is equivalent to a d-dimensional section of an k-cube of dimension $d \approx \log k$ (equivalently $k_E(C) \leq \exp(cd)$ for any d-dimensional E)

Consider a section of an n-cube of dimension $d >> \log n$: does it admit a further section of large dimension (i.e. $>> 1$) that is equivalent to a cube ?

Operator spaces

Definition

("Non-commutative Banach spaces") An operator space is a subspace of $B(\mathcal{H})$, i.e. we are given

$$E \subset B(H)$$

"Operator space Theory" is now well developed after Ruan's1987 thesis cf. Effros-Ruan, Blecher-Paulsen...

Let

$$u: \ E \to F$$

be a linear map between operator spaces. We denote for any given $N \geq 1$

$$u_N = Id \otimes u : \ M_N(E) \to M_N(F)$$

$$[a_{ij}] \mapsto [u(a_{ij})]$$

$$(\ u_N = Id \otimes u : \ M_N \otimes E \to M_N \otimes F$$

$$\sum a_k \otimes x_k \mapsto \sum a_k \otimes u(x_k) \ )$$

Recall that

$$\|u\|_{cb} = \sup_{N \geq 1} \|u_N\|.$$

Minimal Tensor Product

Given

$$E \subset B(H) \qquad F \subset B(\mathcal{H})$$

we define

$$E \otimes_{\min} F \subset B(H \otimes_2 \mathcal{H})$$

("spatial" or "minimal" tensor product)

Note: This norm will be used everywhere !

$$B = B(H) \quad \text{or} \quad B = M_N \quad \text{if } \dim(H) = N$$

Quantization: Scalars replaced by operators

Banach space structure on E: a norm on $\mathbb{C} \otimes E$: ($E \simeq \mathbb{C} \otimes E$)

$$\left\| \sum c_k \otimes x_k \right\| \quad c_k \in \mathbb{C} \;\; x_k \in E$$

Operator space structure on E: a norm on $B \otimes E$

$$\left\| \sum b_k \otimes x_k \right\| \quad b_k \in B \;\; x_k \in E$$

Here

$$\|.\| = \|.\|_{\min}$$

Matricial ("Quantized") Banach-Mazur distance

Assuming $E \simeq F$, we set

$$d_N(E,F) = \inf\{\|u_N\| \|(u^{-1})_N\|\}$$

where the inf runs over all the isomorphisms $u: \ E \to F$.
We set $d_N(E,F) = \infty$ if E, F are not isomorphic.
Also, if E, F are completely isomorphic, we set

$$d_{cb}(E,F) = \inf\{\|u\|_{cb} \|u^{-1}\|_{cb}\}$$

where the inf runs over all the complete isomorphisms

$$u: \ E \to F.$$

If E, F finite dim by compactness:

$$d_{cb}(E,F) = \sup_{N \geq 1} d_N(E,F)$$

Recall (for E finite dim. Banach space)

$$k_E(C) = \inf\{k \mid E \overset{C}{\subset} \ell_\infty^k\}$$

Matricial version of $k_E(C)$

Let E be a finite dimensional operator space. Fix $C > 0$. We denote by $k_E(N, C)$ the smallest integer k such that there is an operator subspace $F \subset M_k$ such that

$$d_N(E, F) \leq C.$$

In short:

$$k_E(N, C) = \inf\{k \mid E \overset{N,C}{\subset} M_k\}$$

Subexponential operator spaces

Definition

We say that an operator space X is C-subexponential if

$$\limsup_{N\to\infty} \frac{\log k_E(N, C)}{N} = 0.$$

for any finite dimensional subspace $E \subset X$.
Note: If X itself is finite dimensional, it suffices to consider $E = X$.
We will denote by $C(X)$ the smallest C such that X is C-subexponential.

An operator space X is called C-exact if for any finite dimensional subspace $E \subset X$ and any $c > C$ there is a k and $F \subset M_k$ such that $d_{cb}(E, F) < c$. We denote by $ex(X)$ the smallest such C.

Lemma

An operator space X is C-exact iff

$$\sup_{N \geq 1} k_E(N, C) < \infty.$$

for any finite dimensional subspace $E \subset X$.

Proof: by a standard compactness argument

Let $Y^{(N)}$ be a random $N \times N$-matrix with i.i.d. complex Gaussian entries with mean zero and second moment equal to $N^{-1/2}$, let $(Y_j^{(N)})$ be a sequence of i.i.d. copies of $Y^{(N)}$.

Theorem (Implicit in Haagerup-Thorbjørnsen 1999)

For any $a_1, \cdots, a_r \in B(H)$, let $S = \sum_1^r a_j \otimes Y_j^{(N)}$.
Assume $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \leq 1$.
For any integer $p \geq 1$, let $\Sigma = \mathbb{E}(S^* S)^p$. Then:

$$\mathbb{E}(S^* S)^p = \Sigma \otimes 1 \leq \mathbb{E}(Y^{(N)^*} Y^{(N)})^p \otimes 1.$$

Consequently, if $\dim(H) = k$ we have

$$(\mathbb{E}\,\mathrm{tr}|S|^{2p})^{1/2p} \leq (k\mathbb{E}\,\mathrm{tr}|Y^{(N)}|^{2p})^{1/2p}. \tag{1}$$

where the trace on the left is on $\ell_2^N \otimes H$ and the one on the right is on ℓ_2^N.

Remark

Actually, (using Buchholz 2001) more generally, if $\max\{\|(\sum a_j^* a_j)^{1/2}\|_{2p}, \|(\sum a_j a_j^*)^{1/2}\|_{2p}\} \le 1$, then

$$\mathbb{E}\|S\|_{2p}^{2p} \le \mathbb{E}\|Y^{(N)}\|_{2p}^{2p}.$$

Obviously best possible.
Interpretation: Khintchine type inequality for Gaussian random matrices with best possible constant.

Application of concentration of measure

Lemma

For any $\varepsilon > 0$, there is a constant γ_ε such that for any integer k, any $N \geq 1$ and any $a_1, \cdots, a_r \in M_k$ we have

$$\mathbb{E}\left\|\sum_1^r a_j \otimes Y_j^{(N)}\right\| \leq$$

$$(1+\varepsilon)\left(2+\gamma_\varepsilon\left(\frac{\log(ek)}{N}\right)^{1/2}\right)\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}.$$

$\forall \varepsilon > 0$, $\exists \gamma_\varepsilon$ such that $\forall a_1, \cdots, a_r \in M_k$ we have

$$\mathbb{E}\| \sum_1^r a_j \otimes Y_j^{(N)} \| \leq$$

$$(1+\varepsilon)(2+\gamma_\varepsilon(\frac{\log(ek)}{N})^{1/2}) \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}.$$

This leads us to

Theorem

Let E be a C-subexponential operator space i.e.

$$\frac{\log k_E(N, C)}{N} \to 0$$

Then for any $a_1, \cdots, a_r \in E$ we have

$$\overline{\lim_{N\to\infty}} \mathbb{E} \left\| \sum_1^r a_j \otimes Y_j^{(N)} \right\| \leq 2C \max\{ \left\|(\sum a_j^* a_j)^{1/2}\right\|, \left\|(\sum a_j a_j^*)^{1/2}\right\| \}.$$

Proof.

Fix $c > C$. Consider $u : \ E \to F$ with $F \subset M_k$, $k = k_E(N, C)$ and $\|u_N\| \|{u^{-1}}_N\| \leq c$.
By homogeneity we may assume $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \leq 1$. Let $b_j = u(a_j)$. We may assume $r \leq N$. Then we have

$$\max\{\|(\sum b_j^* b_j)^{1/2}\|, \|(\sum b_j b_j^*)^{1/2}\|\} \leq \|u_r\| \leq \|u_N\|,$$

and also

$$\|\sum_1^r a_j \otimes Y_j^{(N)}\| \leq \|{u^{-1}}_N\| \|\sum_1^r b_j \otimes Y_j^{(N)}\|.$$

This gives us

$$\|\sum_1^r a_j \otimes Y_j^{(N)}\| \leq \|{u^{-1}}_N\| \|u_N\| (1+\varepsilon)(2 + \gamma_\varepsilon (\frac{\log(k)+1}{N})^{1/2},$$

Notation: For $a = (a_1, \cdots, a_r)$ with $a_j \in E$ we denote

$$\|a\|_{RC} = \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\},$$

$$\|a\|_R = \|(\sum a_j a_j^*)^{1/2}\|$$

and

$$\|a\|_C = \|(\sum a_j^* a_j)^{1/2}\|,$$

so that

$$\|a\|_{RC} = \max\{\|a\|_R, \|a\|_C\}.$$

Corollary

Let E, F be subexponential operator spaces with respective constants $C(E), C(F)$. Then any c.b. linear map $u: E \to F^*$ satisfies for any r, any $a = (a_1, \cdots, a_r) \in E^r$ and any $b = (b_1, \cdots, b_r) \in F^r$

$$|\sum \langle u(a_j), b_j \rangle| \leq 4C(E)\ C(F) \|u\|_{cb} \|a\|_{RC} \|b\|_{RC}.$$

It is easy to check that if E is C-subexponential, the minimal tensor product $K(\ell_2) \otimes_{\min} E$ (of E with the set $K(\ell_2)$ of all compact operators on ℓ_2) is also C-subexponential
Indeed, this follows from the fact that, if E is C-subexponential, $\forall n \quad M_n(E)$ is C-subexponential because

$$k_{M_n(E)}(C, N) \leq k_E(C, N/n)$$

By known arguments (there is a very recent much simpler proof by Regev and Vidick RV), and combining that with JP, we obtain:

Corollary

Let E, F be subexponential operator spaces with respective constants $C(E), C(F)$. Then any c.b. linear map $u: E \to F^*$ with $\|u\|_{cb} \leq 1$ satisfies for any r, any $a = (a_1, \cdots, a_r) \in E^r$ and any $t_j > 0$ and any $b = (b_1, \cdots, b_r) \in F^r$

$$|\sum \langle u(a_j), b_j \rangle| \leq c(\|(a_j)\|_R \|(b_j)\|_C + \|(t_j a_j)\|_C \|(t_j^{-1} b_j)\|_R).$$

with

$$c = 4C(E)\, C(F).$$

Assuming $E \subset B(H)$ and $F \subset B(K)$ Hahn-Banach $\Rightarrow \exists$ states f_1, f_2, g_1, g_2 such that $\forall (a, b) \in E \times F$

$$|\langle u(a), b \rangle| \leq c \left(\sqrt{f_1(aa^*)} \sqrt{g_1(b^* b)} + \sqrt{f_2(a^* a)} \sqrt{g_2(bb^*)} \right)$$

We now compare the subexponential case with the general case. Unfortunately, there is a gap between the two estimates

–subGaussian///subexponential–

that we do not see how to fill.

Lemma

Let E be a d-dimensional operator space, then for any $\delta > 0$ we have

$$k_E(N, 1+\delta) \leq N(1+\delta^{-1})^{dN^2}.$$

Therefore, for any operator space X, any finite dimensional subspace $E \subset X$ we have

$$\forall C > 1 \quad \limsup_{N \to \infty} \frac{\log k_E(N, C)}{N^2} < \infty.$$

Underlying reason: $\dim(M_N(E)) = dN^2$!

Remark

A possible **variant** could be to define X as (C, C')-subexponential if for any finite dimensional $E \subset X$ we have

$$\limsup_{N\to\infty} N^{-1} \log k_E(N, C) \leq C'.$$

We conclude by examining some examples. It turns out to be easy to show that the known examples of non-exact operator spaces are also not subexponential. Thus we unfortunately must leave as **an open problem (and a conjecture) the existence of subexponential spaces that are *not* exact.**

Examples

Minimal operator space $E \subset A \simeq \ell_\infty \subset B(H)$
Equivalently $\forall F \; \forall u : \; F \to E$ we have $\|u\|_{cb} = \|u\|$
Example:$E = \ell_\infty$

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Maximal operator space (dual to minimal)
Equivalently $\forall F \; \forall u : \; E \to F$ we have $\|u\|_{cb} = \|u\|$
Example:$E = \mathrm{span}\{U_j\}$ (universal sequence of unitaries)
Note : $E \simeq \ell_1$ (with “maximal" operator space structure)

Proposition

Let E be any d-dimensional space with its maximal operator space structure (e.g. $E = \ell_1^d$). Then

$$C(E) \geq c\sqrt{d}$$

where $c > 0$ is a constant independent of d.

Proof: We transplant from exact to subexponential an argument from [Junge-P 1994].

Remark

In the converse direction, for any d-dimensional operator space E we have

$$C(E) \leq ex(E)$$

and it is known ([P 1996]) that

$$ex(E) \leq d^{1/2}$$

The Operator Hilbert space OH

$$OH_d = \mathrm{span}[T_1, \cdots, T_d]$$

$$\forall a_j \in B(H) \quad \| \sum a_j \otimes T_j \| = \| \sum a_j \otimes \bar{a}_j \|^{1/2}$$

Remark

We claim that

$$d^{1/4}/2 \le C(OH_d) \le d^{1/4}.$$

$\limsup_{N\to\infty} \mathbb{E} \left(\left\| \sum_1^d Y_j^{(N)} \otimes \overline{Y_j^{(N)}} \right\|^{1/2} \right) \le 2C(OH_d) d^{1/4}$, and

since $\left\| \sum_1^d Y_j^{(N)} \otimes \overline{Y_j^{(N)}} \right\| \ge \sum_1^d \mathrm{tr} |Y_j^{(N)}|^2 \approx d$ we obtain

$$d^{1/2} = \limsup_{N\to\infty} \mathbb{E} \left((\sum_1^d \mathrm{tr} |Y_j^{(N)}|^2)^{1/2} \right) \le 2C(OH_d) d^{1/4}$$

Remark

Similarly

$$d^{1/2}/2 \leq C(R_d + C_d) \leq d^{1/2}.$$

Lemma

If E is ℓ_1^d equipped with its maximal operator space structure then for any $C > 1$ there is $\delta > 0$ depending only on C such that for any d, N we have

$$k_E(N, C) \geq \exp \delta N d \tag{2}$$

If $E = R_d + C_d$ or ℓ_2^d equipped with its maximal operator space structure (resp. $E = OH_d$), this still holds (resp. we have $k_E(N, C) \geq \exp \delta N d^{1/2}$) for all $N \geq d$.

Regev and Vidick's quick proof of the OS- GT

O. Regev and T. Vidick have found a very quick, simple and more quntitative proof of the os version of GT proved by Haagerup-Musat for c.b. bilinear forms on C^*-algebras, and it applies also to the previous result on bilinear forms on exact or subexponential spaces

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Note:

$$\text{Let } u: \ E \times F \to \mathbb{C}$$

$$\|u\|_{cb} \leq 1$$

$$\text{IFF}$$

$$\forall n \, \forall \xi, \eta \in B_{\ell_2^n \otimes_2 \ell_2^n} \quad \|\Phi_{\xi,\eta} \otimes u : M_n(E) \times M_n(F) \to \mathbb{C}\| \leq 1$$

where $\Phi_{\xi,\eta} : \ M_n \times M_n \to \mathbb{C}$ is defined by

$$\Phi_{\xi,\eta}(a, b) = \langle (a \otimes b)\xi, \eta \rangle$$

Regev and Vidick manage to deduce directly from

$$(\dagger\dagger\dagger)\ \ |\sum\langle u(a_j), b_j\rangle| \le (\|a\|_R^2 + \|a\|_C^2)^{1/2}(\|b\|_R^2 + \|b\|_C^2)^{1/2}$$

assumed satisfied also on $M_n(E) \times M_n(F)$ for all n that for any $t_j > 0$

$$(***)\ \ |\sum\langle u(a_j), b_j\rangle| \le \|(a_j)\|_R\|(b_j)\|_C + \|(t_ja_j)\|_C\|(t_j^{-1}b_j)\|_R.$$

• Using this idea the OS version of GT of Haagerup-Musat (Invent. 2008) is reduced to Haagerup's 1985 non-commutative GT (extending my own 1978 version)

• In the subexponential or exact case, the same idea reduces the version of P-Shlyakhtenko (Invent. 2002) to the Junge-P version (GAFA 1994)

Regev and Vidick's key ingredient:

Just a family of $n \times n$ matrices $E(t)$ indexed by $t > 0$ with non-negative entries.
$\forall t > 0$

$$\sup_i \sum_j L(t)_{ij} \leq 1$$

$$\sup_j \sum_i L(t)_{ij} \leq t^2$$

and lastly

$$|t^{-1}\langle L(t)z_n, z_n\rangle - 1| \leq C(\log n)^{-1} \log(1 + \max\{t, t^{-1}\}), \quad (3)$$

where $z_n = (Z_n)^{-1/2} \sum_1^n j^{-1/2} e_j$ with $Z_n \approx \log n$ defined so that z_n is a unit vector in ℓ_2^n.
Note: $E(t)$, z above are what replaces the use of q-Gaussians for $q = -1$ in H-M and for $q = 0$ in P-S

The definition of $L(t)_{ij}$ is extremely simple: they just define it to be the length of the interval $[i-1, i] \cap [(j-1)t^2, jt^2]$. The verification of all the above properties is then entirely elementary.

Let $\xi_n \in \ell_2 \otimes_2 \ell_2$ be the diagonal operator with the same diagonal entries as z_n so that $\|\xi_n\|_{\ell_2 \otimes_2 \ell_2} = 1$.
Let $\Phi_n : \ M_n \times M_n \to \mathbb{C}$ be the bilinear form defined by

$$\Phi_n(a, b) = \langle (a \otimes b)\xi_n, \xi_n \rangle.$$

Assume that $\|u\|_{cb} \leq 1$ then in particular

$$\|\Phi_n \otimes u : \ M_n(E) \times M_n(F)\| \leq 1.$$

Notation

$$\|ta\|_C = \|(t_j a_j)\|_C, \quad \|t^{-1} a\|_C = \|(t_j^{-1} a_j)\|_C,$$

and same with R

Theorem

Suppose $\Phi_n \otimes u : M_n(E) \times M_n(F) \to \mathbb{C}$ *satisfies* $\dagger\dagger\dagger$ *(e.g. if* E, F *are* C^**-algebras* $\|u\| \leq 1$ *suffices) then* $\forall a_j, b_j \in E, F$ *and* $\forall t_j > 0$

$$|\sum u(a_j, b_j)| \leq (\|a\|_R^2 + \|ta\|_C^2)^{1/2}(\|t^{-1}b\|_R^2 + \|b\|_C^2)^{1/2}$$

$$+C(\log n)^{-1} \sum_j \log(1 + \max\{t_j, t_j^{-1}\})|u(a_j, b_j)|$$

Therefore if $n \to \infty$ *we obtain*

$$|\sum u(a_j, b_j)| \leq (\|a\|_R^2 + \|ta\|_C^2)^{1/2}(\|t^{-1}b\|_R^2 + \|b\|_C^2)^{1/2},$$

and after elementary manipulation we obtain the os-GT in the Haagerup-Musat form:

$$|\sum u(a_j, b_j)| \leq \|a\|_R\|b\|_C + \|ta\|_C\|t^{-1}b\|_R.$$

Proof.

$\forall m = (p, q, j)$, let $X_m \in M_n \otimes E$, $Y_m \in M_n \otimes F$ defined by:

$$X_m = L(t_j)_{p,q} e_{p,q} \otimes a_j \qquad Y_m = t_j^{-1} {L(t_j)_{p,q}}^{1/2} e_{p,q} \otimes b_j.$$

$$\|X\|_C = \|ta\|_C, \ \|X\|_R = \|a\|_R, \ \|Y\|_C = \|b\|_C, \ \|Y\|_R = \|t^{-1}b\|_R.$$

$$|\sum_m [\Phi_n \otimes u](X_m, Y_m)| \le (\|X\|_R^2 + \|X\|_C^2)^{1/2} (\|Y\|_R^2 + \|Y\|_C^2)^{1/2}$$

and hence

$$|\sum_m [\Phi_n \otimes u](X_m, Y_m)| \le (\|x\|_R^2 + \|tx\|_C^2)^{1/2} (\|t^{-1}y\|_R^2 + \|y\|_C^2)^{1/2}$$

But now (4) allows us to conclude...

$$|t^{-1}\langle L(t)z_n, z_n\rangle - 1| \le C(\log n)^{-1} \log(1 + \max\{t, t^{-1}\}), \qquad (4)$$

THE END !!