Ricci curvature lower bounds
and
branching geodesics

Tapio Rajala
University of Jyväskylä
tapio.m.rajala@jyu.fi

Autumn days in Grenoble:
"Curvature in metric spaces"
Grenoble, Oct 17th, 2013

Plan of the talk

Part I

- Branching geodesics.
- Examine a "toy example" $(\mathbb{R}^2, |\cdot|_p)$.
- Local-to-global property in $CD(K, N)$.

Part II

- $RCD(K, N)$ and strong $CD(K, N)$.
- Essential nonbranching.
- Optimal maps.

Part III (if time permits)

- Existence of geodesics with bounded density in $CD(K, N)$.
- Local Poincaré inequalities from geodesics with bounded density.

Basic assumptions and notation:

- (X, d) is complete, separable and geodesics.
- $\mathfrak{m}$ is locally finite Borel measure with $\mathrm{spt}(\mathfrak{m}) = X$.
- $\mathrm{Geo}(X)$ denotes the space of all constant speed geodesics on X parametrized by $[0, 1]$, i.e. for every $\gamma \in \mathrm{Geo}(X)$ we have $\mathsf{d}(\gamma_t, \gamma_s) = |s - t| \mathsf{d}(\gamma_0, \gamma_1)$ for all $0 \leq s \leq t \leq 1$.

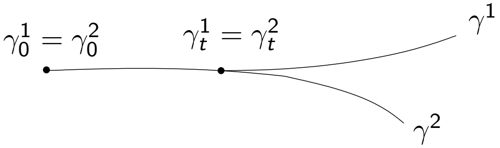
Part I

Branching geodesics,
$(\mathbb{R}^2, |\cdot|_p)$ and
local-to-global property.

Branching geodesics

Definition

Two geodesics $\gamma^1, \gamma^2 \in \mathrm{Geo}(X)$, $\gamma^1 \neq \gamma^2$, branch, if there exists $0 < t < 1$ such that $\gamma^1_s = \gamma^2_s$ for all $s \in [0, t]$.



A space where there are no branching geodesics is called non-branching.

Non-branching spaces

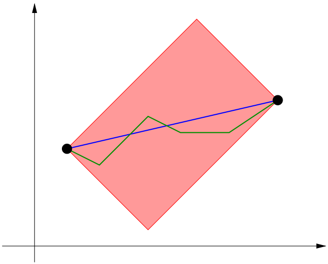
- Riemannian manifolds are non-branching
- Alexandrov spaces with curvature bounded below are non-branching
- Some $CD(K, N)$ spaces are branching
- Are $RCD(K, N)$ spaces non-branching?
- Are Ricci-limits (mGH-limits of Riemannian manifolds with Ricci-curvature bounded below) non-branching?

Basic examples of $CD(K, N)$ spaces

Theorem (Cordero-Erausquin, Sturm and Villani)

The space $(\mathbb{R}^n, ||\cdot||, \mathscr{L}_n)$ *with any norm* $||\cdot||$ *satisfies* $CD(0, n)$.

Let us consider the space $(\mathbb{R}^2, ||\cdot||_\infty, \mathscr{L}_2)$, with $||(x,y)||_\infty = \max(|x|, |y|)$.



This space is very branching.

How does OT in $(\mathbb{R}^2, ||\cdot||_\infty)$ look like?

Suppose you transport a measure μ_0 to μ_1, where μ_0 and μ_1 have small supports and $|x_0 - x_1| >> |y_0 - y_1|$ with $(x_i, y_i) \in \mathrm{spt}(\mu_i)$.

Then the cost of any transport σ is given by

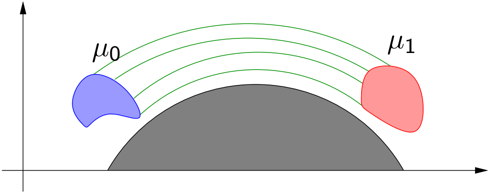
$$\left(\int_{\mathbb{R}^2\times\mathbb{R}^2} ||(x_1, y_1) - (x_2, y_2)||_\infty^2 \, d\sigma\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^2\times\mathbb{R}^2} |x_1 - x_2|^2 \, d\sigma\right)^{\frac{1}{2}}.$$

In other words, any monotone rearrangement in the x-direction is an optimal transport. What is done in the y-direction does not matter.

How does OT in $(\mathbb{R}^2, ||\cdot||_\infty)$ look like?

Thus the fact that $(\mathbb{R}^2, ||\cdot||_\infty, \mathscr{L}_2)$ is $CD(0,2)$ can be seen by combining monotone rearrangements in x and y directions and using the fact that $\mathbb{R}$ is $CD(0,1)$.

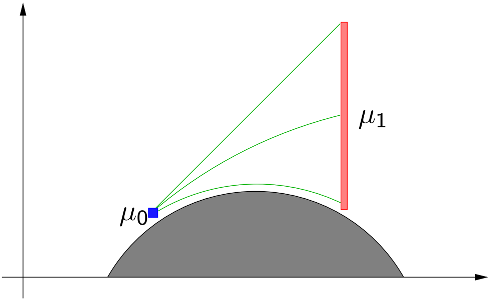
But there are many other ways to transport in the y-direction. We can even try to put obstacles!



Does the planar region with the obstacle satisty $CD(0,2)$?

How does OT in $(\mathbb{R}^2, ||\cdot||_\infty)$ look like?

Does the planar region with the obstacle satisty $CD(0,2)$? No.

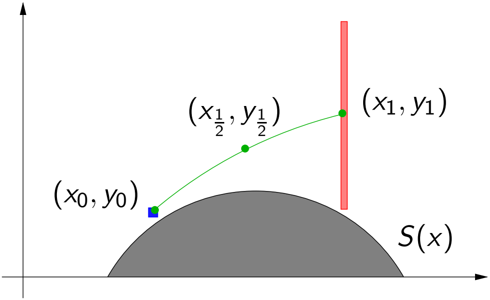


We should have $CD(0,1)$ for the transport in the y-direction, but this is impossible.

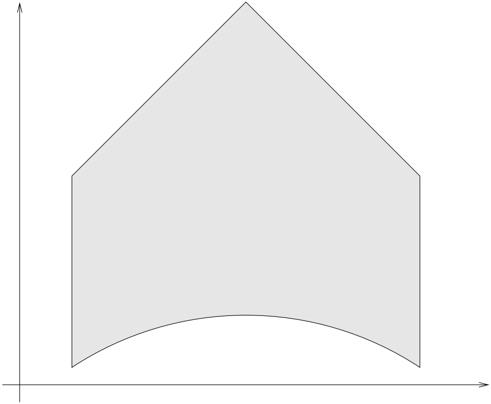
However, the transport in the y-direction can be chosen so that it satisfies $CD(0,2)$. In order to obtain this we define the y-coordinate of the midpoint between (x_0, y_0) and (x_1, y_1) as

$$y_{\frac{1}{2}} = \max\left\{ \frac{y_0 + y_1}{2}, \frac{S(x_0) + S(x_1)}{2} + (x_0 - x_1)^2 \right.$$

$$\left. + \frac{1}{4}\left(\sqrt{y_0 - S(x_0)} + \sqrt{y_1 - S(x_1)}\right)^2 \right\}$$

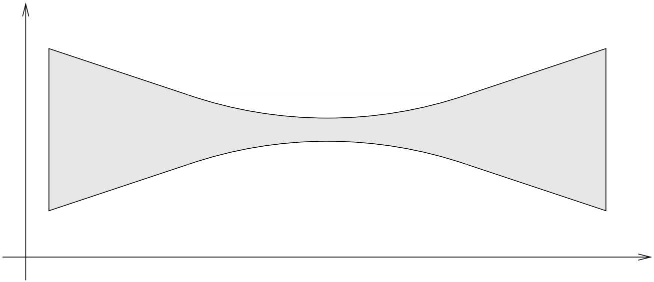


Combining the x and y directions we see that closed subsets like



Satisfy $CD(0, 4)$...

...but on the other hand subsets like



do not satisfy $CD(0,4)$ (or even $CD(K,\infty)$). Hence

Theorem (R. (2013, preprint))

For every $K \in \mathbb{R}$ there exists a compact geodesic metric measure space $(X, \mathrm{d}, \mathfrak{m})$ satisfying $CD(0,4)$ locally, but failing to satisfy $CD(K,\infty)$.

Other results on local-to-global

- Sturm (2006): nonbranching $CD(K, \infty)$ spaces have local-to-global property.
- Villani (2009): nonbranching $CD(0, N)$ spaces have local-to-global property.
- Bacher-Sturm (2010): nonbranching $CD^*(K, N)$ spaces have local-to-global property.
- Ambrosio-Mondino-Savaré (preprint 2013): $RCD^*(K, N)$ spaces have local-to-global property.
- Cavalletti-Sturm (2012): nonbranching $CD^*(K, N)$ self-improves to $MCP(K, N)$.
- Cavalletti (2012, to appear): nonbranching $CD^*(K, N)$ self-improves to $CD(K, N)$ for many other transports. (but not necessarily for all?)
- Cavalletti (preprint yesterday): Existence of optimal maps in the W_1 distance in $RCD^*(K, N)$ spaces. This goes in the direction of proving self-improvement results for $RCD^*(K, N)$.

Related questions remaining

- Question: Does Cavalletti's result hold for all transports in nonbranching $CD^*(K, N)$? If yes, then nonbranching $CD(K, N)$ has local-to-global.
- Question: Does Cavalletti's result hold for all transports in $RCD^*(K, N)$? I.e., does $RCD^*(K, N)$ self-improve to $RCD(K, N)$?
- Question: Does Cavalletti's result hold for all transports in $CD^*(K, N)$? I.e. does $CD^*(K, N)$ self-improve to $CD(K, N)$?

Part II

$RCD(K, N)$ and strong $CD(K, N)$,
essential nonbranching and
optimal maps

$RCD(K, \infty)$

Recall the following

Definition (Ambrosio, Gigli & Savaré (2011, preprint))

A metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has *Riemannian Ricci curvature* bounded below by $K \in \mathbb{R}$, or $RCD(K, \infty)$ for short, if it satisfies $CD(K, \infty)$ and the Sobolev space $W^{1,2}(X)$ is Hilbertian.

Theorem (from a result by Daneri & Savaré (2008))

$RCD(K, \infty)$ *implies strong* $CD(K, \infty)$.

By strong $CD(K, \infty)$ we mean that K-convexity of the entropy holds along every geodesic in $\mathscr{P}_2(X)$.

$RCD(K,\infty) \Rightarrow$ essential non-branching

Theorem (R. & Sturm (2012, to appear))

Strong $CD(K,\infty)$ spaces are essentially non-branching. In particular $RCD(K,\infty)$ spaces are essentially non-branching.

Definition

A space $(X, \mathrm{d}, \mathfrak{m})$ is called *essentially non-branching* if for every $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ that are absolutely continuous with respect to $\mathfrak{m}$ we have that any $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics. (Meaning that $\boldsymbol{\pi}(\Gamma) = 1$ for some $\Gamma \subset \mathrm{Geo}(X)$ so that there are no two branching geodesics in Γ.)

Corollary (of essential non-branching and Gigli's result)

There exist optimal transport maps in strong $CD(K,\infty)$ spaces between $\mu_0, \mu_1 \ll \mathfrak{m}$.

Remarks

- Ohta (2013, to appear): Strong $CD(K,\infty)$ does not imply non-branching.
- Question: Are $RCD(K,\infty)$ spaces non-branching?
- Question: Are Ricci-limits non-branching?
- Gigli, R. & Sturm (2013, preprint): existence of optimal maps in $RCD^*(K,N)$ spaces with $N<\infty$ from $\mu_0 \ll \mathfrak{m}$ to μ_1.

Ohta's example

For $D \subset \mathbb{R}^n$ bounded convex domain one can define a distance

$$\mathrm{d}_{\mathcal{H}}(x,y) := \frac{1}{2}\log\left(\frac{|x'-y|\cdot|x-y'|}{|x'-x|\cdot|y-y'|}\right)$$

$x' = x + s(y-x), y' = x + t(y-x) \in \partial D$, $s < 0 < t$.
The space $(D, \mathrm{d}_{\mathcal{H}}, \mathcal{L}_n)$ is $CD(K,N)$
(with $N > n$ and $K = -n + 1 - \frac{(n+1)^2}{N-n}$).

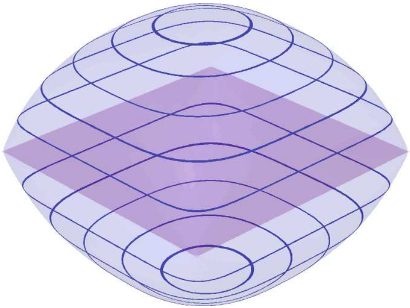
Now, take for example

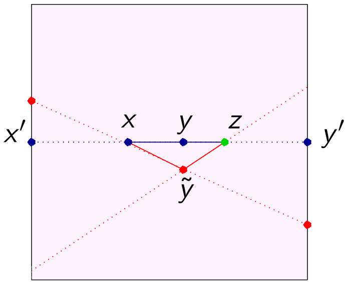
$$D = \left\{x^{2/z} + y^{2/z} + z^2 < 1\right\} \subset \mathbb{R}^3$$

with $E = D \cap \{z = 0\} = (-1,1)^2 \times \{0\}$. Geodesics in E can branch – other geodesics can't. Hence the space is strong $CD(K,N)$ and essentially nonbranching, but not nonbranching.

Ohta's example

$$d_{\mathcal{H}}(x, y) := \frac{1}{2} \log \left(\frac{|x' - y| \cdot |x - y'|}{|x' - x| \cdot |y - y'|} \right)$$





$$D = \left\{ x^{2/z} + y^{2/z} + z^2 < 1 \right\} \subset \mathbb{R}^3 \qquad E = (-1, 1)^2 \times \{0\}$$

$RCD(K,\infty) \Rightarrow$ essential non-branching

Let's see the main idea behind the proof of

Theorem (R. & Sturm (2012, to appear))

Strong $CD(K,\infty)$ spaces are essentially non-branching. In particular $RCD(K,\infty)$ spaces are essentially non-branching.

Definition

A space $(X, \mathsf{d}, \mathfrak{m})$ is called *essentially non-branching* if for every $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ that are absolutely continuous with respect to $\mathfrak{m}$ we have that any $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics. (Meaning that $\boldsymbol{\pi}(\Gamma) = 1$ for some $\Gamma \subset \mathrm{Geo}(X)$ so that there are no two branching geodesics in Γ.)

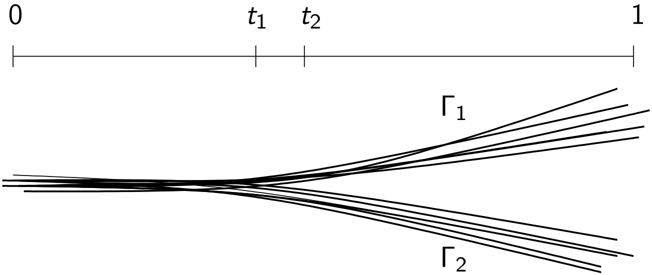
“Proof” of essential nonbranching in strong $CD(K,N)$ spaces
Suppose that the claim is not true so that there exists a measure $\boldsymbol{\pi}$ that is not concentrated on non-branching geodesics. By restricting the measure $\boldsymbol{\pi}$ we may assume that there are $0 < t_1 < t_2 < 1$ with $|t_1 - t_2|$ small and two sets of geodesics Γ_1, Γ_2 so that

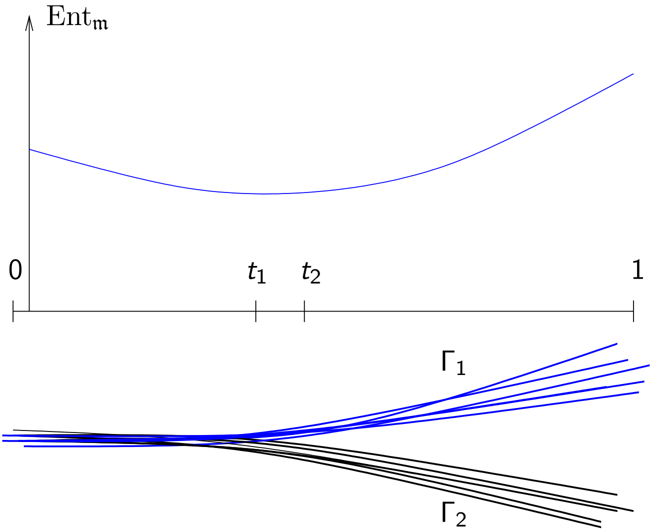
$$(e_t)_\# \boldsymbol{\pi}|_{\Gamma_1} = (e_t)_\# \boldsymbol{\pi}|_{\Gamma_2} \qquad \text{for all } t \in [0, t_1]$$

and

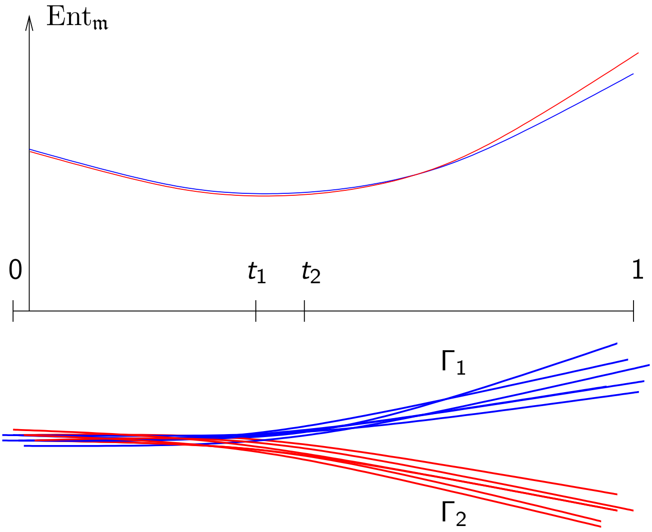
$$(e_t)_\# \boldsymbol{\pi}|_{\Gamma_1} \perp (e_t)_\# \boldsymbol{\pi}|_{\Gamma_2} \qquad \text{for all } t \in [t_2, 1]$$

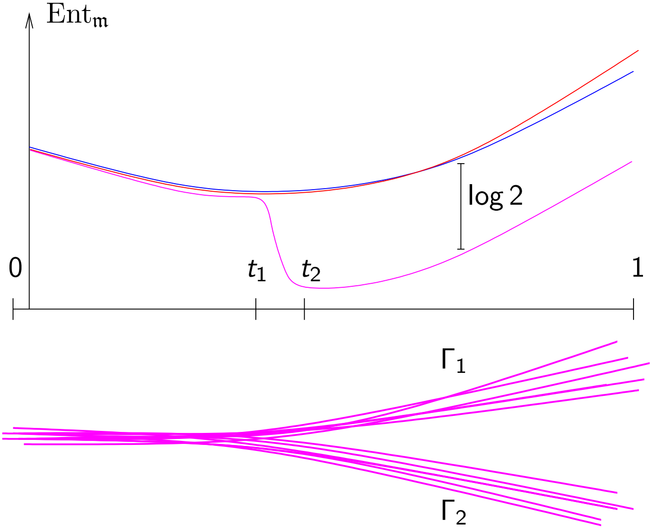
for all $t \in [t_2, 1]$.


0
t_1
t_2
1
Γ_1
Γ_2



$\mathrm{Ent}_{\mathfrak{m}}$
0
t_1
t_2
1
Γ_1
Γ_2


Ent$_{\mathfrak{m}}$
0
t_1
t_2
1
Γ_1
Γ_2

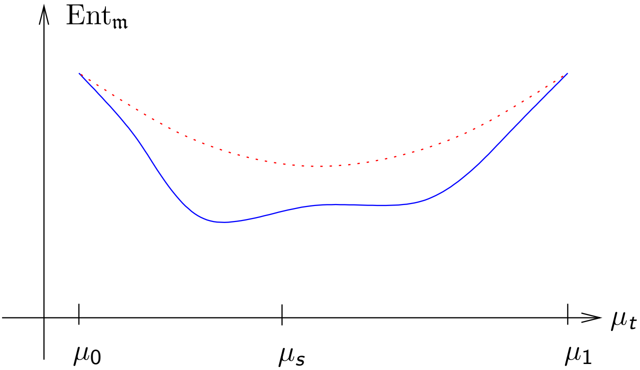


$\mathrm{Ent}_{\mathfrak{m}}$
log 2
0
t_1
t_2
1
Γ_1
Γ_2

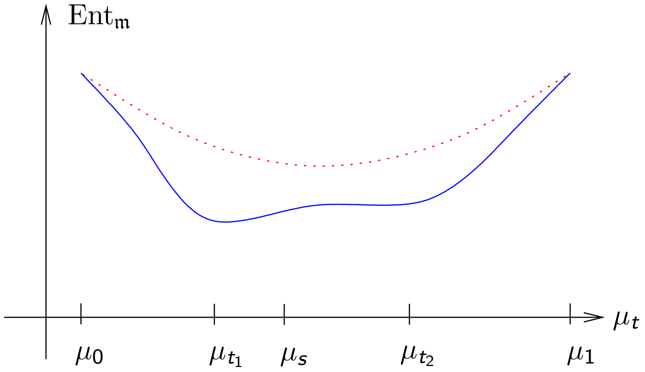
Part III

Existence of geodesics with bounded density in $CD(K, N)$ and local Poincaré inequalities from geodesics with bounded density

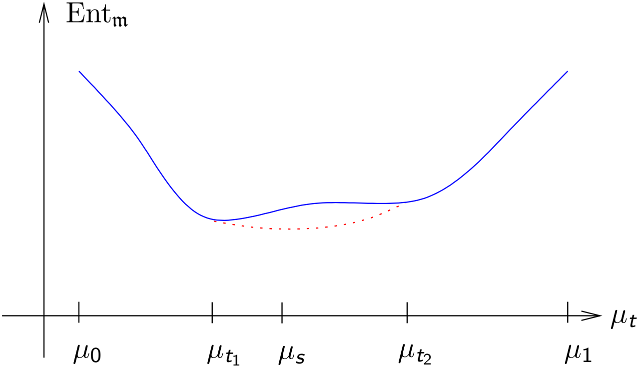
Improving geodesics by iterated minimization



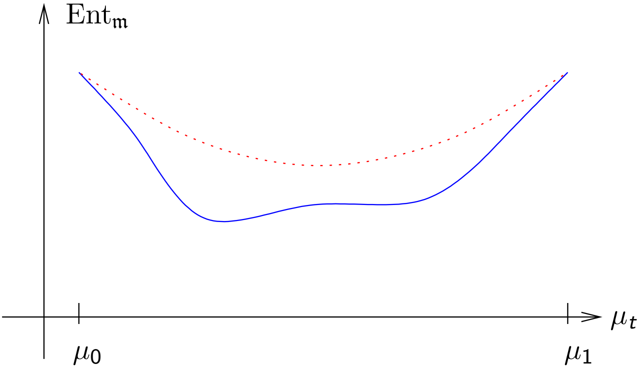
Improving geodesics by iterated minimization



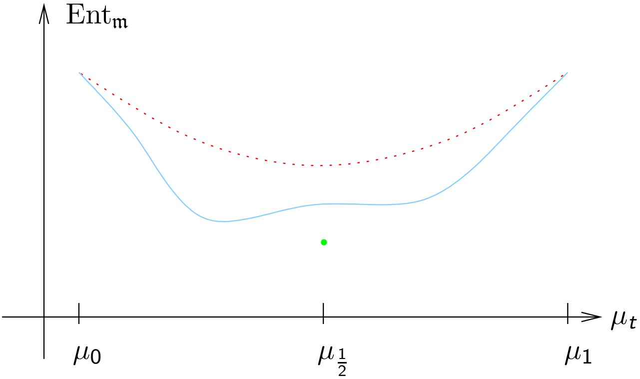
Improving geodesics by iterated minimization



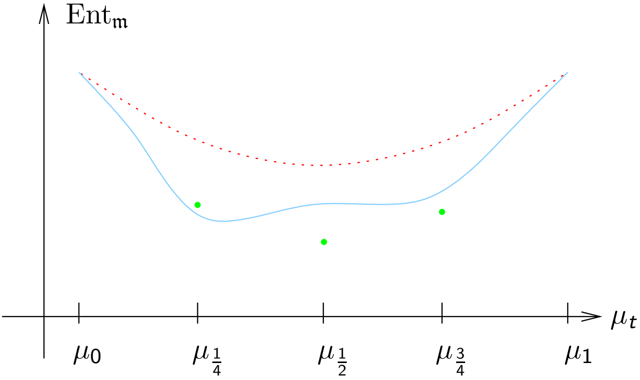
Improving geodesics by iterated minimization



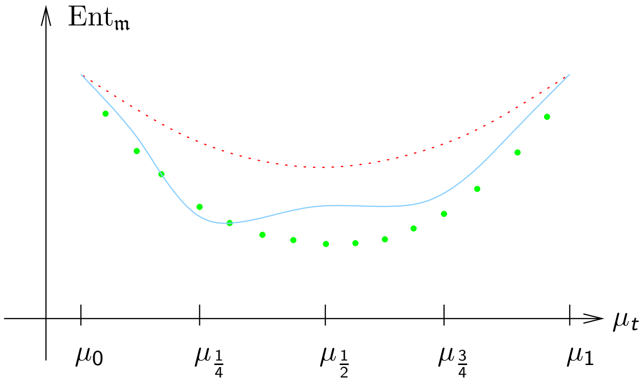
Improving geodesics by iterated minimization



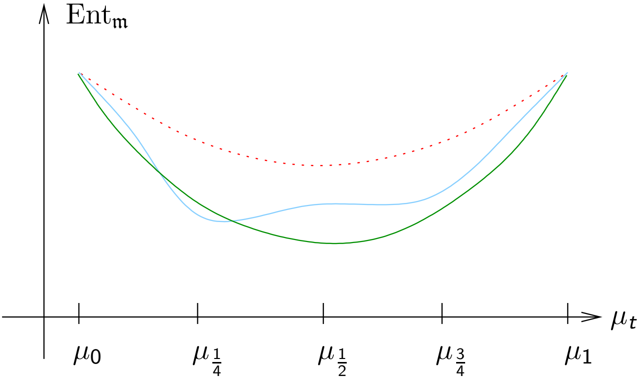
Improving geodesics by iterated minimization



Improving geodesics by iterated minimization

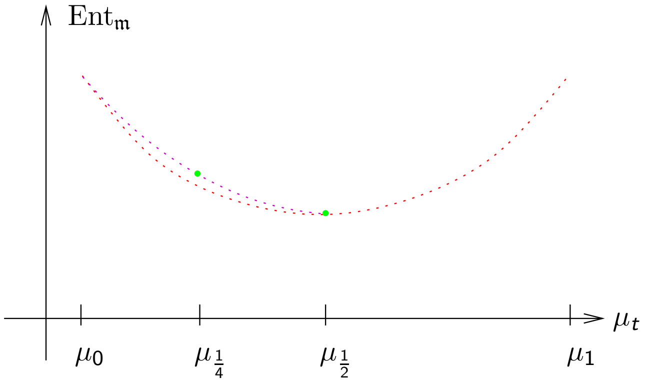


Improving geodesics by iterated minimization



Note: Iterating $CD(K,N)$ gives directly only $CD^*(K,N)$

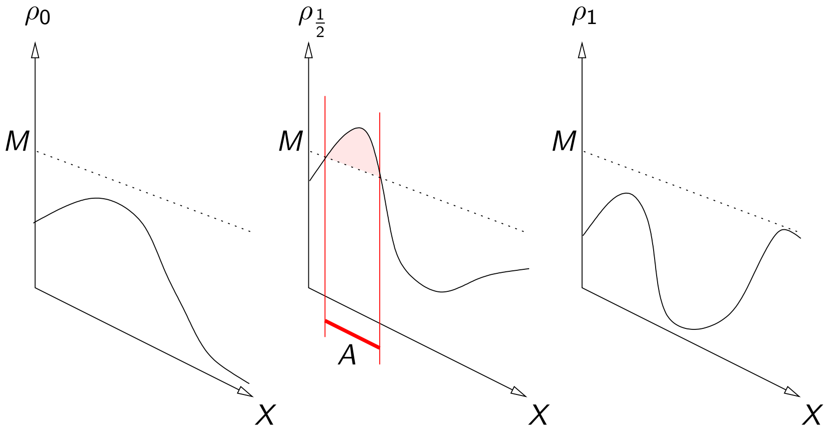
In $CD(K,N)$ spaces, minimizing the (Rényi) entropy $\int_X \rho^{1-1/N}\,\mathrm{d}\mathfrak{m}$ does not always produce a geodesic along which the inequalities required by $CD(K,N)$-condition hold.



Improving geodesics by iterated minimization

Not only does the new geodesic satisfy the K-convexity inequality between any three times $0 < t_1 < s < t_2 < 1$, but also the measures along the geodesic have bounded densities (under some assumptions on the initial and final measures). This is true for instance if we start with two measures μ_0 and μ_1 having bounded densities and bounded supports.

Why is $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\} =: M$?.



Consider the curve $\Gamma \in \mathcal{P}(\mathrm{Geo}(X))$ between the marginals corresponding to the part of the measure which we want to redistribute along which $\mathrm{Ent}_{\mathfrak{m}}$ is displacement convex. We have

$$\mathrm{Ent}_{\mathfrak{m}}(\Gamma_{\frac{1}{2}}) \leq \frac{1}{2}\mathrm{Ent}_{\mathfrak{m}}(\Gamma_0) + \frac{1}{2}\mathrm{Ent}_{\mathfrak{m}}(\Gamma_1) \leq \log M.$$

On the other hand, by Jensen's inequality we always have

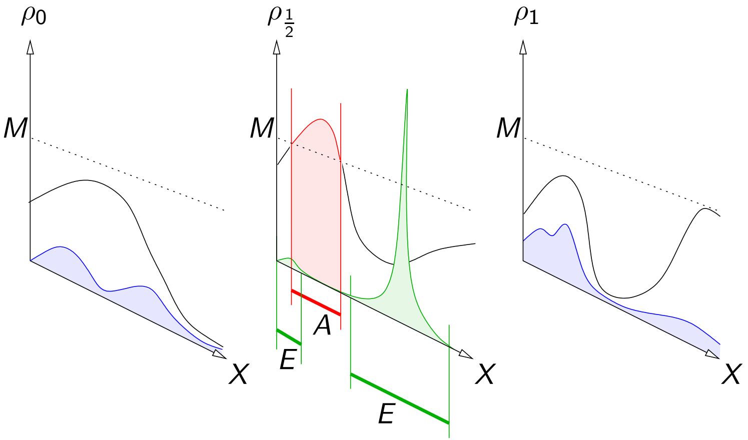
$$\begin{aligned} \mathrm{Ent}_{\mathfrak{m}}(\Gamma_{\frac{1}{2}}) &= \int_E \rho_{\frac{1}{2}} \log \rho_{\frac{1}{2}} \, \mathrm{d}\mathfrak{m} \\ &\geq m(E) \left(\fint_E \rho_{\frac{1}{2}} \, \mathrm{d}\mathfrak{m} \right) \log \left(\fint_E \rho_{\frac{1}{2}} \, \mathrm{d}\mathfrak{m} \right) \geq \log \frac{1}{\mathfrak{m}(E)}, \end{aligned}$$

where $E = \{x \in X \ : \ \rho_{\frac{1}{2}}(x) > 0\}$ and $\Gamma_t = \rho_t \mathfrak{m}$. Thus

$$\mathfrak{m}(E) \geq \frac{1}{M}.$$

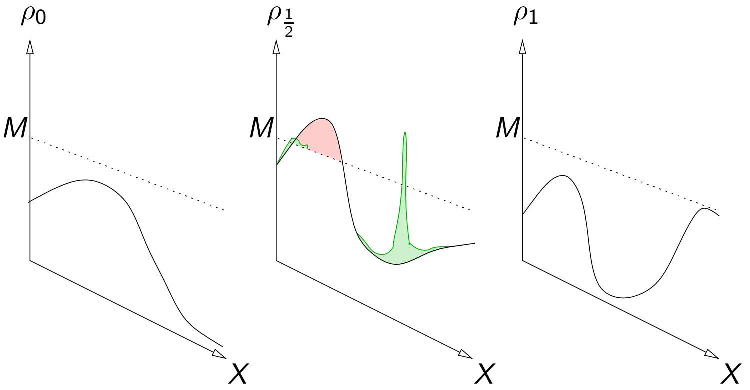
This is why $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\}$.

The $CD(K,\infty)$ condition gives a new well spread midpoint for the high-density part of the old midpoint.



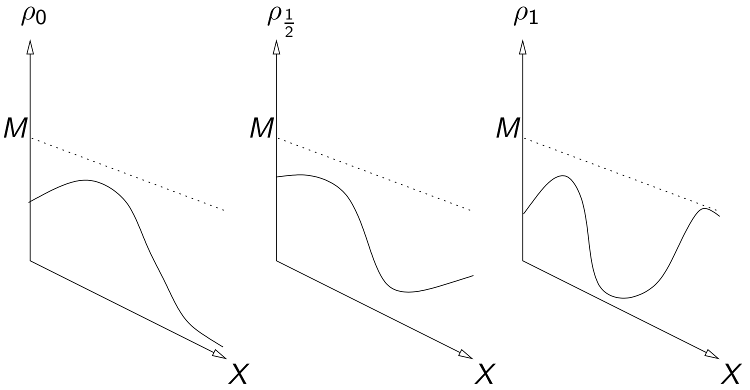
This is why $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\}$.

Taking a weighted combination of this new midpoint measure and the old one lowers the entropy.



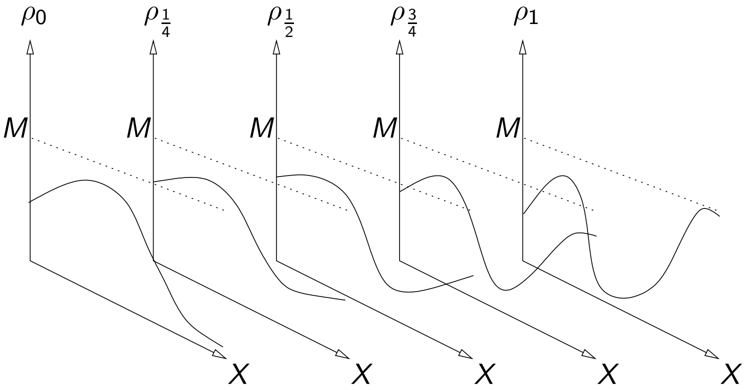
This is why $||\rho_{1/2}||_\infty \leq \max\{||\rho_0||_\infty, ||\rho_1||_\infty\}$.

Therefore at the minimum of the entropy among the midpoints we have the density bound.



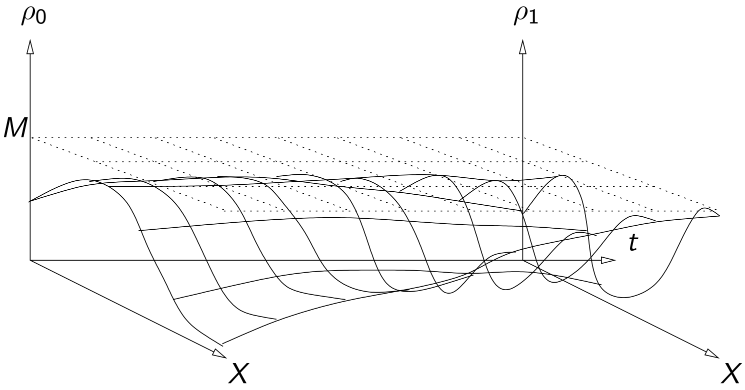
The rest of the geodesic.

When we continue taking minimizers in the next level midpoints the bound is preserved.



The rest of the geodesic.

Finally we end up with a complete geodesic with the density bound.



local Poincaré inequality in non-branching $MCP(K, N)$

Theorem (Lott & Villani, von Renesse, Sturm, Hinde & Petersen, Cheeger & Colding)

Suppose that $(X, \mathrm{d}, \mathfrak{m})$ *is a nonbranching* $MCP(K, N)$ *space with* $K \in \mathbb{R}$. *Then the weak local Poincaré inequality*

$$\fint_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \,\mathrm{d}\mathfrak{m} \leq C(N, K, r) r \fint_{B(x,2r)} g \,\mathrm{d}\mathfrak{m}$$

holds for any measurable function u *defined on* X, *any upper gradient* g *of* u *and for each point* $x \in X$ *and radius* $r > 0$.

local Poincaré inequality in $CD(K,\infty)$

Theorem (R. (2012))

Suppose that $(X, \mathrm{d}, \mathfrak{m})$ is a $CD(K,\infty)$ space (in the sense of Sturm) with $K \in \mathbb{R}$. Then the weak local Poincaré inequality

$$\int_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \, \mathrm{d}\mathfrak{m} \le 4re^{K^- r^2} \int_{B(x,2r)} g \, \mathrm{d}\mathfrak{m}$$

holds for any measurable function u defined on X, any upper gradient g of u and for each point $x \in X$ and radius $r > 0$.

- Observe that we do not have average integrals in this theorem.

Proof of the local Poincaré inequality

Let us prove the $CD(0,\infty)$ case for simplicity. We have to show that

$$\int_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \, \mathrm{d}\mathfrak{m} \le 4r \int_{B(x,2r)} g \, \mathrm{d}\mathfrak{m}.$$

Abbreviate $B = B(x,r)$ and denote

$$M = \inf \left\{ a \in \mathbb{R} : \mathfrak{m}(\{u > a\}) \le \frac{\mathfrak{m}(B)}{2} \right\}.$$

Split the ball B into two Borel sets B^+ and B^- so that $B = B^+ \cup B^-$, $B^+ \cap B^- = \emptyset$, $\mathfrak{m}(B^+) = \mathfrak{m}(B^-)$ and

$$u(x) \le M \le u(y) \qquad \text{for all } x \in B^-, y \in B^+.$$

Proof of the local Poincaré inequality

Let $(\mu_t)_{t=0}^1$ be a geodesic between $\frac{1}{\mathfrak{m}(B^+)}\mathfrak{m}|_{B^+}$ and $\frac{1}{\mathfrak{m}(B^-)}\mathfrak{m}|_{B^-}$ along which we have the density bound (writing $\mu_t = \rho_t \mathfrak{m}$)

$$\rho_t(y) \leq \frac{2}{\mathfrak{m}(B)}$$

for all $t \in [0,1]$ at $\mathfrak{m}$-almost every $y \in X$. Let $\boldsymbol{\pi}$ be a corresponding measure on the set of geodesics.

Proof of the local Poincaré inequality

From $u(z) \le M \le u(y)$ for all $(z, y) \in B^- \times B^+$ we get

$$|u(\gamma(0)) - u(\gamma(1))| = |u(\gamma(0)) - M| + |M - u(\gamma(1))|$$

for $\boldsymbol{\pi}$-almost every $\gamma \in \mathrm{Geo}(X)$. Therefore

$$\begin{aligned}
&\int_{\mathrm{Geo}(X)} |u(\gamma(0)) - u(\gamma(1))| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= \int_{\mathrm{Geo}(X)} |u(\gamma(0)) - M| \,\mathrm{d}\boldsymbol{\pi}(\gamma) + \int_{\mathrm{Geo}(X)} |M - u(\gamma(1))| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= \frac{2}{\mathfrak{m}(B)} \int_{B^+} |u(z) - M| \,\mathrm{d}\mathfrak{m}(z) + \frac{2}{\mathfrak{m}(B)} \int_{B^-} |M - u(z)| \,\mathrm{d}\mathfrak{m}(z) \\
&= \frac{2}{\mathfrak{m}(B)} \int_{B} |u(z) - M| \,\mathrm{d}\mathfrak{m}(z).
\end{aligned}$$

Proof of the local Poincaré inequality

$$\int_{B(x,r)} |u - \langle u \rangle_{B(x,r)}| \, \mathrm{d}\mathfrak{m} \le \frac{1}{\mathfrak{m}(B)} \iint_{B \times B} |u(z) - u(y)| \, \mathrm{d}\mathfrak{m}(z) \, \mathrm{d}\mathfrak{m}(y)$$

$$\le \frac{1}{\mathfrak{m}(B)} \iint_{B \times B} (|u(z) - M| + |M - u(y)|) \, \mathrm{d}\mathfrak{m}(z) \, \mathrm{d}\mathfrak{m}(y)$$

$$= 2 \int_B |u(z) - M| \, \mathrm{d}\mathfrak{m}(z) = \mathfrak{m}(B) \int_{\mathrm{Geo}(X)} |u(\gamma(0)) - u(\gamma(1))| \, \mathrm{d}\boldsymbol{\pi}(\gamma)$$

$$\le 2r\mathfrak{m}(B) \int_{\mathrm{Geo}(X)} \int_0^1 g(\gamma(t)) \, dt \, \mathrm{d}\boldsymbol{\pi}(\gamma)$$

$$= 2r\mathfrak{m}(B) \int_0^1 \int_X g(z) \rho_t(z) \, \mathrm{d}\mathfrak{m}(z) \, \mathrm{d}t$$

$$\le 4r \int_0^1 \int_{B(x,2r)} g(z) \, \mathrm{d}\mathfrak{m}(z) \, \mathrm{d}t = 4r \int_{B(x,2r)} g \, \mathrm{d}\mathfrak{m}.$$

Thank you!

Merci!

Kiitos!