Suprema of Chaos Processes and the RIP of structured random matrices

Holger Rauhut
Hausdorff Center for Mathematics
& Institute for Numerical Simulation
University of Bonn

Phenomena in high dimensions in geometric analysis, random matrices, and computational geometry
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Joint work with Felix Krahmer and Shahar Mendelson

- Compressive Sensing
- Partial Random Circulant Matrices (Random Convolutions)
- Suprema of Chaos Processes
- Time-Frequency Structured Random Matrices

Reconstruct $\mathbf{x} \in \mathbb{C}^N$ from its vector $\mathbf{y}$ of m measurements

$$\mathbf{y} = A\mathbf{x}, \qquad A \in \mathbb{C}^{m \times N},$$

when $m \ll N$, under additional knowledge that $\mathbf{x}$ is s-sparse,

$$\|\mathbf{x}\|_0 := \#\{\ell : x_\ell \neq 0\} \leq s.$$

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More generally, $\mathbf{x}$ is compressible in the sense that its best s-term approximation error $\sigma_s(\mathbf{x})_p$ decreases quickly in s, where

$$\sigma_s(\mathbf{x})_p = \inf_{\mathbf{z}:\|\mathbf{z}\|_0 \leq s} \|\mathbf{x} - \mathbf{z}\|_p.$$

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Preferably we would like to have a fast algorithm that performs the reconstruction.

ℓ_0-minimization:

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Problem: ℓ_0-minimization is NP hard!

ℓ_1 minimization:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 = \sum_{j=1}^{N} |x_j| \quad \text{subject to} \quad A\mathbf{x} = \mathbf{y}$$

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Convex relaxation of ℓ_0-minimization problem.

Efficient minimization methods available.

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Alternatives:
Greedy Algorithms (Matching Pursuits)
Iterative Algorithms

Recovery Conditions for ℓ_1-minimization

- Null space property: necessary and sufficient for exact recovery of all s-sparse vectors via ℓ_1-minimization with A.

  $\|\mathbf{v}_S\|_1 \leq \rho \|\mathbf{v}_{S^c}\|_1$ for all $\mathbf{v} \in \ker A, S \subset [N], |S| = s,$ some $\rho < 1$.

  Implies also stability of reconstruction.
- Restricted isometry property (RIP): sufficient for exact (and stable) recovery of all s-sparse vectors via ℓ_1-minimization and other recovery algorithms (see next slide).
- Dual certificates: Sufficient (and sometimes necessary) conditions on $\mathbf{x}$ and A for exact (and stable) recovery via ℓ_1-minimization.
  Leads to nonuniform recovery guarantees (see later).

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Definition

The restricted isometry constant δ_s of a matrix $A \in \mathbb{C}^{m \times N}$ is defined as the smallest δ_s such that

$$(1 - \delta_s)\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1 + \delta_s)\|\mathbf{x}\|_2^2$$

for all s-sparse $\mathbf{x} \in \mathbb{C}^N$.

Requires that all s-column submatrices of A are well-conditioned.

RIP implies recovery by ℓ_1-minimization

Theorem (Candès, Romberg, Tao 2004 – Candès 2008 – Foucart, Lai 2009 – Foucart 2009 – Li, Mo 2011)

Assume that the restricted isometry constant of $A \in \mathbb{C}^{m \times N}$ satisfies

$$\delta_{2s} < 0.4931.$$

Then ℓ_1-minimization reconstructs every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ from $y = Ax$.

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Theorem (Candès, Romberg, Tao 2004 – Candès 2008 – Foucart, Lai 2009 – Foucart 2009 – Li, Mo 2011)

Let $A \in \mathbb{C}^{m \times N}$ *with* $\delta_{2s} < 0.4931$*.Let* $x \in \mathbb{C}^N$*, and assume that noisy data are observed,* $y = Ax + \eta$ *with* $\|\eta\|_2 \leq \sigma$*. Let* $x^\#$ *by a solution of*

$$\min_z \|z\|_1 \quad \text{such that} \quad \|Az - y\|_2 \leq \sigma.$$

Then

$$\|x - x^\#\|_2 \leq C \frac{\sigma_s(x)_1}{\sqrt{s}} + D\sigma$$

and

$$\|x - x^\#\|_1 \leq C\sigma_s(x)_1 + D\sqrt{s}\sigma$$

for constants $C, D > 0$*, that depend only on* δ_{2s}*.*

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Open problem: Give explicit matrices $A \in \mathbb{C}^{m \times N}$ with small $\delta_{2s} \leq 0.49$ for "large" s.

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Goal: $\delta_s \leq \delta$, if

$$m \geq C_\delta s \ln^\alpha(N),$$

for constants C_δ and α.

Deterministic matrices known, for which $m \geq C_{\delta,k} s^2$ suffices if $N \leq m^k$.

Small improvement by Bourgain et al. (2010): $m \geq C_\delta s^{2-\epsilon}$, $\epsilon > 0$, under additional assumptions on m, N.

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Way out: consider random matrices.

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RIP for Gaussian and Bernoulli matrices

Gaussian: entries of A independent $\mathcal{N}(0,1)$ random variables
Bernoulli : entries of A independent Bernoulli ± 1 distributed rv

Theorem

Let $A \in \mathbb{R}^{m \times N}$ be a Gaussian or Bernoulli random matrix and assume

$$m \geq C\delta^{-2}(s \ln(eN/s) + \ln(2\varepsilon^{-1}))$$

for a universal constant $C > 0$. Then with probability at least $1 - \varepsilon$ the restricted isometry constant of $\frac{1}{\sqrt{m}}A$ satisfies $\delta_s \leq \delta$.

Consequence: Recovery via ℓ_1-minimization with probability exceeding $1 - e^{-cm}$ provided

$$m \geq Cs \ln(eN/s).$$

Bound is optimal as follows from lower bound for Gelfand widths of ℓ_p-balls, $0 < p \leq 1$. (Gluskin, Garnaev 1984 — Foucart, Pajor, Rauhut, Ullrich 2010)

Why structure?

- Applications impose structure due to physical constraints, limited freedom to inject randomness.
- Fast matrix vector multiplies (FFT) in recovery algorithms, unstructured random matrices impracticable for large scale applications.

Random Partial Fourier Matrices

$F \in \mathbb{C}^{N \times N}$ discrete Fourier matrix, $F_{jk} = e^{2\pi i jk/N}$.
$\Omega \subset [N]$ random subset of cardinality m (row indices).
$R_\Omega : \mathbb{C}^N \to \mathbb{C}^\Omega$: restriction operator.
$A = \frac{1}{\sqrt{m}} R_\Omega F$: random partial Fourier matrix, application corresponds to random subsampling of the Fourier transform.

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A satisfies RIP with high probability (Candés, Romberg, Tao 2006 — Rudelson, Vershynin 2008 — Rauhut 2009):
$\delta_s = \delta_s(A) \leq \delta$ with probability at least $1 - N^{-\ln^3 s}$ provided

$$m \geq C\delta^{-2} s \ln^3(s) \ln(N).$$

Generalization to random sampling in bounded orthonormal systems, including Legendre/Jacobi polynomials (Rauhut, Ward 2010).

Circulant matrix: For $\mathbf{b} = (b_0, b_1, \ldots, b_{N-1}) \in \mathbb{C}^N$ let $\Phi = \Phi(\mathbf{b}) \in \mathbb{C}^{N \times N}$ be the matrix with entries $\Phi_{i,j} = b_{j-i \mod N}$,

$$\Phi(\mathbf{b}) = \begin{pmatrix} b_0 & b_1 & \cdots & \cdots & b_{N-1} \\ b_{N-1} & b_0 & b_1 & \cdots & b_{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_{N-1} & b_0 \end{pmatrix}.$$

Partial random circulant matrices

Let $\Theta \subset [N]$ arbitrary of cardinality m.
R_Θ: operator that restricts a vector $\mathbf{x} \in \mathbb{C}^N$ to its entries in Θ.

Restrict $\Phi(\mathbf{b})$ to the rows indexed by Θ:
Partial circulant matrix: $\Phi^\Theta(\mathbf{b}) = R_\Theta \Phi(\mathbf{b}) \in \mathbb{C}^{m \times N}$

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$\mathbf{y} = R_\Theta \Phi(\mathbf{b})\mathbf{x} = R_\Theta(\mathbf{b} * \mathbf{x})$

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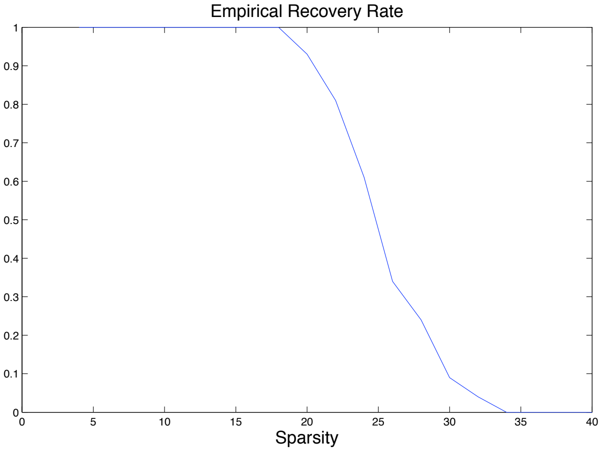
Matrix vector multiplication via the FFT!

We choose the vector $\mathbf{b} \in \mathbb{C}^N$ at random, in particular, as Rademacher sequence $\mathbf{b} = \epsilon$, that is, $\epsilon_\ell = \pm 1$.

Performance of $\Phi^\Theta(\epsilon)$ in compressive sensing?

Sparse recovery via ℓ_1-minimization with partial random circulant matrix $\Phi^{\Theta}(\epsilon) \in \mathbb{R}^{m \times N}$, $N = 500$, $m = 100$.



Nonuniform recovery result for circulant matrices

Theorem (Rauhut 2009)

Let $\Theta \subset [N]$ be an arbitrary (deterministic) set of cardinality m. Let $\mathbf{x} \in \mathbb{C}^N$ be s-sparse such that the signs of its non-zero entries form a Rademacher (or Steinhaus) sequence. Choose $\epsilon \in \mathbb{R}^N$ to be a Rademacher sequence. Let $\mathbf{y} = \Phi^{\Theta}(\epsilon)\mathbf{x} \in \mathbb{C}^m$. If

$$m \geq 57 s \ln^2(17N^2/\varepsilon)$$

then $\mathbf{x}$ can be recovered from $\mathbf{y}$ via ℓ_1-minimization with probability at least $1 - \varepsilon$.

RIP estimate for partial random circulant matrices

Theorem (Krahmer, Mendelson, Rauhut 2012)

Let $\Theta \subset [N]$ be an arbitrary (deterministic) set of cardinality m. Choose $\epsilon \in \mathbb{R}^N$ to be a Rademacher sequence. Assume that

$$m \geq C\delta^{-2} s \ln^2(s) \ln^2(N).$$

Then with probability at least $1 - N^{-\ln^2(s)\ln(N)}$ the restricted isometry constants of $\frac{1}{\sqrt{m}}\Phi^{\Theta}(\epsilon)$ satisfy $\delta_s \leq \delta$.

Generalizes to subgaussian generator.

Previous bounds:

Haupt, Bajwa, Raz (2008): $m \geq C_\delta s^2 \ln N$.

Rauhut, Romberg, Tropp (2010): $m \geq C_\delta s^{3/2} \ln^{3/2}(N)$.

Random sets Θ, Romberg (2009): $m \geq C_\delta s \ln^6 N$.

Consequence for Johnson-Lindenstrauss embeddings

Combination with result of F. Krahmer and R. Ward (2010) connecting the RIP and Johnson-Lindenstrauss embeddings yields:

Theorem

Fix $\eta, \delta \in (0,1)$, and consider a finite set $Q \subset \mathbb{R}^N$ of cardinality $|Q| = p$. Choose

$$m \geq C_\eta \delta^{-2} \log(p)(\log(\log p))^2(\log N)^2.$$

Let $\Phi \in \mathbb{C}^{m \times N}$ be a partial circulant matrix generated by a Rademacher vector. Furthermore, let $\epsilon \in \mathbb{R}^N$ be a Rademacher vector independent of Φ. Then with probability exceeding $1 - \eta$,

$$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|\Phi D_\epsilon \mathbf{x}\|_2^2 \leq (1+\delta)\|\mathbf{x}\|_2^2 \quad \text{for all } \mathbf{x} \in Q.$$

Here D_ϵ denotes the diagonal matrix with diagonal ϵ.

Recall that δ_s is smallest constant such that

$$(1-\delta_s)\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1+\delta_s)\|\mathbf{x}\|_2^2 \quad \text{for all } s-\text{sparse } \mathbf{x}.$$

Equivalently, with $T_s = \{\mathbf{x} \in \mathbb{C}^N : \|\mathbf{x}\|_2 \leq 1, \|\mathbf{x}\|_0 \leq s\}$

$$\delta_s = \sup_{\mathbf{x} \in T_s} \left| \|A\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2 \right|.$$

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For partial random circulant matrices,

$$A\mathbf{x} = \frac{1}{\sqrt{m}} R_\Omega(\epsilon * \mathbf{x}) = \frac{1}{\sqrt{m}} R_\Omega(\mathbf{x} * \epsilon) =: V_{\mathbf{x}}\epsilon$$

with appropriate $V_{\mathbf{x}} \in \mathbb{R}^{m\times N}$. Furthermore, $\mathbb{E}\|V_{\mathbf{x}}\epsilon\|_2^2 = \|\mathbf{x}\|_2^2$.
Therefore, δ_s is supremum of a chaos process,

$$\delta_s = \sup_{\mathbf{x}\in T_s} \left| \|V_{\mathbf{x}}\epsilon\|_2^2 - \mathbb{E}\|V_{\mathbf{x}}\epsilon\|_2^2 \right|.$$

Let $\mathcal{B}$ be a subset of a vector space with norm $\|\cdot\|$. Diameter

$$d_{\|\cdot\|}(\mathcal{B}) = \sup_{B \in \mathcal{B}} \|B\|.$$

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A sequence of subsets $T_r \subset \mathcal{A}$, $r \in \mathbb{N}_0$, is called admissible if $|T_0| = 1$, $|T_r| \leq 2^{2^r}$, $r \geq 1$.
For $\alpha > 0$ define the γ_α-functional

$$\gamma_\alpha(\mathcal{B}, \|\cdot\|) = \inf \sup_{B\in\mathcal{B}} \sum_{r=0}^{\infty} 2^{r/\alpha} d(B, T_r), \quad d(B, T_r) = \inf_{B_r \in T_r} \|B - B_r\|,$$

where the infimum is over all admissible sequences (T_r).

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Estimate by Dudley-type integral

$$\gamma_\alpha(\mathcal{B}, \|\cdot\|) \leq C \int_0^{d_{\|\cdot\|}(\mathcal{B})} \left(\log N(\mathcal{B}, \|\cdot\|, u)\right)^{1/\alpha} du,$$

where $N(\mathcal{B}, \|\cdot\|, u)$ denotes the smallest number of balls of radius u in the norm $\|\cdot\|$ required to cover $\mathcal{B}$.

Theorem (Krahmer, Mendelson, Rauhut 2012)

Let $\mathcal{B}$ be a symmetric set of matrices and ϵ a Rademacher vector. Then

$$\mathbb{E} \sup_{B \in \mathcal{B}} \left| \|B\epsilon\|_2^2 - \mathbb{E}\|B\epsilon\|_2^2 \right| \\ \leq C_1 \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2})^2 + C_2 d_{\|\cdot\|_F}(\mathcal{B}) \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}).$$

Here, $\|\cdot\|_F = \sqrt{\mathrm{Tr}(B^* B)}$ denotes the Frobenius norm.

Tail bound

Theorem (Krahmer, Mendelson, Rauhut 2012)

Let $\mathcal{B} \subset \mathbb{C}^{m \times n}$ with $\mathcal{B} = -\mathcal{B}$ and ϵ be a Rademacher vector. Then

$$\mathbb{P}\left(\sup_{B \in \mathcal{B}} \left| \|B\varepsilon\|_2^2 - \mathbb{E}\|B\varepsilon\|_2^2 \right| \geq C_1 E + t\right) \leq 2\exp\left(-C_2 \min\left\{\frac{t^2}{V^2}, \frac{t}{U}\right\}\right),$$

where

$$E := \gamma_2(\mathcal{B}, \|\cdot\|_{2\to2})\left(\gamma_2(\mathcal{B}, \|\cdot\|_{2\to2}) + d_{\|\cdot\|_F}(\mathcal{B})\right)$$
$$V := d_{\|\cdot\|_{2\to2}}(\mathcal{B})(\gamma_2(\mathcal{B}, \|\cdot\|_{2\to2}) + d_{\|\cdot\|_F}(\mathcal{B})), \qquad U := d^2_{\|\cdot\|_{2\to2}}(\mathcal{B}).$$

Symmetry assumption $\mathcal{B} = -\mathcal{B}$ only made for convenience.
Generalizes to random vectors with independent mean-zero, variance 1, subgaussian entries.

Rewrite process as

$$\sup_{B\in\mathcal{B}} \left| \|B\epsilon\|_2^2 - \mathbb{E}\|B\epsilon\|_2^2 \right| = \sup_{B\in\mathcal{B}} \left| \sum_{j\neq k} \epsilon_j \epsilon_k (B^*B)_{j,k} \right|$$

Chaos process indexed by $\mathcal{D} = \{B^*B : B \in \mathcal{B}\}$.
General estimate (Talagrand, 1993)

$$\mathbb{E} \sup_{D\in\mathcal{D}} \Big| \sum_{j\neq k} \epsilon_j \epsilon_k D_{j,k} \Big| \leq C_1 \gamma_2(\mathcal{D}, \|\cdot\|_F) + C_2 \gamma_1(\mathcal{D}, \|\cdot\|_{2\to 2}).$$

Relation to previous estimate of Talagrand

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This bound was used in the previous RIP estimate due to Rauhut, Romberg, Tropp (2010). The appearance of the γ_1-functional results in the exponent $3/2$.

Bound $\gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2})$ for $\mathcal{B} = \{V_{\mathbf{x}}, \mathbf{x} \in T_s\}$.

Relate to the analysis of random partial Fourier matrices via the convolution theorem, and bound Dudley-type integral using covering number estimates due to Rudelson and Vershynin (2008).

Proof sketch for Bound of Chaos Processes

Decoupling (with ϵ' a Rademacher vector independent of ϵ)

$$\mathbb{E} C_{\mathcal{B}} := \mathbb{E} \sup_{B \in \mathcal{B}} \left| \|B\epsilon\|_2^2 - \mathbb{E}\|B\epsilon\|_2^2 \right| \leq 4\mathbb{E} \sup_{B \in \mathcal{B}} \left| \langle B\epsilon, B\epsilon' \rangle \right|$$

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Chaining: admissible sequence (T_r),
$\pi_r B = \arg\min_{A \in T_r} \|A - B\|_{2\to 2}$,

$$\sup_{A \in \mathcal{B}} \left| \langle B\epsilon, B\epsilon' \rangle \right| \leq \sum_{r=0}^{\infty} |\langle (\pi_{r+1}B - \pi_r B)\epsilon, \pi_{r+1}B\epsilon' \rangle|$$
$$+ \sum_{r=0}^{\infty} |\langle \pi_r B\epsilon, (\pi_{r+1}B - \pi_r B)\epsilon' \rangle| + |\langle \pi_0 B\epsilon, \pi_0 B\epsilon' \rangle|$$

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Condition on ϵ' in the first term. Note that

$$|\langle(\pi_{r+1}B - \pi_r B)\epsilon, \pi_{r+1}B\epsilon'\rangle| = |\langle\epsilon, (\pi_{r+1}B - \pi_r B)^*(\pi_{r+1}B)\epsilon'\rangle|,$$
$$\|(\pi_{r+1}B - \pi_r B)^*(\pi_{r+1}B)\epsilon'\|_2 \leq \|\pi_{r+1}B - \pi_r B\|_{2\to 2}\|\pi_r B\epsilon'\|_2$$
$$\leq \|\pi_{r+1}B - \pi_r B\|_{2\to 2}N_{\mathcal{B}},$$

where $N_{\mathcal{B}} = \sup_{B\in\mathcal{B}}\|B\epsilon'\|_2$.

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Condition on ϵ' in the first term. Note that

$$|\langle (\pi_{r+1}B - \pi_r B)\epsilon, \pi_{r+1}B\epsilon' \rangle| = |\langle \epsilon, (\pi_{r+1}B - \pi_r B)^*(\pi_{r+1}B)\epsilon' \rangle|,$$

$$\|(\pi_{r+1}B - \pi_r B)^*(\pi_{r+1}B)\epsilon'\|_2 \leq \|\pi_{r+1}B - \pi_r B\|_{2\to 2} \|\pi_r B\epsilon'\|_2$$

$$\leq \|\pi_{r+1}B - \pi_r B\|_{2\to 2} N_{\mathcal{B}},$$

where $N_{\mathcal{B}} = \sup_{B\in\mathcal{B}} \|B\epsilon'\|_2$

Proof Ingredients II

With Hoeffding's inequality, a union bound over all elements of $T_r \times T_{r+1}$ and all levels r, and integration we obtain

$$\mathbb{E} C_{\mathcal{B}} \lesssim \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) \mathbb{E} N_{\mathcal{B}} + d_F(\mathcal{B}) d_{2\to 2}(\mathcal{B}).$$

Furthermore,

$$\begin{aligned}(\mathbb{E} N_{\mathcal{B}})^2 &\leq \mathbb{E} \sup_{B\in\mathcal{B}} \|B\epsilon\|_2^2 \leq \mathbb{E} \sup_{B\in\mathcal{B}} \left| \|B\epsilon\|_2^2 - \mathbb{E}\|B\epsilon\|_2^2 \right| + d_F^2(\mathcal{B}) \\ &= \mathbb{E} C_{\mathcal{B}} + d_F^2(\mathcal{B}) \lesssim \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) \mathbb{E} N_{\mathcal{B}} + d_F^2(\mathcal{B}).\end{aligned}$$

Theorefore,

$$\mathbb{E} N_{\mathcal{B}} \lesssim \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) + d_F(\mathcal{B})$$

and

$$\mathbb{E} C_{\mathcal{B}} \lesssim \gamma_2^2(\mathcal{B}, \|\cdot\|_{2\to 2}) + d_F(\mathcal{B}) \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) + d_F(\mathcal{B}) d_{2\to 2}(\mathcal{B}).$$

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$$= \mathbb{E} C_{\mathcal{B}} + d_F^2(\mathcal{B}) \lesssim \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) \mathbb{E} N_{\mathcal{B}} + d_F^2(\mathcal{B}).$$

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Tail bound follows from moment estimate, for $p \geq 1$,

$$\left(\mathbb{E} C_{\mathcal{B}}^p\right)^{1/p} \lesssim \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2})\left(\gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) + d_F(\mathcal{B})\right) + \sqrt{p} d_{2\to 2}(\mathcal{B})\left(\gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) + d_F(\mathcal{B})\right) + p d_{2\to 2}^2(\mathcal{B}).$$

Gabor Systems in Finite Dimension

Translation and Modulation on $\mathbb{C}^m$

$$(T^k g)_j = g_{(j-k) \mod m} \quad \text{and} \quad (M^\ell g)_j = e^{2\pi i \ell j/m} g_j.$$

Time-frequency shifts

$$\pi(\lambda) = M^\ell T^k, \quad \lambda = (k, \ell) \in \{0, \ldots, m-1\}^2.$$

For $g \in \mathbb{C}^m$ define Gabor synthesis matrix ($\omega = e^{2\pi i/m}$)

$$\Psi_g = (\pi(\lambda)g)_{\lambda \in \{0,\ldots,m-1\}^2}$$

$$= \left(\begin{array}{cccc|ccc|cc} g_0 & g_{m-1} & \cdots & g_1 & g_0 & \cdots & g_1 & \cdots & g_1 \\ g_1 & g_0 & \cdots & g_2 & \omega g_1 & \cdots & \omega g_2 & \cdots & \omega^{m-1} g_2 \\ g_2 & g_1 & \cdots & g_3 & \omega^2 g_2 & \cdots & \omega^2 g_3 & \cdots & \omega^{2(m-1)} g_3 \\ g_3 & g_2 & \cdots & g_4 & \omega^3 g_3 & \cdots & \omega^3 g_4 & \cdots & \omega^{3(m-1)} g_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ g_{m-1} & g_{m-2} & \cdots & g_0 & \omega^{m-1} g_{m-1} & \cdots & \omega^{m-1} g_0 & \cdots & \omega^{(m-1)^2} g_0 \end{array} \right).$$

Performance of $\Psi_g \in \mathbb{C}^{m \times m^2}$ in compressive sensing?
Choice of g?

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g_1 & g_0 & \cdots & g_2 & \omega g_1 & \cdots & \omega g_2 & \cdots & \omega^{m-1} g_2 \\
g_2 & g_1 & \cdots & g_3 & \omega^2 g_2 & \cdots & \omega^2 g_3 & \cdots & \omega^{2(m-1)} g_3 \\
g_3 & g_2 & \cdots & g_4 & \omega^3 g_3 & \cdots & \omega^3 g_4 & \cdots & \omega^{3(m-1)} g_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
g_{m-1} & g_{m-2} & \cdots & g_0 & \omega^{m-1} g_{m-1} & \cdots & \omega^{m-1} g_0 & \cdots & \omega^{(m-1)^2} g_0
\end{array}\right).$$

Performance of $\Psi_g \in \mathbb{C}^{m \times m^2}$ in compressive sensing?
Choice of g?

Applications in: Radar, sonar, wireless communications.
Simplified discrete model for radar (Herman, Strohmer 2009)

Emitted signal: $g \in \mathbb{C}^m$.
Objects scatters g and radar device receives the contribution

$$x_\lambda \pi(\lambda) g = x_{k,\ell} M^\ell T^k g.$$

T^k corresponds to delay, i.e., distance of object
M^ℓ corresponds to Doppler shift, i.e., speed of the object
$x_{k,\ell}$ reflectivity of object

Received signal is superposition of contribution of all scatteres:

$$y = \sum_{\lambda \in \Lambda} x_\lambda \pi(\lambda) g = \Psi_g x.$$

Usually few scatterers so that $x \in \mathbb{C}^{m^2}$ can be assumed sparse.

Random choice of $g \in \mathbb{C}^m$,

$$g = \frac{1}{\sqrt{m}}\epsilon$$

where ϵ is either a Rademacher or Steinhaus vector,

- Steinhaus: entries ϵ_ℓ are independent and uniformly distributed on torus $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$.

Then Ψ_g is structured random matrix!

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Theorem (Pfander, Rauhut 2007)

Let $x \in \mathbb{C}^N$, $N = m^2$, be s-sparse. Choose $g \in \mathbb{C}^m$ as a random Steinhaus sequence. Assume that

$$m \geq Cs \log(N/\epsilon).$$

Then with probability at least $1 - \epsilon$ ℓ_1-minimization recovers x from $y = \Psi_g x$.

Theorem (Krahmer, Mendelson, Rauhut 2012)

Let $\Psi_g \in \mathbb{C}^{m \times N}$, $N = m^2$, be generated by a random draw of a Steinhaus or Rademacher vector. If, for $\delta \in (0,1)$,

$$m \geq C\delta^{-2} s \log^2 s \log^2 N,$$

then with probability at least $1 - N^{-\log N \log^2 s}$ the restricted isometry constant of Ψ_g satisfies $\delta_s \leq \delta$.

Improves previous estimate (Pfander, Rauhut, Tropp – 2011), which roughly requires

$$m \geq C_\delta s^{3/2} \log^3 N.$$

Congratulations Alain!