Winter School
Combinatorial and algorithmic aspects of convexity
Paris, Institut Henri Poincaré, January 19-23, 2015.

Algorithmic aspects of convexity. Santosh Vempala's lecture II: Convex optimization

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The topic of this lecture is convex optimization and a more general view of convex programming. We want to minimize a convex function f in a convex body K. This depends on the presentation of K and f. Recall from the last lecture that, using binary search, we reduced to the feasibility problem: "Find $x \in K$ or K is empty".

We assume that the convex set K is presented by an oracle. We shall consider two kinds of oracles:

- Separation Oracle: we ask if a point x is in K. If $x \in K$, the oracle answers "yes". If $x \notin K$ the oracle returns a half-space H such that $K \subset H$ and $x \notin K$. In addition, we are given two numbers $r, R \in \mathbb{R}$ such that there exists a point x_0 (unknown) satisfying
$$x_0 + rB_n \subset K \subset RB_n.$$
- Membership Oracle: we ask if a point x is in K. If $x \in K$, the oracle answers "yes". If $x \notin K$, the oracle answers "no". Here in addition to the oracle we need an initial given point $x_0 \in K$, and two radius $r, R \in \mathbb{R}$ so that
$$x_0 + rB_n \subset K \subset RB_n.$$

We say that an algorithm is efficient, if it has complexity
$$poly(n, \log \frac{R}{r})$$

An excellent reference for these questions is the book "Geometric algorithms and convex optimization" by Grötschel, Lovász and Schrijver.

Examples 0.1. Here are three examples of convex sets given by an oracle.

1. Linear programming: we are given an $m \times n$ matrix A, a vector $b \in \mathbb{R}^m$ and
$$K = \{x \in \mathbb{R}^n; Ax \leq b\} := \{x \in \mathbb{R}^n; (Ax)_i \leq b_i, 1 \leq i \leq m\}$$
is a convex polyhedron.
2. Semi-definite programming: we are given m symmetric matrices $A_1, \ldots, A_m$ and m real numbers $b_1, \ldots, b_m$ and one looks for a semidefinite positive symmetric matrix X such that
$$\langle A_i, X \rangle := Tr(A_i X) = b_i.$$
3. Perfect matching. Recall that a perfect matching of an undirected graph $G = (V, E)$ is a subset $M \subset E$ such that every vertex $i \in V$ belongs to exactly one edge $e \in M$. The algorithmic search for a perfect matching may be done by linear programming via the introduction of the perfect matching polytope $PM(G)$ which is the set of $(x_{i,j})_{i,j \in V}$ such that $\sum_{(i,j)\in E} x_{i,j} = 1$ for all $i \in V$, $0 \leq x_{i,j} \leq 1$ and $\sum_{(i,j)\in S} x_{i,j} \leq \frac{|S|-1}{2}$ for all $S \subset E$, with $|S|$ odd.

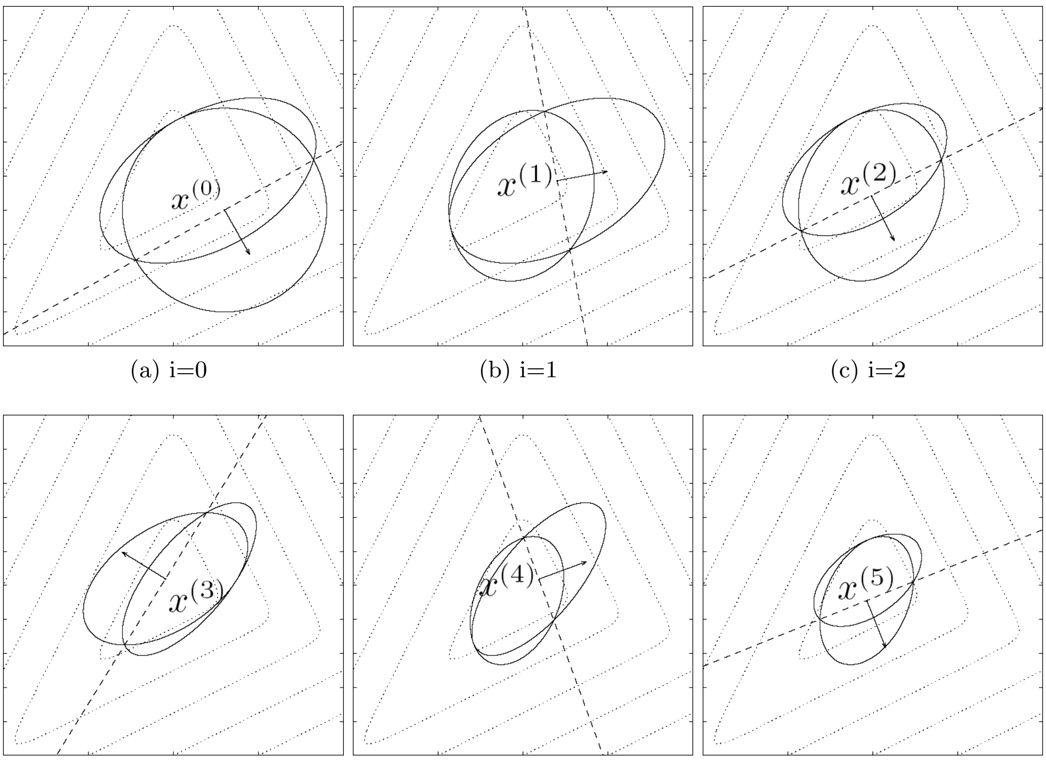
Remark 0.2. R can be exponentially large and r can be exponentially small, so we cannot afford a "search".

1 Ellipsoid Algorithm

An ellipsoid is the affine image of the unit ball B_2^n. It is uniquely defined by a point z, its center, and a positive definite matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$E(z, A) = \{x \in \mathbb{R}^n : (x - z)^T A^{-1} (x - z) \leq 1\} = z + A^{\frac{1}{2}}(B_2^n).$$

For any ellipsoid E, we denote by $z(E)$ its center. We define an algorithm which produces a sequence of ellipsoids E_i with centers $z_i = z(E_i)$.
We start with $E_0 = RB_2^n$ and $z_0 = 0$. Then at each step we process the following loop:
We ask the oracle: "Does $z_i \in K$?".
- if yes, return z_i.
- if no, the oracle gives a unit vector a_i such that $a_i^T x \leq a_i^T z_i$ for all $x \in K$. Then let E_{i+1} be the unique minimum volume ellipsoid containing $E_i \cap \{x; a_i^T x \leq a_i^T z_i\}$ and let z_{i+1} be its center.
Repeat t times. If no feasible point, declare "K is empty".



(a) i=0 (b) i=1 (c) i=2

(d) i=3 (e) i=4 (f) i=5

Figure 1: Steps of the Ellipsoid Algorithm

Lemma 1.1. *With the previous notations, one has*

$$Vol(E_{i+1}) \leq e^{-\frac{t}{2(n+1)}} Vol(E_i).$$

Proof. We want to find the equation of the minimum volume ellipsoid E_{i+1} containing a given half-ellipsoid E_i. By an affine transform, we may assume that E_i is the Euclidean ball B_2^n of radius one and that the halfspace is $\{x; x_1 > 0\}$. For any $z \in [0, 1]$, consider the minimum volume ellipsoid $E(z)$, centered at $(z, 0, \cdots, 0)$ containing $B_2^n \cap \{x; x_1 > 0\}$. Then by symmetry, $E(z)$ has one semi-axis of length $1 - z$ with direction e_1 and $n-1$ other semi-axes of identical length that we denote by b. The relation between z and b is given by

$$((0,1) - (z,0))^T \begin{pmatrix} \frac{1}{(1-z)^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} ((0,1) - (z,0)) = 1$$

Hence, one has

$$\frac{z^2}{(1-z)^2} + \frac{1}{b^2} = 1.$$

Thus

$$b = \frac{1-z}{\sqrt{1-2z}}.$$

The volume of the ellipsoid $E(z)$ is $Vol(E(z)) = (1-z)b^{n-1}Vol(B_2^n)$. Let

$$V(z) = (1-z)b^{n-1} = (1-z)\frac{(1-z)^{n-1}}{(1-2z)^{\frac{n-1}{2}}} = \frac{(1-z)^n}{(1-2z)^{\frac{n-1}{2}}}.$$

Then

$$V'(z) = \frac{(1-2z)^{\frac{n-1}{2}}(-n)(1-z)^{n-1} - \left(\frac{n-1}{2}\right)(-2)(1-2z)^{\frac{n-3}{2}}(1-z)^n}{(1-2z)^{n-1}}.$$

Therefore the function $Vol(E(z))$ reaches its minimum at a point z satisfying $n(1-2z) = (n-1)(1-z)$, that is $z = 1/(n+1)$. Hence the minimum volume ellipsoid containing $B_2^n \cap \{x; x_1 > 0\}$ has its center at distance $1/(n+1)$ from the origin. Finally we get

$$\begin{aligned}
\frac{Vol(E_i+1)}{Vol(E_i)} &= V(z) = \frac{(1-\frac{1}{n+1})^n}{(1-\frac{2}{n+1})^{\frac{n-1}{2}}} = \left(\frac{n}{n+1}\right)^n \left(\frac{n+1}{n-1}\right)^{\frac{n-1}{2}} \\
&= \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} \times \frac{n}{n+1} \\
&= \left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \left(1 - \frac{1}{n+1}\right) \\
&\leq e^{\frac{1}{2(n+1)}} e^{-\frac{1}{n+1}} = e^{-\frac{1}{2(n+1)}},
\end{aligned}$$

where we used in the last inequality that $1 + x \leq e^x$ for every $x \in \mathbb{R}$. □

Theorem 1.2. *If a convex body is given by a separation oracle then the ellipsoid algorithm produces a point in the convex body after at most*

$$t = \mathcal{O}(n^2 \log\left(\frac{R}{r}\right)) \textit{ iterations.}$$

Proof. From the lemma, before that the algorithms stops, one has

$$Vol(rB_n) \leq Vol(E_{t+1}) \leq e^{-\frac{t}{2(n+1)}} Vol(E_0) = e^{-\frac{t}{2(n+1)}} R^n Vol(B_2^n)$$

Thus

$$t \leq 2(n+1)\log\left(\frac{R}{r}\right)^n = \mathcal{O}(n^2 \log\left(\frac{R}{r}\right)).$$

□

A variant of the ellipsoid algorithm may also be used for "rounding", *i.e.* finding an ellipsoid E whose center is denoted by z and a shrinking factor λ such that

$$(1-\lambda)z + \lambda E \subset K \subset E.$$

Theorem 1.3. Rounding (Lovász) *With at most $\mathcal{O}(n^2 \log\left(\frac{R}{r}\right))$ calls to the oracle, we may find such a rounding ellipsoid with a shrinking factor $\lambda = (\sqrt{n}(n+1))^{-1}$.*

Proof. We describe an algorithm which produces a sequence of ellipsoids (E_i) containing K and which eventually produces the target rounding ellipsoid. We start with $E_0 = RB_2^n$. At each step, we apply the following procedure to the ellipsoid E_i.

Given an ellipsoid E containing K with center z, with orthonormal basis of eigenvectors $v_1, \cdots, v_n$ and semi-axes $a_1, \cdots, a_n$, we call the oracle $2n$ times to ask if the points $z \pm a_i v_i/(n+1)$ belong to K.

• if the answer is YES for all $2n$ points then we deduce that

$$\operatorname{conv}\left(z \pm \frac{a_i v_i}{n+1}\right) = z + \operatorname{conv}\left(\pm\frac{a_i v_i}{n+1}\right) \subset K.$$

But in the same way that $\frac{B_2^n}{\sqrt{n}} \subset B_1^n$, one has $\frac{E-z}{\sqrt{n}(n+1)} \subset \operatorname{conv}(\pm\frac{a_i v_i}{n+1})$ thus we get

$$(1-\lambda)z + \lambda E = z + \lambda(E - z) \subset K \subset E,$$

with $\lambda = (\sqrt{n}(n+1))^{-1}$.

• if the answer is NO for at least one of the $2n$ points then one has (say) $z + \frac{a_1 v_1}{n+1} \notin K$ and we get a separating hyperplane which cuts the ellipsoid E through this point. Then we may show that the ellipsoid E' of minimum volume containing this "half" ellipsoid satisfies that $Vol(E') \leq e^{-c/n} Vol(E)$.

Exactly as before, because of the volume drop, running the algorithm, we get that one cannot be in case 2 more than $\mathcal{O}(n^2 \log\left(\frac{R}{r}\right))$ times. Hence after at most this number of steps we are in case 1 and we obtain the rounding ellipsoid. □

Theorem 1.4. Lower bound for feasibility. *Any feasibility algorithm needs at least $\Omega(n \log\left(\frac{R}{r}\right))$ calls to the oracle for determining a point in a convex body K given by a separation oracle and two constants $0 < r < R$ which are the radius of an inscribed and a circumscribed ball, in the worst case.*

Proof. For example, if K is an unknown cube of radius r in a grid of radius R. At each cut, we eliminate in the best case half of the cubes, so we shall need at least $\log_2\left(\left(\frac{R}{r}\right)^n\right) = n \log_2\left(\frac{R}{r}\right)$ oracle calls. □

We have seen a bound of $\mathcal{O}(n^2 \log\left(\frac{R}{r}\right))$ queries for the ellipsoid algorithm. Can we find an algorithm that achieve the best bound given above. The answer is "Yes" and the algorithm is described in the next section.

2 A centroid based algorithm

Given a high-dimensional convex body K, we would like to pick a point z such that for any cut of the body by a half-space, the piece containing z is big. A reasonable choice for z is the centroid, i.e.

$$z = centroid(K) = \frac{1}{Vol(K)} \int_{x \in K} x \, dx$$

We introduce the following centroid based algorithm which construct a sequence of points z_i and polytopes P_i which contain K.
We start with $P_0 = RB_\infty^n$ and $z_0 = 0$. At each step, we ask the oracle: "Does $z_i \in K$?".
- if yes, return z_i.
- if no, the oracle gives a unit vector a_i such that $a_i^T x \leq a_i^T z_i$ for all $x \in K$. Then let $P_{i+1} = P_i \cap \{x; a_i^T x \leq a_i^T z_i\}$ and let z_{i+1} be the centroid of P_{i+1}.

This choice guarantees to get at least half of the volume for any origin symmetric body. We now want to know how much we are guaranteed to get for a general convex body, and what body gives the worst case. Actually, Grunbaum's theorem states that the circular cone is the worst case if we choose the centroid. Thus the previous algorithm removes in the worst case $\frac{1}{e}$ of the total mass of the container P_i, which means good news as it leads to fast algorithm.

Theorem 2.1 (Grunbaum)**.** *Let K be a convex body and H be a half-space containing the centroid z of K, then*

$$\mathrm{vol}(H \cap K) \geq \frac{1}{e}\mathrm{vol}(K).$$

From Grunbaum's theorem, using again volume considerations, we get the following theorem.

Theorem 2.2. *With the previous notations*

$$t = \mathcal{O}(n \log \left(\frac{R}{r}\right)) \text{ iterations suffices.}$$

For the sake of completeness, we nevertheless prove Grunbaum's theorem.

Proof. of Grunbaum's theorem. Without loss of generality, we change coordinates by an affine transformation so that the centroid is the origin and the half-space H used to cut is $\{x_1 \geq 0\}$, then perform the following operations:

1. Replace every $(n-1)$-dimensional slice K_r with an $(n-1)$-dimensional ball with the same volume to get K_H (symmetrized for orthogonal H), which is convex by Lemma 2.3.

2. Turn K_H into a cone, such that the ratio gets smaller by Lemma 2.4.

□

Lemma 2.3. *K_H is convex.*

Proof. Let $K_H^r = K_H \cap \{x_1 = r\}$ be a parallel slice in the new body. The radius of K_H^r is proportional to $Vol(K^r)^{\frac{1}{n-1}}$. By applying Brunn-Minkowski inequality, we get that $Vol(K^r)^{\frac{1}{n-1}}$ is a concave function in r, thus K_H is convex. □

Lemma 2.4. *We can turn K_H into a cone while decreasing the ratio.*

Proof. Let $K_H^+ = K_H \cap \{x_1 \geq 0\}$, $K_H^- = \{x_1 \leq 0\}$. We now make a cone C_+ by picking a point having x_1 coordinate positive on the x_1-axis, and $Vol(C) = Vol(K_H^+)$. We extend the cone in the $x_1 \leq 0$ region, so that the volume of the extended part C_- is equal to $Vol(K_H^-)$. We name the resulting cone C. Now because the construction of C only moved mass from left to right, we deduce that the centroid of C must lie in H. Let H' be the

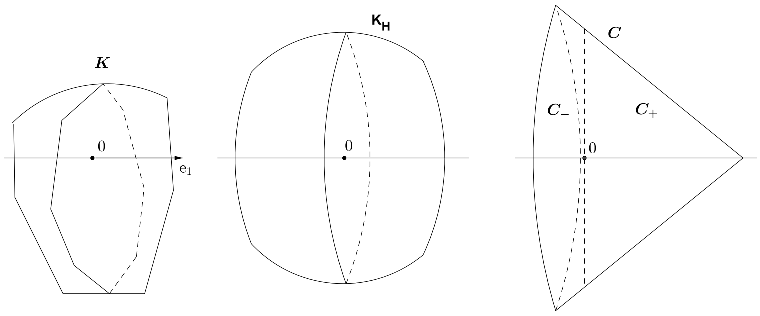


Figure 2: From left to right: K, its Schwartz symmetrization K_H with respect to e_1 and the final cone C.

translation of H along the x_1-axis so that it contains the centroid of C in its bounding hyperplane. Then

$$\frac{\operatorname{vol}(H \cap K)}{\operatorname{vol}(K)} = \frac{\operatorname{vol}(H \cap C)}{\operatorname{vol}(C)} \geq \frac{\operatorname{vol}(H' \cap C)}{\operatorname{vol}(C)} = \left(\frac{n}{n+1}\right)^n \geq \frac{1}{e}.$$

$\square$

The problem is that computing the centroid is an NP-hard problem, thus it is not plausible to find the centroid even for polytopes and deterministic algorithms. In order to conclude a feasible algorithm, in the previous algorithm we replace the centroid by an approximate centroid

$$z_{i+1} := \frac{1}{m} \sum_{j=1}^{m} x_j, \quad x_1, \ldots, x_m \text{ random points in } P_{i+1}.$$

Next theorem is a corrected bound of $\operatorname{vol}(K \cap H)$ as in Theorem 2.1 when we consider $z = \frac{1}{m} \sum_{j=1}^{m} x_j$, x_j random uniform points from K, instead of the centroid.

Theorem 2.5. *Let K be an n-dimensional convex body and $z := \frac{1}{m} \sum_{j=1}^{m} x_j$, where x_j are m random uniform points from K. Then*

$$\mathbb{E}(\min_{H \ni z} \operatorname{vol}(K \cap H)) \geq \left(\frac{1}{e} - \sqrt{\frac{n}{m}}\right) \operatorname{vol}(K).$$

Corollary 2.6. *With probability $\geq 1 - \delta$ the algorithm will provide an answer after $T = O(n \log \frac{R}{r\delta})$ queries.*

In order to show Theorem 2.5 we may assume that K is in isotropic position as the quotient keeps constant under affine maps.

Lemma 2.7. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a marginal isotropic log-concave density of an n-dimensional convex body K, i.e.,*

$$\int f(y)dy = 1, \quad \int y f(y)dy = 0 \quad \textit{and} \quad \int y^2 f(y)dy = 1.$$

Then

$$\max f(y) \leq \frac{n}{n+1} \sqrt{\frac{n}{n+2}} < 1.$$

Proof of Theorem 2.5. Let us assume that K in isotropic position. Then we have

$$\mathbb{E}_K(z \cdot z^\top) = \frac{1}{m}\mathbb{E}_K(x_j \cdot x_j^\top) = \frac{n}{m}$$

as $\mathbb{E}_K(x_j \cdot x_j^\top) = n$ for any such random uniform point in an isotropic set K. Thus $\mathbb{E}_K(||z||_2) \leq \sqrt{\mathbb{E}_K(||z||_2^2)} = \sqrt{n/m}$. On the other hand, let H be any halfspace containing z. We may assume that $H = \{x; x_1 \geq a\}$, with $a \geq 0$. One has $\|z\|_2 \geq z_1 \geq a$. Letting

$$f(t) := \frac{\mathrm{vol}_{n-1}(\{x \in K; x_1 = t\})}{\mathrm{vol}_n(K)}$$

the marginal density of K (which is also isotropic and log-concave) then by Lemma 2.7, Theorem 2.1 we conclude

$$\mathrm{vol}(K \cap H) = \int_{y \geq a} f(y)dy = \int_{y \geq 0} f(y)dy - \int_{0 \leq y \leq a} f(y)dy \geq \frac{1}{e} - a \geq \frac{1}{e} - ||z||_2.$$

Taking the expectation, we conclude that

$$\mathbb{E}(\min_{H \ni z} \mathrm{vol}(K \cap H)) \geq \mathbb{E}\left(\frac{1}{e} - ||z||_2\right) \geq \frac{1}{e} - \sqrt{\frac{n}{m}}.$$

□

Proof of Lemma 2.7. Let us assume after a suitable rigid motion that the marginals are taken with respect to the hyperplane $H = \mathrm{e}_1^\perp$. Let $a\mathrm{e}_1 = y^* \in \mathbb{R}$, $a \geq 0$, be such that $f(y^*) = \max f(y)$. We first apply Schwarz symmetrization to K with respect to $\mathrm{lin}\{\mathrm{e}_1\}$ and get K_H. We now modify K_H (and f change accordingly) and call K' the final result, in the following way: call $K_1 = K_H \cap \{x : x^\top \mathrm{e}_1 \leq 0\}$, $K_2 = K_H \cap \{x : 0 \leq x^\top \mathrm{e}_1 \leq a\}$ and $K_3 = K_H \cap \{x : x^\top \mathrm{e}_1 \geq a\}$. Replace K_1 by the circular piramid $K_1' = \mathrm{conv}(b\mathrm{e}_1 \cup (K_1 \cap K_2))$ with $b \leq 0$ be such that $\mathrm{vol}(K_1) = \mathrm{vol}(K_1')$. Replace K_2 by the truncated cone $K_2' = \mathrm{conv}((K_1 \cap K_2) \cup (K_2 \cap K_3))$ and finally exchange K_3 by the circular piramid $K_3' = \mathrm{conv}(c\mathrm{e}_1 \cup (K_2 \cap K_3))$ and $c \geq 0$ be such that $\mathrm{vol}(K_3') = \mathrm{vol}(K_3) + \mathrm{vol}(K_2) - \mathrm{vol}(K_2')$.

Let us observe that the mass moves away from the center of mass and spread in the direction $\mathrm{lin}\{\mathrm{e}_1\}$. Indeed, calling the map of the change $y \to y + g(y)$ with $yg(y) \geq 0$, and $I(K) = \int_K (y - \mathbb{E}_K(y))^2 f_K(y)dy$ (the moment of inertia), then

$$\begin{aligned} I(K') &= \mathbb{E}_{K'}(y^2) - \mathbb{E}_{K'}(y)^2 \\ &= \mathbb{E}_K((y + g(y))^2) - \mathbb{E}_K(y + g(y))^2 \\ &= I(K) + Var(g(y)) + 2\mathbb{E}_K(yg(y)) \geq I(K). \end{aligned}$$

Thus the moment of inertia will only increase if we modify K' in the following way: instead of K_1', we continue K_2' on the left until it reaches the axis. We have replaced $K_1 \cup K_2'$ by a single cone with the same slope as K_2'. It has added mass on the left. We compensate it by moving the point c, which defines K_3', to the right. Then the whole mass and the variance have increased, the center of mass and the maximum have not moved. Rescaling down in the line $\mathrm{lin}\{\mathrm{e}_1\}$ as well as rescaling up in H the maximum of its marginal increases. With a similar move we finally arrive at a cone. Calling h, and resp. A, the height of the last cone, and resp. the area of its base, we get that for its marginal f

$$\max f = \frac{n}{n+1}\sqrt{\frac{n}{n+2}},$$

□

3 Optimization with a membership oracle

Our next goal is to find a suitable algorithm for optimizing a linear function on a convex body known only by a membership oracle. Recall what it means. We are given an initial point $x_0 \in K$, and two radius $r, R \in \mathbb{R}$ so that

$$x_0 + rB_n \subset K \subset RB_n.$$

We ask if a point x is in K. If $x \in K$, the oracle answers "yes". If $x \notin K$, the oracle answers "no".

Data: $z_0 \in K$, $c \in \mathbb{R}^n$, $\varepsilon > 0$, $K_0 = K$, Membership oracle
Result: Approximating $\min_{x \in K} c^\top x$
initialization;
if $c^\top z_i \leq \min_K c^\top x + \varepsilon$ **then**
  DONE;
else
  sample K;
  $K_{i+1} = K_i \cap \{x \in \mathbb{R}^n : c^\top x \leq c^\top z_i\}$;
  let z_{i+1} be the centroid of K_{i+1};
end

This algorithm lead to a number of iterations of order $m = O(n \log \frac{1}{\varepsilon})$ (and a complexity of $O^*(n^5)$). Let us also remark that the distance to $\arg\min_K c^\top x$ decreases by a factor of $(1 - \frac{1}{n})$ at least. Assuming a factor $(1 - \frac{1}{\sqrt{n}})$ on the outer radius R on each of those steps and sampling with a density getting us closer to the optimal solution, we add the following lines to the algorithm: for $i = 1, \ldots, m$ do

- $T_i = R(1 - \frac{1}{\sqrt{n}})^i$;
- Sample from K_{i+1} with density $f_i(x) \approx e^{-\frac{c^\top x}{T_i}} \chi_K(x)$.

Theorem 3.1. *For $m = O(\sqrt{n} \log \frac{1}{\varepsilon})$ iterations it suffices with probability $\geq 1 - \delta$ that $c^\top x_m \leq \min_K c^\top x + \varepsilon$.*

Next lemma shows the truth of Theorem 3.1 and the algorithm provided above.

Lemma 3.2. *Let $f(x) = e^{-\frac{c^\top x}{T}} \chi_K(x)$ be a density on K, where K is an n-dimensional convex body, $c \in \mathbb{R}^n$ and $T > 0$. Then*

$$\mathbb{E}(c^\top x) \leq \min_K c^\top x + nT.$$

Proof. After a suitable rigid motion we can assume that $c = (1, 0, \ldots, 0)$ and $x^* = a\mathrm{e}_1 \in K$ be such that $K \subset \{x \in \mathbb{R}^n : c^\top x^* \leq c^\top x\}$ (see Figure 3). Let us denote by $v = \mathbb{E}(c^\top x)$ and

$$K' = \{x \in \mathbb{R}^n : x = x^* + \alpha(y - x^*),\, y \in K \cap H(v),\, \alpha \geq 0\}$$

where $H(v) = v\mathrm{e}_1 + \mathrm{e}_1^\perp$. Then $v = \mathbb{E}(c^\top x) \leq \mathbb{E}_{K'}(c^\top x)$ and

$$\mathbb{E}_{K'}(c^\top x) = \frac{\int_0^\infty y e^{-\frac{y}{T}} \mathrm{vol}(K' \cap H(y)) dy}{\int_0^\infty e^{-\frac{y}{T}} \mathrm{vol}(K' \cap H(y)) dy}.$$

Using $\mathrm{vol}(K' \cap H(y)) = (y/v)^{n-1} \mathrm{vol}(K' \cap H(v))$ we finally get that

$$\mathbb{E}_{K'}(c^\top x) = \frac{\int_0^\infty y^n e^{-\frac{y}{T}} dy}{\int_0^\infty y^{n-1} e^{-\frac{y}{T}} dy} = \frac{n! T^n}{(n+1)! T^{n-1}} = nT.$$

The last computation follows from $\int_0^\infty y^n e^{-\frac{y}{T}} dy = n! T^{n+1}$. $\square$

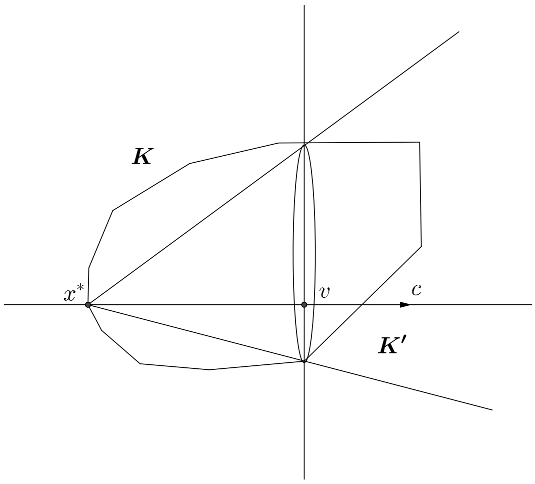


Figure 3: The original set K and the infinite cone K' containing the direction c.

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