Winter School
Combinatorial and algorithmic aspects of convexity
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Algorithmic aspects of convexity. Santosh Vempala's lecture V.

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1 Preliminaries (more or less mentioned in the previous lecture)

We consider a state space $\mathcal{K}$ and a σ-algebra $\mathcal{A}$ on the subsets of $\mathcal{K}$. For $u \in \mathcal{K}$ and $A \in \mathcal{A}$, let $P_u(A)$ be the "one-step probability", which tells us the probability of being in A after taking one step from u. We have a starting distribution Q_0 on $\mathcal{K}$ which gives us a probability $Q_0(A)$ of starting in the set $A \in \mathcal{A}$.

With this setup, a Markov chain is a sequence of points $w_0, w_1, w_2, \dots$ such that $\mathbb{P}(w_0 \in A) = Q_0(A)$ and

$$\mathbb{P}(w_{i+1} \in A | w_0, \dots, w_i) = \mathbb{P}(w_{i+1} \in A | w_i) = P_{w_i}(A),$$

for each $A \in \mathcal{A}$. A distribution Q is called stationary if, for each $A \in \mathcal{A}$,

$$Q(A) = \int_{\mathcal{K}} P_u(A)\, dQ(u).$$

The Markov chain is called time reversible if, for each $A, B \in \mathcal{A}$,

$$Q(A) \int_A P_u(B)\, dQ(u) = Q(B) \int_B P_u(A)\, dQ(u).$$

For any $A \in \mathcal{A}$, the ergodic flow $\Phi(A)$ is the probability of transitioning from A to $\mathcal{K} \setminus A$,

$$\Phi(A) = \int_A P_u(\mathcal{K} \setminus A)\, dQ(u).$$

We define, for each $A \in \mathcal{A}$,

$$\phi(A) = \frac{\Phi(A)}{\min\{Q(A), Q(\mathcal{K} \setminus A)\}}.$$

Then the conductance of $\mathcal{K}$ is $\phi = \min_{A \subset \mathcal{K}} \phi(A)$, where Q is a stationary distribution.

Now let $\mathcal{K} = K$ be a convex body in $\mathbb{R}^n$ and $\mathcal{A}$ the set of measurable subsets of K. The local conductance at $u \in K$ is

$$\ell(u) = 1 - P_u(u) = \frac{vol((u + \delta B_n) \cap K)}{vol(\delta B_n)}$$

We have seen that the conductance of the ball walk in K can be exponentially small. We can bypass this problem by defining the α-extension of K, $K' = K + \alpha B_n$. Then for $\alpha > 2\delta\sqrt{n}$ we have $\ell(u) \geqslant 1/8$, for every $u \in K'$.

If Q_i is the distribution of the i-th step of the random walk, set

$$d_{TV}(Q_t, Q) = \sup_{A \in \mathcal{A}} (Q_t(A) - Q(A))$$

and

$$M(Q_t, Q) = \sup_{A \in \mathcal{A}} \frac{Q_t(A)}{Q(A)}.$$

The following theorem was proved in lecture 4.

Theorem 1.1.

$$d_{TV}(Q_t, Q) \leqslant \sqrt{M(Q_0, Q)} \big(1 - \phi^2/2\big)^t.$$

We want to prove that the random walk we defined is rapidly mixing, that is, that conductance ϕ is bounded from below by an inverse polynomial in the dimension. Specifically, our goal is to prove

Theorem 1.2. *If $D = \operatorname{diam}(K)$ and for every $u \in K$ the local conductance of the ball walk with δ steps is at least ℓ, then*

$$\phi \geqslant \frac{\ell^2 \delta}{16\sqrt{n}D}.$$

Proof. Consider an arbitrary measurable $S_1 \subseteq K$ and set $S_2 = K \setminus S_1$. We will prove that

$$\Phi(S_1) = \int_{S_1} P_u(S_2)\, dQ(u) \geqslant \frac{\ell^2 \delta}{16\sqrt{n}D} \cdot \min\{vol(S_1), vol(S_2)\}$$

Observe that, since the distribution Q is stationary, we have

$$\int_{S_1} P_u(S_2)\, dQ(u) = \int_{S_2} P_u(S_1)\, dQ(u).$$

Next we define the sets

$$S_1' = \{u \in S_1 : P_u(S_2) < \ell/4\},$$

$$S_2' = \{u \in S_2 : P_u(S_1) < \ell/4\}$$

and

$$S_3' = K \setminus (S_1' \cup S_2').$$

Then

$$\begin{aligned}
\int_{S_1} P_u(S_2)\, dQ(u) &= \frac{1}{2}\left(\int_{S_1} P_u(S_2)\, dQ(u) + \int_{S_2} P_u(S_1)\, dQ(u)\right) \\
&\geqslant \frac{1}{2} \int_{S_3'} \frac{\ell}{4}\, dQ(u).
\end{aligned}$$

We have thus proved that $\Phi(S_1) \geqslant \ell/8 \cdot Q(S_3')$. We will see how we derive the wanted result from the following theorem, a variant of which will be proved in the next section.

Theorem 1.3. *Given a partition $\{S_1', S_2', S_3'\}$ of a convex body K in $\mathbb{R}^n$,*

$$Q(S_3') \geqslant \frac{2}{D} d(S_1,' S_2) \cdot \min\{Q(S_1'), Q(S_2')\},$$

where $D = \operatorname{diam}(K)$, and $d(S_1', S_2')$ the usual Euclidean distance between the sets S_1', S_2'.

We will also use the fact that S_1' and S_2' are "far apart". The following Lemma was proved in Lecture 4.

Lemma 1.4. *Let $u, v \in K$, such that $\ell(u), \ell(v) \geqslant \ell$. Then*

$$\|u - v\| \leqslant \frac{t\delta}{\sqrt{n}} \Longrightarrow d_{TV}(P_u, P_v) \leqslant t + 1 - \ell.$$

Now let $v \in S_1'$, $u \in S_2'$. By Lemma 1.4, we have

$$d_{TV}(P_u, P_v) > 1 - \frac{\ell}{2} \Longrightarrow \|u - v \;\geqslant \frac{\ell\delta}{2\sqrt{n}}.$$

Assume that $Q(S_1') < 1/2 \cdot Q(S_1)$. Then

$$\int_{S_1} P_u(S_2)\, dQ(u) \geqslant \frac{\ell}{4} \int'_{S_1 \setminus S_1} dQ(u) \geqslant \frac{\ell}{8} vol(S_1),$$

which proves what was wanted. We are thus left with the case $Q(S_i') \geqslant 1/2 \cdot Q(S_i)$, $i = 1, 2$. Then, by Theorem 1.3,

$$\begin{aligned} \Phi(S_1) &\geqslant \frac{\ell}{8} \cdot \frac{2}{D} \cdot \frac{\ell\delta}{2\sqrt{n}} \min\{Q(S_1'), Q(S_2')\} \\ &\geqslant \frac{\ell^2}{8} \cdot \frac{\delta}{D\sqrt{n}} \cdot \frac{1}{2} \min\{Q(S_1), Q(S_2)\}, \end{aligned}$$

which implies that

$$\phi \geqslant \frac{\ell^2 \delta}{16 D \sqrt{n}},$$

concluding the proof of the Theorem. □

Remarks 1.5. (i) The random walk mixing rate is of order $O(1/\phi^2)$, so by Theorem 1.2 we have an upper bound $O(n^4 D^2)$.
(ii) Another method would be to bound the average local conductance $\mathbb{E}_K(\ell(u)) \geqslant c$, so that, choosing $\delta = c'/\sqrt{n}$, we wouuld have a mixing rate of order $O(n^2 D^2)$.
(iii) By considering the example of a cylinder of height D and base radius 1, $\delta = 1/\sqrt{n}$, we gain the bound $\Omega(n^2 D^2)$.

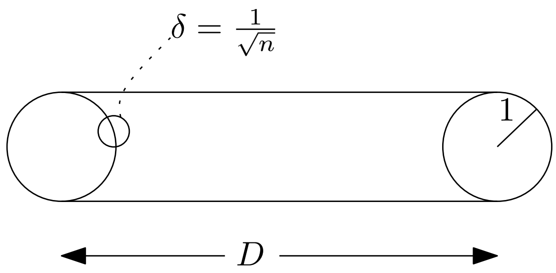

Figure 1: The example with the cylinder of height D and base radius 1.

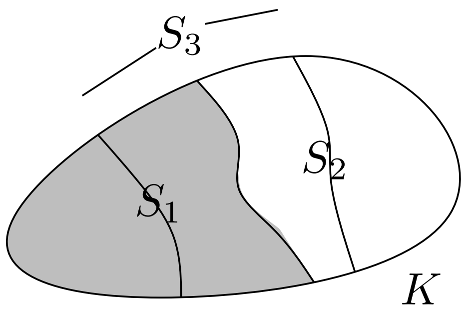

Figure 2: A partition of a convex body.

2 The localization lemma and an isoperimetric inequality

We now state and prove the isoperimetric inequality of Theorem 1.3 in the more general context of log-concave functions.

Theorem 2.1. *Let f be a log-concave function, $K := \mathrm{supp}(f)$, $D := \mathrm{diam}(K)$, and Π_f the induced probability distribution. Then for any partition $\{S_1, S_2, S_3\}$ of K,*

$$\Pi_f(S_3) \geqslant \frac{2d(S_1, S_2)}{D} \min\{\Pi_f(S_1), \Pi_f(S_2)\}.$$

We write ϕ_f for the best constant $c > 0$ such that

$$\Pi_f(S_3) \geqslant c \cdot \min\{\Pi_f(S_1), \Pi_f(S_2)\}.$$

By a Theorem of Kannan, Lovász and Simonovits [1], we have

$$\phi_f \geqslant \frac{c}{\sqrt{\mathbb{E}_{\Pi_f}(\|X - \overline{X}\|^2)}} = \frac{c}{\sqrt{\sum \lambda_i(A)}},$$

where $\overline{X} = \mathbb{E}_{\Pi_f}(X)$, A is the covariance matrix of Π_f and $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \ldots \geqslant \lambda_n(A)$ the eigenvalues of A. In the same paper, KLS prove the upper bound $\phi_f \leqslant \frac{c}{\sqrt{\lambda_1(A)}}$ and conjecture that ϕ_f is actually of the order $\Theta(1/\sqrt{\lambda_1(A)})$.

Conjecture 2.2 (Kanan, Lovász, Simonovits, [1])**.**

$$\phi_f \leqslant \frac{c}{\sqrt{\lambda_1(A)}}.$$

We also formulate the following (weaker) conjecture, in terms of the Frobenius norm of A, which we discuss at the end of these notes.

Conjecture 2.3.

$$\phi_f \geqslant \frac{c}{(\sum \lambda_i(A)^2)^{1/4}} = \frac{c}{\sqrt{\|A\|_F}}.$$

We now proceed to the proof of our main result.

Proof of Theorem 2.1. Let $c = 2d(S_1, S_2)/D$ and suppose that $\{S_1, S_2, S_3\}$ is a partition of K such that

$$\int_{S_3} f < c \int_{S_1} f \qquad \text{and} \qquad \int_{S_3} f < c \int_{S_2} f,$$

or equivalently that $\int_{\mathbb{R}^n} g > 0$ and $\int_{\mathbb{R}^n} h > 0$, where

$$(1)\quad g(x) = \begin{cases} cf(x), & \text{if } x \in S_1 \\ 0, & \text{if } x \in S_2 \\ -f(x), & \text{if } x \in S_3 \end{cases} \qquad \text{and} \qquad h(x) = \begin{cases} 0, & \text{if } x \in S_1 \\ cf(x), & \text{if } x \in S_2 \\ -f(x), & \text{if } x \in S_3 \end{cases}$$

To prove th theorem, we reduce the assertion to the one-dimensional case, through the following localization lemma.

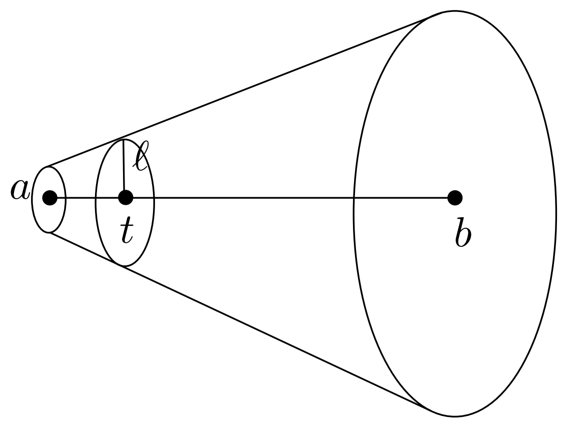

Figure 3: The truncated cone construction. Connection of n-dimensional integration with one dimensional.

Lemma 2.4. *Let $g, h : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous integrable functions such that $\int_{\mathbb{R}^n} g > 0$ and $\int_{\mathbb{R}^n} h > 0$. Then there exist two points $a, b \in \mathbb{R}^n$ and an affine function ("needle") $l : [0,1] \to \mathbb{R}_+$ such that*

$$\int_0^1 l(t)^{n-1} g((1-t)a+tb)\, dt > 0 \qquad \textit{and} \qquad \int_0^1 l(t)^{n-1} h((1-t)a+tb)\, dt > 0.$$

Sketch of the Proof. The proof of the Lemma can be roughly divided into three steps.

Step 1. Let A be a $(n-2)$-dimensional affine subspace of $\mathbb{R}^n$. For each such

A, there is a halfspace H (bisecting halfspace) with A contained in its bounding hyperplane, such that

$$\int_H g = \frac{1}{2}\int_{\mathbb{R}^n} g,$$

while at the same time (replacing H by its complementary halfspace, if necessary)

$$\int_H g > 0 \qquad \text{and} \qquad \int_H h > 0.$$

Now let $A_1, A_2, \ldots$ be a sequence of such $(n-2)$-dimensional subspaces with rational coorinates, and consider $K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots$ a respective sequence of convex bodies, where each K_{i+1} is obtained from K_i by cutting it into two by a bisecting hyperplane P_i through A_i, and choosing the appropriate half. Then $L = \bigcap_{i=1}^{\infty} K_i$ is at most 1-dimensional.

Step 2. Without loss of generality, let $a = o$ and $b = e_1$. Define

$$Z_t = \{x \in \mathbb{R}^n : x_1 = t\}.$$

By the Brunn-Minkowski inequality, for each $i = 1, 2, \ldots$ the function

$$\psi_i = \left(\frac{vol(K_i \cap Z_t)}{vol(K_i)}\right)^{\frac{1}{n-1}}$$

is concave. Moreover, $\psi_i(t) \leqslant n^{\frac{1}{n-1}}$ and for each $0 \leqslant s \leqslant t \leqslant 1$,

$$\frac{s}{t}\psi_i(t) \leqslant \psi_i(s) \leqslant \frac{1-s}{1-t}\psi_i(t).$$

Thus there is a limiting concave function ψ such that

$$\int \psi_i(t)^{n-1} g \longrightarrow \int \psi(t)^{n-1} g.$$

Since $\int \psi_i(t)^{n-1} g > 0$ for each $i = 1, 2, \ldots$, it follows that the assertion of the Lemma holds for the function ψ in the place of l.

Step 3. The final step in the proof is to obtain the affine function l from the concave function ψ of Step 2. The technical details were not presented in class. □

To conclude the proof of the Theorem, we proceed as follows: Partition $[0, 1]$ into $\{Z_1, Z_2, Z_3\}$, where

$$Z_i = \{t \in [0,1] : (1-t)a + tb \in S_i\}, \qquad i = 1, 2, 3.$$

Apply Lemma 2.4 to the functions g, h of (1) [1].

[1] Actually the functions g and h as defined in (1) are not lower semi-continuous. However this can be achieved by expanding S_1 and S_2 slightly so as to make them open sets, and making the support of f an open set. Since we are proving strict inequalities, we do not lose anything by these modifications.

Rewriting g and h in terms of our original function f we get that

$$\int_{Z_3} l(t)^{n-1} f((1-t)a+tb)\,dt < c \int_{Z_1} l(t)^{n-1} f((1-t)a+tb)\,dt$$

and

$$\int_{Z_3} l(t)^{n-1} f((1-t)a+tb)\,dt < c \int_{Z_2} l(t)^{n-1} f((1-t)a+tb)\,dt.$$

The functions f and $l(\cdot)^{n-1}$ are both log-concave, so the same holds for their product $F(t) = l(t)^{n-1} f((1-t)a+tb)$. We will have reached a contradiction as soon as we prove that

$$\int_{Z_3} F \geqslant 2d(Z_1, Z_2) \cdot \min\{\int_{Z_1} F, \int_{Z_2} F\}.$$

Assume at first that Z_1, Z_3, Z_2 are successive intervals, and write u (resp. v) for the right (resp. left) endpoint of Z_1 (resp. Z_2), so that $d(Z_1, Z_2) = |u-v|$. Without loss of generality suppose that $F(u) \leqslant F(v)$. Then

$$\begin{aligned}\int_{Z_3} F &\geqslant F(u) \cdot |u-v| = |u-v| \cdot F(u) \cdot |1-0| \\ &\geqslant |u-v| \cdot \int_{Z_1} F(u)\,dt \geqslant d(Z_1, Z_2) \cdot \int_{Z_1} F\end{aligned}$$

that proves our claim, disregarding the factor 2 (the full proof would require a little bit more of a struggle).

In the general setting, consider a maximal interval $(r, s) \subseteq Z_3$. Then, by our previous argument, the integral of F over Z_3 is at least c times the smaller of the integrals to its left $[0, r]$ and to its right $[s, 1]$. If all of Z_1 or Z_2 is contained in one of these intervals, we are done. If not, then set $U = [0, r] \vee [s, 1]$. Since Z_1, Z_2 are seperated by at least $d(S_1, S_2)/D$, there is an interval of Z_3 of length $d(S_1, S_2)/D$ between $U \cap Z_1$ and $U \cap Z_2$. Repeating the process for this interval yields the required result. □

Open problem 1. *Given convex body $K \in \mathbb{R}^n$ for every distance function $d_x(y) = ||x-y||_{E_x}$ convex in $x \in K$ all hyperplane partitions are within constant of the optimum Ψ.*

3 Hit-and-run

We now focus on the analysis of the hit-and-run random walk.

Theorem 3.1. *[3] For the hit-and-run random walk it holds $\Phi \leq \frac{c}{nD}$ staring from any point in a convex body K.*

The following definition of the cross-ratio is used

$$d_K(u, v) = \frac{|u-v|\,|p-q|}{|p-u|\,|v-q|} = (p : v : u : q)$$

for the proof of the result. In what follows we list a set of results such as the localization lemma for hit-and-run without proofs. For more information see [3].

Theorem 3.2. *It holds*

$$\pi_f(s_3) \geq d_k(s_1, s_2) \min\{\pi_f(s_1), \pi_f(s_2)\}$$

Lemma 3.3. *For* $0 \leq u \leq w$ *it holds*

$$\frac{(e^v - e^u)\,(e^w - 1)}{(e^u - 1)\,(e^w - e^v)} \geq \frac{(v-u)\,w}{u\,(w-v)}$$

Theorem 3.4. *For function* $h : K \to \mathbb{R}_+$

$$h(x) \leq \frac{1}{3} \min\{1, d_K(u, v)\}$$

for all $u \in S_1, v \in S_2$ *and* $x \in chord(u, v)$.

Finally, for a S_1, S_2, S_3 partition of K it holds $\pi(S_3) \geq E_K(h(x))\pi(S_1)\pi(S_2)$.

4 Open problems

Open problem 2. *Find simpler algorithms to round convex bodies. For example analyse the following*

```
walk for 2T steps in K
1. compute the covariance matrix of the trace X_2T of the walk
2. if exist an eigenvalue l>2
   then make isotropic and goto 1
   else return K
```

Open problem 3. *Analyse the coordinate directions hit-and-run. That is instead from picking a random direction from the unit ball pick a random direction from the set* $\{-e_i, e_i\}, \forall i \in [n]$.

Open problem 4. *Given a polytope* $K := \{x \ : \ Ax \leq b\} \subset \mathbb{R}^n$ *is there any deterministic approximation algorithm that computes* $(1+\epsilon)vol(K)$ *in time polynomial in* $n, 1/\epsilon$*?*

Open problem 5. *If the ball walk starts from* $E_K(x)$*, does it mix in polynomial time?*

Open problem 6. *Is there any statistical tests that guarantees that the distribution* Q_t *is close to* Q*? Is the conductance* $E_{Q_t}(||X||^2)$ *monotonically increasing?*

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