1 Preliminaries (more or less mentioned in the previous lecture)

We consider a state space $K$ and a $\sigma$-algebra $A$ on the subsets of $K$. For $u \in K$ and $A \in A$, let $P_u(A)$ be the "one-step probability", which tells us the probability of being in $A$ after taking one step from $u$. We have a starting distribution $Q_0$ on $K$ which gives us a probability $Q_0(A)$ of starting in the set $A \in A$.

With this setup, a Markov chain is a sequence of points $w_0, w_1, w_2, \ldots$ such that $P(w_0 \in A) = Q_0(A)$ and

$$P(w_{i+1} \in A | w_0, \ldots, w_i) = P(w_{i+1} \in A | w_i) = P_{w_i}(A),$$

for each $A \in A$. A distribution $Q$ is called stationary if, for each $A \in A$,

$$Q(A) = \int_K P_u(A) \, dQ(u).$$

The Markov chain is called time reversible if, for each $A, B \in A$,

$$Q(A) \int_A P_u(B) \, dQ(u) = Q(B) \int_B P_u(A) \, dQ(u).$$

For any $A \in A$, the ergodic flow $\Phi(A)$ is the probability of transitioning from $A$ to $K \setminus A$,

$$\Phi(A) = \int_A P_u(K \setminus A) \, dQ(u).$$

We define, for each $A \in A$,

$$\phi(A) = \frac{\Phi(A)}{\min\{Q(A), Q(K \setminus A)\}}.$$

Then the conductance of $K$ is $\phi = \min_{A \subset K} \phi(A)$, where $Q$ is a stationary distribution.

Now let $K = K$ be a convex body in $\mathbb{R}^n$ and $A$ the set of measurable subsets of $K$. The local conductance at $u \in K$ is

$$\ell(u) = 1 - P_u(u) = \frac{\text{vol}(u + \delta B_n) \cap K)}{\text{vol}(\delta B_n)}.$$

We have seen that the conductance of the ball walk in $K$ can be exponentially small. We can bypass this problem by defining the $\alpha$-extension of $K$, $K' = K + \alpha B_n$. Then for $\alpha > 2\delta \sqrt{n}$ we have $\ell(u) \geq 1/8$, for every $u \in K'$.
If $Q_t$ is the distribution of the $i$-th step of the random walk, set
\[ d_{TV}(Q_t, Q) = \sup_{A \in \mathcal{A}} (Q_t(A) - Q(A)) \]
and
\[ M(Q_t, Q) = \sup_{A \in \mathcal{A}} \frac{Q_t(A)}{Q(A)}. \]
The following theorem was proved in lecture 4.

**Theorem 1.1.**
\[ d_{TV}(Q_t, Q) \leq \sqrt{M(Q_0, Q)} (1 - \phi^2/2)^t. \]

We want to prove that the random walk we defined is rapidly mixing, that is, that conductance $\phi$ is bounded from below by an inverse polynomial in the dimension. Specifically, our goal is to prove

**Theorem 1.2.** If $D = \text{diam}(K)$ and for every $u \in K$ the local conductance of the ball walk with $\delta$ steps is at least $\ell$, then
\[ \phi \geq \frac{\ell^2 \delta}{16 \sqrt{\pi D}}. \]

**Proof.** Consider an arbitrary measurable $S_1 \subseteq K$ and set $S_2 = K \setminus S_1$. We will prove that
\[ \Phi(S_1) = \int_{S_1} P_u(S_2) dQ(u) \geq \frac{\ell^2 \delta}{16 \sqrt{\pi D}} \cdot \min\{\text{vol}(S_1), \text{vol}(S_2)\} \]
Observe that, since the distribution $Q$ is stationary, we have
\[ \int_{S_1} P_u(S_2) dQ(u) = \int_{S_2} P_u(S_1) dQ(u). \]
Next we define the sets
\[ S'_1 = \{u \in S_1 : P_u(S_2) < \ell/4\}, \]
\[ S'_2 = \{u \in S_2 : P_u(S_1) < \ell/4\} \]
and
\[ S'_3 = K \setminus (S'_1 \cup S'_2). \]
Then
\[ \int_{S_1} P_u(S_2) dQ(u) = \frac{1}{2} \left( \int_{S_1} P_u(S_2) dQ(u) + \int_{S_2} P_u(S_1) dQ(u) \right) \geq \frac{1}{2} \int_{S'_3} \frac{\ell}{4} dQ(u). \]

We have thus proved that $\Phi(S_1) \geq \ell/8 \cdot Q(S'_3)$. We will see how we derive the wanted result from the following theorem, a variant of which will be proved in the next section.
Theorem 1.3. Given a partition \( \{S'_1, S'_2, S'_3\} \) of a convex body \( K \) in \( \mathbb{R}^n \),

\[
Q(S'_3) \geq \frac{2}{D} d(S'_1, S'_2) \cdot \min\{Q(S'_1), Q(S'_2)\},
\]

where \( D = \text{diam}(K) \), and \( d(S'_1, S'_2) \) the usual Euclidean distance between the sets \( S'_1, S'_2 \).

We will also use the fact that \( S'_1 \) and \( S'_2 \) are ”far apart”. The following Lemma was proved in Lecture 4.

Lemma 1.4. Let \( u, v \in K \), such that \( \ell(u), \ell(v) \geq \ell \). Then

\[
\|u - v\| \leq \frac{t\delta}{\sqrt{n}} \Rightarrow d_{TV}(P_u, P_v) \leq t + 1 - \ell.
\]

Now let \( v \in S'_1, u \in S'_2 \). By Lemma 1.4 we have

\[
d_{TV}(P_u, P_v) > 1 - \frac{\ell}{2} \Rightarrow \|u - v\| \geq \frac{\ell\delta}{2\sqrt{n}}.
\]

Assume that \( Q(S'_i) < 1/2 \cdot Q(S_i) \). Then

\[
\int_{S_1} P_u(S_2) dQ(u) = \frac{\ell}{4} \int_{S_1 \setminus S_1} dQ(u) \geq \frac{\ell}{8} \text{vol}(S_1),
\]

which proves what was wanted. We are thus left with the case \( Q(S'_i) \geq 1/2 \cdot Q(S_i), i = 1, 2 \). Then, by Theorem 1.3

\[
\Phi(S_1) \geq \frac{\ell}{8} \cdot \frac{2}{D} \cdot \frac{\ell\delta}{2\sqrt{n}} \min\{Q(S'_1), Q(S'_2)\}
\]

\[
\geq \frac{\ell^2}{8} \cdot \frac{\delta}{D\sqrt{n}} \cdot \frac{1}{2} \min\{Q(S_1), Q(S_2)\},
\]

which implies that

\[
\phi \geq \frac{\ell^2\delta}{16D\sqrt{n}},
\]

concluding the proof of the Theorem.

Remarks 1.5. (i) The random walk mixing rate is of order \( O(1/\phi^2) \), so by Theorem 1.2 we have an upper bound \( O(n^4D^2) \).

(ii) Another method would be to bound the average local conductance \( \mathbb{E}_K(\ell(u)) \geq c \), so that, choosing \( \delta = c'/\sqrt{n} \), we would have a mixing rate of order \( O(n^2D^2) \).

(iii) By considering the example of a cylinder of height \( D \) and base radius 1, \( \delta = 1/\sqrt{n} \), we gain the bound \( \Omega(n^2D^2) \).
δ = \frac{1}{\sqrt{n}}

Figure 1: The example with the cylinder of height D and base radius 1.

\[ S_1 \quad S_3 \quad S_2 \]

\[ K \]

Figure 2: A partition of a convex body.

2 The localization lemma and an isoperimetric inequality

We now state and prove the isoperimetric inequality of Theorem 1.3 in the more general context of log-concave functions.

**Theorem 2.1.** Let \( f \) be a log-concave function, \( K := \text{supp}(f), D := \text{diam}(K) \), and \( \Pi_f \) the induced probability distribution. Then for any partition \( \{ S_1, S_2, S_3 \} \) of \( K \),

\[
\Pi_f(S_3) \geq \frac{2d(S_1, S_2)}{D} \min\{\Pi_f(S_1), \Pi_f(S_2)\}.
\]

We write \( \phi_f \) for the best constant \( c > 0 \) such that

\[
\Pi_f(S_3) \geq c \cdot \min\{\Pi_f(S_1), \Pi_f(S_2)\}.
\]

By a Theorem of Kannan, Lovász and Simonovits \([1]\), we have

\[
\phi_f \geq \frac{c}{\sqrt{\text{Var}f(\|X - X\|^2)}} = \frac{c}{\sqrt{\sum \lambda_i(A)}},
\]

where \( X = \text{Var}f(\|X - X\|^2) \), \( A \) is the covariance matrix of \( \Pi_f \) and \( \lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A) \) the eigenvalues of \( A \). In the same paper, KLS prove the upper bound \( \phi_f \leq \frac{c}{\sqrt{\lambda_1(A)}} \) and conjecture that \( \phi_f \) is actually of the order \( \Theta(1/\sqrt{\lambda_1(A)}) \).

**Conjecture 2.2** (Kanan, Lovász, Simonovits, \([1]\)).

\[
\phi_f \leq \frac{c}{\sqrt{\lambda_1(A)}}.
\]
We also formulate the following (weaker) conjecture, in terms of the Frobenius norm of $A$, which we discuss at the end of these notes.

**Conjecture 2.3.**

$$\phi_f \geq \frac{c}{(\sum \lambda_i(A)^2)^{1/4}} = \frac{c}{\sqrt{\|A\|_F}}.$$  

We now proceed to the proof of our main result.

**Proof of Theorem 2.1.** Let $c = 2d(S_1, S_2)/D$ and suppose that $\{S_1, S_2, S_3\}$ is a partition of $K$ such that

$$\int_{S_3} f < c \int_{S_1} f$$

and

$$\int_{S_3} f < c \int_{S_2} f,$$

or equivalently that $\int_{\mathbb{R}^n} g > 0$ and $\int_{\mathbb{R}^n} h > 0$, where

(1) $g(x) = \begin{cases} cf(x), & \text{if } x \in S_1 \\ 0, & \text{if } x \in S_2 \end{cases}$

and $h(x) = \begin{cases} cf(x), & \text{if } x \in S_2 \\ -f(x), & \text{if } x \in S_3 \end{cases}$

To prove the theorem, we reduce the assertion to the one-dimensional case, through the following localization lemma.

**Lemma 2.4.** Let $g, h : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous integrable functions such that $\int_{\mathbb{R}^n} g > 0$ and $\int_{\mathbb{R}^n} h > 0$. Then there exist two points $a, b \in \mathbb{R}^n$ and an affine function ("needle") $l : [0, 1] \to \mathbb{R}_+$ such that

$$\int_0^1 l(t)^{n-1} g((1-t)a+tb) dt > 0$$

and

$$\int_0^1 l(t)^{n-1} h((1-t)a+tb) dt > 0.$$  

**Sketch of the Proof.** The proof of the Lemma can be roughly divided into three steps.

**Step 1.** Let $A$ be a $(n-2)$-dimensional affine subspace of $\mathbb{R}^n$. For each such
A, there is a halfspace $H$ (bisecting halfspace) with $A$ contained in its bounding hyperplane, such that

$$\int_H g = \frac{1}{2} \int_{\mathbb{R}^n} g,$$

while at the same time (replacing $H$ by its complementary halfspace, if necessary)

$$\int_H g > 0 \quad \text{and} \quad \int_H h > 0.$$

Now let $A_1, A_2, \ldots$ be a sequence of such $(n-2)$-dimensional subspaces with rational coordinates, and consider $K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots$ a respective sequence of convex bodies, where each $K_{i+1}$ is obtained from $K_i$ by cutting it into two by a bisecting hyperplane $P_i$ through $A_i$, and choosing the appropriate half. Then $L = \bigcap_{i=1}^{\infty} K_i$ is at most 1-dimensional.

**Step 2.** Without loss of generality, let $a = o$ and $b = e_1$. Define

$$Z_t = \{x \in \mathbb{R}^n : x_1 = t\}.$$

By the Brunn-Minkowski inequality, for each $i = 1, 2, \ldots$ the function

$$\psi_i = \left(\frac{\text{vol}(K_i \cap Z_t)}{\text{vol}(K_i)}\right)^{\frac{1}{n-1}}$$

is concave. Moreover, $\psi_i(t) \leq n \frac{1}{s-t}$ and for each $0 \leq s \leq t \leq 1$,

$$\frac{s}{t} \psi_i(t) \leq \psi_i(s) \leq \frac{1-s}{1-t} \psi_i(t).$$

Thus there is a limiting concave function $\psi$ such that

$$\int \psi_i(t)^{n-1} g \to \int \psi(t)^{n-1} g.$$

Since $\int \psi_i(t)^{n-1} g > 0$ for each $i = 1, 2, \ldots$, it follows that the assertion of the Lemma holds for the function $\psi$ in the place of $l$.

**Step 3.** The final step in the proof is to obtain the affine function $l$ from the concave function $\psi$ of Step 2. The technical details were not presented in class.

To conclude the proof of the Theorem, we proceed as follows: Partition $[0, 1]$ into $\{Z_1, Z_2, Z_3\}$, where

$$Z_i = \{t \in [0, 1] : (1-t)a + tb \in S_i\}, \quad i = 1, 2, 3.$$

Apply Lemma 2.4 to the functions $g, h$ of (4)\footnote{Actually the functions $g$ and $h$ as defined in (4) are not lower semi-continuous. However this can be achieved by expanding $S_1$ and $S_2$ slightly so as to make them open sets, and making the support of $f$ an open set. Since we are proving strict inequalities, we do not lose anything by these modifications.}.
Rewriting $g$ and $h$ in terms of our original function $f$ we get that
\[
\int_{Z_3} l(t)^{n-1} f((1 - t)a + tb) \, dt < c \int_{Z_1} l(t)^{n-1} f((1 - t)a + tb) \, dt
\]
and
\[
\int_{Z_3} l(t)^{n-1} f((1 - t)a + tb) \, dt < c \int_{Z_2} l(t)^{n-1} f((1 - t)a + tb) \, dt.
\]

The functions $f$ and $l(\cdot)^{n-1}$ are both log-concave, so the same holds for their product $F(t) = l(t)^{n-1} f((1 - t)a + tb)$. We will have reached a contradiction as soon as we prove that
\[
\int_{Z_3} F \geq 2d(Z_1, Z_2) \cdot \min\{ \int_{Z_1} F, \int_{Z_2} F \}.
\]
Assume at first that $Z_1, Z_3, Z_2$ are successive intervals, and write $u$ (resp. $v$) for the right (resp. left) endpoint of $Z_1$ (resp. $Z_2$), so that $d(Z_1, Z_2) = |u - v|$. Without loss of generality suppose that $F(u) \leq F(v)$. Then
\[
\int_{Z_1} F \geq F(u) \cdot |u - v| = |u - v| \cdot F(u) \cdot |1 - 0| \\
\geq |u - v| \cdot \int_{Z_1} F(u) \, dt \geq d(Z_1, Z_2) \cdot \int_{Z_1} F
\]
that proves our claim, disregarding the factor 2 (the full proof would require a little bit more of a struggle).

In the general setting, consider a maximal interval $(r, s) \subseteq Z_3$. Then, by our previous argument, the integral of $F$ over $Z_3$ is at least $c$ times the smaller of the integrals to its left $[0, r]$ and to its right $[s, 1]$. If all of $Z_1$ or $Z_2$ is contained in one of these intervals, we are done. If not, then set $U = [0, r] \cup [s, 1]$. Since $Z_1, Z_2$ are separated by at least $d(S_1, S_2)/D$, there is an interval of $Z_3$ of length $d(S_1, S_2)/D$ between $U \cap Z_1$ and $U \cap Z_2$. Repeating the process for this interval yields the required result.

**Open problem 1.** Given convex body $K \in \mathbb{R}^n$ for every distance function $d_x(y) = ||x - y||_{E_x}$ convex in $x \in K$ all hyperplane partitions are within constant of the optimum $\Psi$.

### 3 Hit-and-run

We now focus on the analysis of the hit-and-run random walk.

**Theorem 3.1.** For the hit-and-run random walk it holds $\Phi \leq \frac{c}{nD}$ staring from any point in a convex body $K$.

The following definition of the cross-ratio is used
\[
d_K(u, v) = \frac{|u - v| |p - q|}{|p - u| |v - q|} = (p : v : u : q)
\]
for the proof of the result. In what follows we list a set of results such as the localization lemma for hit-and-run without proofs. For more information see [3].
Theorem 3.2. It holds
\[ \pi_f(s_3) \geq d_k(s_1, s_2) \min\{\pi_f(s_1), \pi_f(s_2)\} \]

Lemma 3.3. For \(0 \leq u \leq w\) it holds
\[ \begin{align*}
( e^w - e^x ) ( e^w - 1 ) & \geq (v - u) w \\
( e^u - 1 ) ( e^w - e^x ) & \geq u (w - v)
\end{align*} \]

Theorem 3.4. For function \( h : K \to \mathbb{R}_+ \)
\[ h(x) \leq \frac{1}{3} \min\{1, d_K(u, v)\} \]
for all \( u \in S_1, v \in S_2 \) and \( x \in \text{chord}(u, v) \).

Finally, for a \( S_1, S_2, S_3 \) partition of \( K \) it holds \( \pi(S_3) \geq E_K(h(x)) \pi(S_1) \pi(S_2) \).

4 Open problems

Open problem 2. Find simpler algorithms to round convex bodies. For example analyse the following
walk for 2T steps in \( K \)
1. compute the covariance matrix of the trace \( X_{\cdot 2T} \) of the walk
2. if exist an eigenvalue \( \lambda > 2 \)
   then make isotropic and goto 1
   else return \( K \)

Open problem 3. Analyse the coordinate directions hit-and-run. That is instead from picking a random direction from the unit ball pick a random direction from the set \( \{-e_i, e_i\}, \forall i \in [n] \).

Open problem 4. Given a polytope \( K := \{ x : Ax \leq b \} \subset \mathbb{R}^n \) is there any deterministic approximation algorithm that computes \( (1 + \epsilon) \text{vol}(K) \) in time polynomial in \( n, 1/\epsilon \)?

Open problem 5. If the ball walk starts from \( E_K(x) \), does it mix in polynomial time?

Open problem 6. Is there any statistical tests that guarantees that the distribution \( Q_t \) is close to \( Q \)? Is the conductance \( E_{Q_t}(||X||^2) \) monotonically increasing?

References

