Expansion and anti-concentration properties of random digraphs

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Introduction and Notations

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- d-regular directed graph on n vertices ⇔ each vertex has exactly d in-neighbors and d out-neighbors.

  Adjacency matrix: "d" double stochastic matrix.

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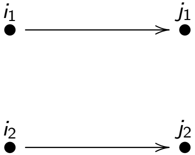
- Question: How to estimate cardinalities?

Switching method



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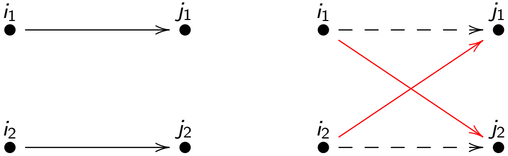

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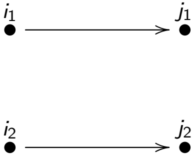

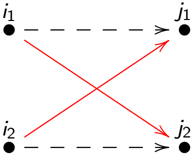

$$\begin{array}{c} \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{cc} j_1 & j_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{cc} j_1 & j_2 \end{array} \\ \begin{pmatrix} 0 & \color{red}{1} \\ \color{red}{1} & 0 \end{pmatrix} \end{array}$$

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   $H_2 \in R(\Gamma_1)$, bound $|R^{-1}(H_2)|$ from above: $|R^{-1}(H_2)| \leq \beta$.

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4. Deduce that $|\Gamma_1| \leq \frac{\beta}{\alpha}|\Gamma_2|$.

Connectivity of large sets

Theorem 1 (LLTTY'15)

There exist absolute positive constants c, C such that the following holds. Let $2 \leq d \leq n/24$ and let natural numbers ℓ and r satisfy

$$\frac{n}{4} \geq r \geq \ell \geq \frac{Cn \ln(en/r)}{d}.$$

Then

$$\mathbb{P}\left\{\bigcup\{I \not\to J\}\right\} \leq \exp\left(-\frac{cr\ell d}{n}\right),$$

where the union is taken over all $I, J \subset [n]$ with $|I| \geq \ell$ and $|J| \geq r$.

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- Undirected setting: Bollobàs'81, Mckay'87, Frieze-Luczak'92, Krivelevich-Sudakov-Vu-Wormald'01,Cooper-Frieze-Reed-Riordan'02.

Proof

- $I = [\ell]$ and $J = [r]$. Fix $2e^2 n/d \leq p \leq \ell$ and $2e^2 n/d \leq k < r$.

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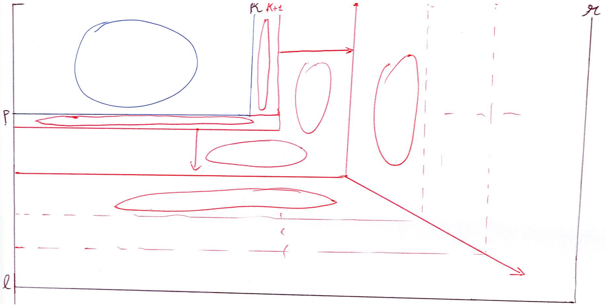
- **Lemma:** $|\{[p] \not\to [k+1]\}| \leq \exp\left(-\frac{pd}{2e^2n}\right) \cdot |\{[p] \not\to [k]\}|$



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Applying $(r - r_0)$ times

$$|\Gamma(\ell, r)| \leq \exp\left(-\frac{\ell(r - r_0)d}{2e^2 n}\right) \cdot |\Gamma(\ell, r_0)|$$
$$\leq \exp\left(-c\frac{\ell r d}{n}\right) \cdot |\mathcal{D}_{n,d}|.$$

Proof of Lemma

Suppose $p > 2d^3$, let $q \in \{1, \ldots, d\}$.

Goal: Compare size of $\Gamma_1 = \{[p] \not\to [k+1]\}$ and $\Gamma_2 = \{[p] \not\to [k]\}$.

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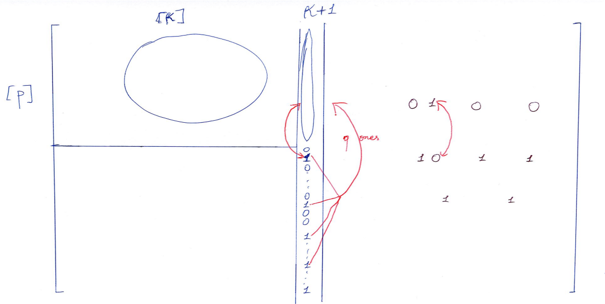
- Move q ones to rows in $[p]$ disjoint with these q rows ($\exists$ at least $p - q(d-1)^2 \geq p/2$ rows).



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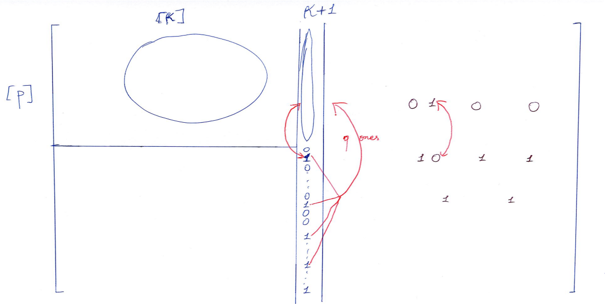

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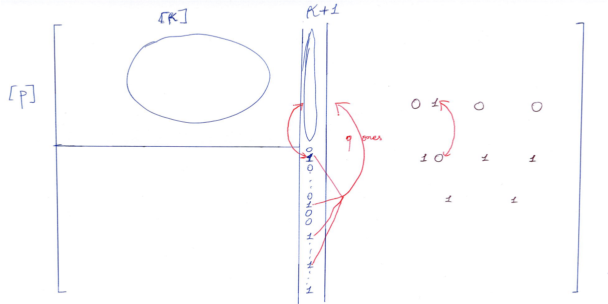

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We deduce $|\Gamma_1| \leq \left(\frac{2eqn}{pd}\right)^q |\Gamma_2|$.

Take $q := \frac{pd}{2e^2 n}$,

$$|\{[p] \not\to [k+1]\}| \leq \exp\left(-\frac{pd}{2e^2 n}\right) \cdot |\{[p] \not\to [k]\}|.$$

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  Question: Is $|\mathcal{N}_G^{in}(J)|$ concentrated around any of the extreme bounds?

Theorem 2 (LLTTY'15)

Let $8 \leq d \leq n/12$, $\varepsilon \in (0,1)$ *and* $k \leq c\varepsilon n/d$. *Then with probability* $1 - \exp\left(-c\varepsilon^2 dk \ln\left(e\varepsilon n/dk\right)\right)$ *we have*

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Remark

- *If* $i, j \in [n]$ *are two vertices. With high probability,*

$$|\mathcal{N}_G^{in}\{i,j\}| \geq 2(1-\varepsilon)d.$$

This means there are at most $2\varepsilon d$ *common in-neighbors to* i *and* j.

- *Cook'14: Concentration inequalities for codegrees when* $d \gg \log n$.

- Denote $\partial J = \mathcal{N}_G^{in}(J) \setminus J$. The u-vertex isoperimetric number

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- When $u = 1/2$, we call it the vertex isoperimetric number. Several estimates in the undirected setting: Bollobàs'88, Friedman'08, Kolesnik-Wormald'14.

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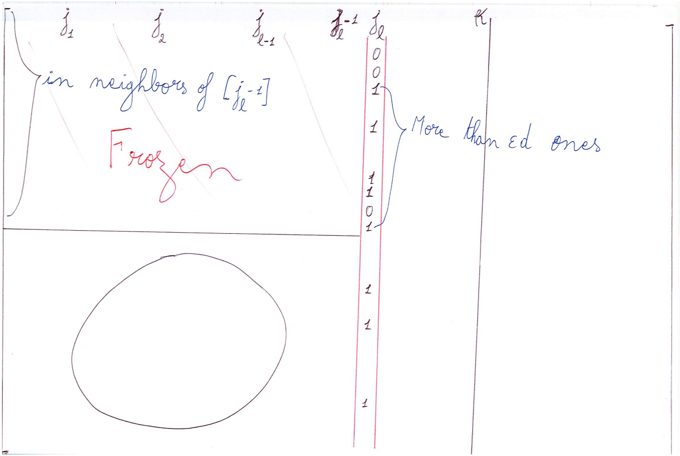
$\Rightarrow \exists j_1, \ldots, j_s$ with $s := \varepsilon k/2$, such that $\forall \ell \le s$

$$\left| C^{in}(j_\ell, [j_\ell - 1]) \right| = \left| \{i :\ i \to j_\ell \text{ and } i \to [j_\ell - 1]\} \right| \ge \varepsilon d/2.$$

Lemma: $\mathbb{P}\{|C^{in}(j_\ell, [j_\ell - 1])| \geq \varepsilon d \mid [j_\ell - 1] \text{ Frozen}\} \leq \left(\frac{e\, j_\ell}{\epsilon n}\right)^{\varepsilon d}$.



Anti-concentration

$\xi_1, \ldots, \xi_n$ iid ± 1 Bernoulli variables. Define

$$f : \mathbb{R}^n \to \mathbb{R} \quad \text{by} \quad a \mapsto \langle \xi, a \rangle = \sum_{i=1}^{n} a_i \xi_i .$$

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Theorem (Littlewood-Offord'40, Erdös'43)

For any $a \in \mathbb{R}^n$ with $|a| \geq 1$ (coordinatewise), we have

$$\sup_{c \in \mathbb{R}} \mathbb{P}\{f(a) = c\} \leq O\Big(\frac{1}{\sqrt{n}}\Big).$$

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Kleitman, Halasz, Stanley, Sarkozy-Szemeredi, Frankl-Furedi, Tao-Vu, Rudelson-Vershynin, Friedland-Sodin, Nguyen-Vu, Friedland-Giladi-Guédon, Costello, Meka-Nguyen-Vu.

- Back to our setting: fix $J \subset [n]$ of size $\leq cn/d$. Define

$$\delta_i^J = 1 \text{ if } i \to J \quad \text{and} \quad 0 \text{ otherwise.}$$

Set $\delta^J = (\delta_1^J, \ldots, \delta_n^J) \in \{0,1\}^n$.

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- Theorem 3 (LLTTY'15)

  Let $8 \leq d \leq cn$ and $I, J \subset [n]$ disjoint such that

$$|I| \leq \frac{d|J|}{8} \quad \textit{and} \quad |J| \leq c\frac{n}{d}.$$

  Let $F \subset [n] \times [n]$. For any $v \in \{0,1\}^n$,

$$p := \mathbb{P}\{\delta^J = v \mid E_G^{in}(I) = F\} \leq \exp\left(-cd|J| \ln \frac{n}{d|J|}\right)$$

Remarks

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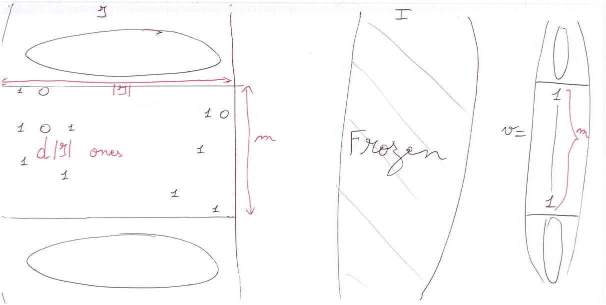
- *When* $|J| \sim n/d$, *then* δ^J *behaves like a uniformly distributed r.v on a "large" subset of the cube.*

Remarks

- $m := |\mathrm{supp}\, v| < \max(d, |J|)$ *or* $m > d|J| \Rightarrow \mathbb{P}\{\delta^J = v\} = 0.$
- *When* $|J| \sim n/d$, *then* δ^J *behaves like a uniformly distributed r.v on a "large" subset of the cube.*
- *Consider* $\tilde{\delta}^J$ *same quantity for Erdös-Renyi model.*

$$\tilde{p} = |J|^m \left(\frac{d}{n}\right)^m \binom{m|J| - m}{d|J| - m} \left(\frac{d}{n}\right)^{d|J|-m} \left(1 - \frac{d}{n}\right)^{|J|(n-d|J|)}$$



Freezing a part of the graph makes the difficulty. Take $I = \emptyset$ and $v \in \{0, 1\}^n$.

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If $m \leq d|J|/2$, then

$$\mathbb{P}\{\delta^J = v\} \leq \mathbb{P}\{|\mathcal{N}_G^{in}(J)| \leq d|J|/2\} \leq \exp\left(-cd|J| \ln\left(\frac{n}{d|J|}\right)\right).$$

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If $m \ge d|J|/2$, set $T := \{w \in \{0,1\}^n, |\mathrm{supp}\, w| = m\}$.

$$\begin{aligned}\mathbb{P}\{\delta^J = v\} &= \mathbb{P}\{\delta^J = w\} \quad \text{(for any } w \in T\text{)} \\ &\le |T|^{-1} \le \exp\left(-cd|J| \ln\left(\frac{n}{d|J|}\right)\right).\end{aligned}$$

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Problem: Can't use the permutation argument when a big part of the graph is frozen.

Solution:

- To deal with $m \leq d|J|/2$, prove the expansion of small sets under conditioning. Similar proof, more technical.

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- To deal with $m \leq d|J|/2$, prove the expansion of small sets under conditioning. Similar proof, more technical.
- When $m \geq d|J|/2$. We use the following:

Lemma

There exists $\sigma := \sigma_I \in S_n$ such that for every $2|I| \leq i_1 \leq \ldots \leq i_k$ and every $\ell \leq k := \frac{d|J|}{4}$

$$\mathbb{P}\{\delta^J_{\sigma_{i_\ell}} = 1 \mid E^{in}_G\big(\sigma\big([2|I|], i_1, \ldots, i_{\ell-1}\big), I^c\big) \text{ frozen}\} \sim \frac{d|J|}{n}.$$

Technical but based on switching. Main idea: show that if $i_\ell \to J$, then it is more likely that it happens with one edge.

Let $v \in \{0,1\}^n$ with $m \geq d|J|/2$.

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Take σ from Lemma. Since $m \geq d|J|/2$ and $2|I| \leq d|J|/4$,

$$\exists\, 2|I| \leq i_1 \leq \ldots \leq i_k, \quad v_{\sigma(i_\ell)} = 1 \quad \forall \ell \leq k := d|J|/4.$$

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Iterate the lemma

$$\mathbb{P}\{\delta^J = v \mid E_G^{in}(I) = F\} \leq \mathbb{P}\{\delta^J_{\sigma(i_\ell)} = 1\, \forall \ell \leq k \mid E_G^{in}(I) = F\}$$

$$\sim \left(\frac{d|J|}{n}\right)^k.$$

Thank you.