A discrete version of Koldobsky's slicing inequality.

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(based on a joint work with Matthew Alexander and Martin Henk)

"Convexity, probability and discrete structures, a geometric view point",
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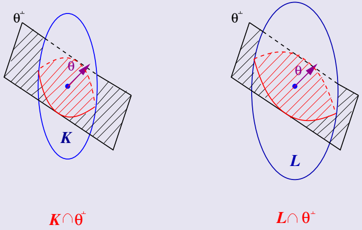
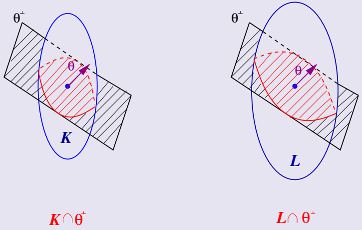


K, L origin symmetric convex bodies in $\mathbb{R}^d$. Let

$$\text{vol}_{d-1}\left(K \cap \boldsymbol{\theta}^{\perp}\right) \leq \text{vol}_{d-1}\left(L \cap \boldsymbol{\theta}^{\perp}\right), \forall \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

Does it follow that

$$\text{vol}_d(K) \leq \text{vol}_d(L)?$$



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Yes, $d\le 4$; No, $d\ge 5$.

K. Ball, J. Bourgain, R. Gardner, A.Giannopoulos, A. Koldobsky, D. Larman, E. Lutwak, M. Papadimitrakis, C. Rogers, T. Schlumprecht, G.Zhang.

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Fix $d \geq 2$. Given two convex origin-symmetric bodies K and L in $\mathbb{R}^d$ such that

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(A.Z., 2005):

The answer to the above problem is independent from the "choice" of measure and depends only on the dimension d (i.e. YES for $d \leq 4$ and NO for $d \geq 5$).

Isomorphic version of the Busemann-Petty problem

Does there exist a constant $\mathcal{L} > 0$, so that if

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then

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This question is equivalent to

Slicing Problem:

Does there exist a constant $\mathcal{L}_1$ such that for any convex symmetric body $K \subset \mathbb{R}^d$

$$\mathrm{vol}_d(K)^{\frac{d-1}{d}} \leq \mathcal{L}_1 \max_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} \mathrm{vol}_{d-1}(K \cap \boldsymbol{\theta}^{\perp})?$$

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Isomorphic version Busemann-Petty problem for arbitrary measures.

Problem:

Does there exist a constant $\mathcal{L}_2 > 0$, so that if $\mu(K \cap \theta^\perp) \le \mu(L \cap \theta^\perp)$ for every $\theta \in \mathbb{S}^{d-1}$, then $\mu(K) \le \mathcal{L}_2 \mu(L)$?

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(A. Koldobsky - A.Z, 2014)

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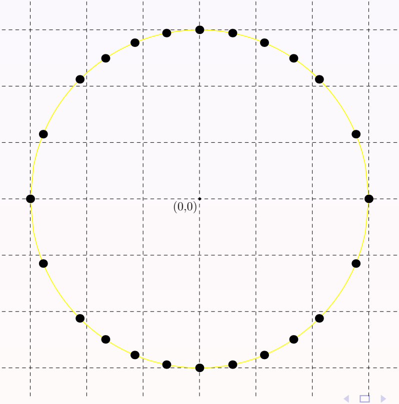
The above is not true without positive density assumption: take an even measure uniformly distributed over $2N$ points on the unit circle, then $\mathcal{L}_3$ will depend on N:



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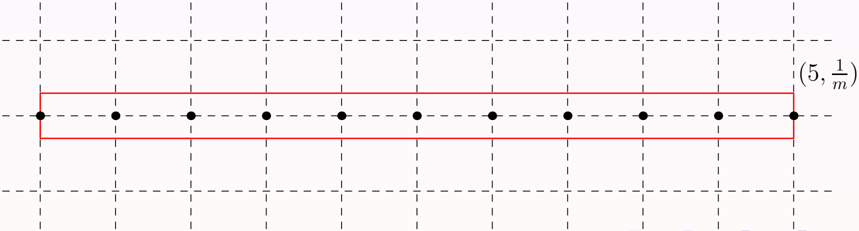
Note: we require containment of d linearly independent lattice points in order to eliminate degenerate cases:



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Does there exist an absolute constant C such that

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Let $K \subset \mathbb{R}^d$ be an origin-symmetric convex body such that $\mathrm{vol}_d(K) \geq 2^d$ then K contains at least one non-zero element of $\mathbb{Z}^d$.

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$$\mathrm{Area}(K) \leq 2\sqrt{\mathrm{Area}(K)} \cdot \#(L_{\boldsymbol{u}} \cap K) \leq 2\sqrt{\mathrm{Area}(K)} \cdot \max_{\theta \in \mathbb{S}^1} \#(K \cap \theta^{\perp}).$$

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Pick's Theorem

Let P be an integral convex polygon, then $A = I + \frac{1}{2}B - 1$ where $A = \mathrm{Area}(P)$ is the area of the polygon, I is the number of lattice points in the interior of P, and B is the number of lattice points on the boundary.

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We now have that

$$\#K = \#P \leq 2\mathrm{Area}(P) + 1 \leq 4\sqrt{\mathrm{Area}(K)} \cdot \max_{\theta \in \mathbb{S}^1} \#(K \cap \theta^\perp) + 1.$$

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Let G be an additive group, $\boldsymbol{N} = (N_1, \ldots, N_d) \in \mathbb{N}^d$ and $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d) \in G^d$. Then a generalized symmetric arithmetic progression $\mathbf{P}$ is a triplet $(\boldsymbol{N}, \boldsymbol{v}, d)$. In addition, define

$$\text{Image}(\mathbf{P}) = [-\boldsymbol{N}, \boldsymbol{N}] \cdot \boldsymbol{v} = \{n_1 \boldsymbol{v}_1 + \ldots + n_d \boldsymbol{v}_d : n_j \in [-N_j, N_j] \cap \mathbb{Z} \text{ for all } 1 \le j \le d\}.$$

The progression is called proper if the map $\boldsymbol{n} \mapsto \boldsymbol{n} \cdot \boldsymbol{v}$ is injective, $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$ is called its basis vectors, and d its rank.

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Let G be an additive group, $\boldsymbol{N} = (N_1, \ldots, N_d) \in \mathbb{N}^d$ and $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d) \in G^d$. Then a generalized symmetric arithmetic progression $\mathbf{P}$ is a triplet $(\boldsymbol{N}, \boldsymbol{v}, d)$. In addition, define

$$\mathrm{Image}(\mathbf{P}) = [-\boldsymbol{N}, \boldsymbol{N}] \cdot \boldsymbol{v} = \{n_1 \boldsymbol{v}_1 + \ldots + n_d \boldsymbol{v}_d : n_j \in [-N_j, N_j] \cap \mathbb{Z} \text{ for all } 1 \le j \le d\}.$$

The progression is called proper if the map $\boldsymbol{n} \mapsto \boldsymbol{n} \cdot \boldsymbol{v}$ is injective, $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$ is called its basis vectors, and d its rank.

Discrete F. John Theorem by T. Tao and V. Vu

Let K be a convex origin-symmetric body in $\mathbb{R}^d$. Then there exists a symmetric, proper, generalized arithmetic progression $\mathbf{P} \subset \mathbb{Z}^d$, such that $\mathrm{rank}(\mathbf{P}) \le d$ and

$$(O(d)^{-3d/2} K) \cap \mathbb{Z}^d \subset Image(\mathbf{P}) \subset K \cap \mathbb{Z}^d,$$

in addition

$$O(d)^{-7d/2} \# K \le \# \mathbf{P}.$$

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First estimate (Yes, the constant is independent of the body):

Let $K \subset \mathbb{R}^d$ be an o-symmetric convex body containing d linearly independent lattice points. Then

$$\# K \le O(d)^{7d/2} \max_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} \left(\#(K \cap \boldsymbol{\theta}^{\perp}) \right) \mathrm{vol}_d(K)^{\frac{1}{d}}.$$

What about $\mathbb{Z}^d$ or try harder and do better.

Main Theorem.

Let $K \subset \mathbb{R}^d$ be an o-symmetric convex body containing d linearly independent lattice points. Then

$$\#K \leq O(1)^d \max_{\theta \in \mathbb{S}^{d-1}} \left(\#(K \cap \theta^{\perp})\right) \mathrm{vol}_d(K)^{\frac{1}{d}}.$$

Main Idea:

Let Γ be a lattice in $\mathbb{R}^d$ and K be a convex body in $\mathbb{R}^d$. Define the successive minima to be

$$\lambda_j = \lambda_j(K, \Gamma) = \min\{\lambda > 0 : \lambda \cdot K \text{ contains } j \text{ linearly independent elements of } \Gamma\}.$$

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Minkowski Second Theorem

Let Γ be a lattice in $\mathbb{R}^d$, K be an origin-symmetric convex body with successive minima λ_i then there exist a set of linearly independent vectors from Γ, $\boldsymbol{v}_1, \dots, \boldsymbol{v}_d$, such that $\boldsymbol{v}_i$ lies on the boundary of $\lambda_i \cdot K$ but the interior of $\lambda_i \cdot K$ does not contain any vectors outside the span of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}$. These vectors are called a directional basis. Moreover,

$$\frac{1}{d!}\prod_{i=1}^{d}\frac{2}{\lambda_i} \leq \frac{\text{vol}(K)}{\det(\Gamma)} \leq \prod_{i=1}^{d}\frac{2}{\lambda_i}.$$

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Note that the vectors $\boldsymbol{v}_1, \dots \boldsymbol{v}_d$ from Minkowski's Second theorem may not necessarily form a basis of Γ.

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$K \subset \mathbb{R}^d$ is **unconditional** if $(\pm x_1, \pm x_2, \dots, \pm x_d) \in K$ for all $x \in K$.

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If $\lambda_d > 1$ then all integer points of K are contained in $K \cap \boldsymbol{e}_d{}^\perp$. Thus, $\lambda_d \le 1$ and, using $\lambda_d \ge \lambda_i$, for all $i = 1, \ldots, d$, we get

$$2\left\lfloor \frac{1}{\lambda_d} \right\rfloor + 1 \le \frac{3}{\lambda_d} \le O(d) \left(\frac{1}{d!} \prod_{i=1}^d \frac{2}{\lambda_i}\right)^{1/d}.$$

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Finally we use Minkowski second theorem to finish the proof:

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Note that for

$$B_1^d = \{\boldsymbol{x} \in \mathbb{R}^d : \sum |x_i| \leq 1\},$$

we have $\#B_1^d = 2d + 1$, $\#(K \cap \boldsymbol{e}_i^{\perp}) = 2(d-1) + 1$ and $\mathrm{vol}_d(K)^{\frac{1}{d}} = O(d)^{-1}$.

A discrete analog of Brunn's theorem?

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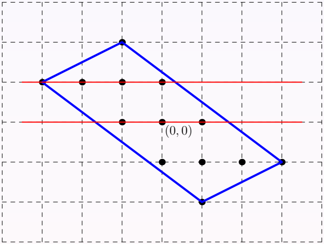

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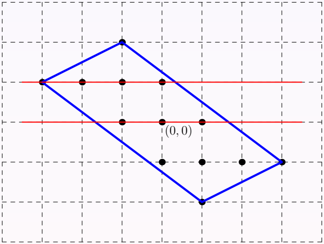

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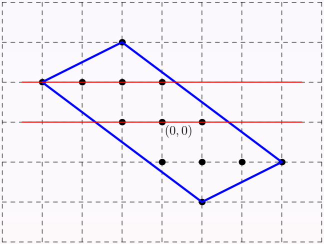

So, there is no direct equivalent of Brunn's theorem in this setting. But...:

A discrete version of the Brunn's theorem

Consider a convex, origin-symmetric body $K \subset \mathbb{R}^d$ and a lattice $\Gamma \subset \mathbb{R}^d$, then

$$\#(K \cap \boldsymbol{\theta}^{\perp} \cap \Gamma) \geq 9^{-(d-1)} \#(K \cap (\boldsymbol{\theta}^{\perp} + t\boldsymbol{\theta}) \cap \Gamma), \text{ for all } t \in \mathbb{R}.$$

Proof of the main result:

We would like to estimate the number of points in $K \cap \mathbb{Z}^d$ using the number of points from $\mathbb{Z}^d$ in a central hyperplane section of K. Our goal is to find a direction for which the lattice width of K is small enough and use the discrete version of Brunn's theorem.

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Let $\lambda_i^* = \lambda_i(K^*, \mathbb{Z}^d)$ be the successive minima of the polar body

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Notice that the minimal lattice width of our body K is $2\lambda_1^*$. Indeed,

$$||\boldsymbol{z}||_{K^*} = \min\{t \in \mathbb{R} : \boldsymbol{z} \in tK^*\} \geq \lambda_1^*,$$

for any $\boldsymbol{z} \in \mathbb{Z}^d \setminus \{0\}$ and there exists a non-zero lattice point $\boldsymbol{w}$ with $||\boldsymbol{w}||_{K^*} = \lambda_1^*$.

We would like to estimate the number of points in $K \cap \mathbb{Z}^d$ using the number of points from $\mathbb{Z}^d$ in a central hyperplane section of K. Our goal is to find a direction for which the lattice width of K is small enough and use the discrete version of Brunn's theorem.

For $\boldsymbol{u} \in \mathbb{Z}^d \setminus \{0\}$ define **the lattice width** of K in the direction $\boldsymbol{u}$ as $2 \max_{x \in K} \boldsymbol{x} \cdot \boldsymbol{u}$.

Let $\lambda_i^* = \lambda_i(K^*, \mathbb{Z}^d)$ be the successive minima of the polar body

$$K^* = \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x} \cdot \boldsymbol{y}| \leq 1, \quad \forall \boldsymbol{y} \in K\}.$$

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Proof of the main result:

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Proof of the main result:

Next we notice that by Minkowski Second Theorem we get

$$\lambda_1^* \leq \left(\prod_{i=1}^{d} \lambda_i^* \right)^{\frac{1}{d}} \leq 2\mathrm{vol}_d(K^*)^{-\frac{1}{d}}$$

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\end{aligned}$$

Most General Statement

Let $\mathrm{G}_{\mathbb{Z}}(i,d)$ be the set of all i-dimensional linear subspaces containing i linearly independent lattice vectors of $\mathbb{Z}^d$, i.e., the set of all i-dimensional lattice hyperplanes.

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Theorem (M. Alexander, M. Henk, A.Z., 2015+)

Let $K \subset \mathbb{R}^d$ be an o-symmetric convex body containing d linearly independent lattice points. Then

$$\#K \leq O(1)^d \, d^{d-m} \max\{\#(K \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\} \, \mathrm{vol}_d(K)^{\frac{d-m}{d}}.$$