Finite volume schemes for the approximation of PDMP

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measures $\mu(t)(i,x)$, marginal law of env. variable with $\boxed{\text{initial law } \mu_0(i,x)}$

$$\boxed{\sum_{i\in E}\int_{\mathbb{R}^d} \mathrm{d}\mu(s)(i,x) = 1}$$

E finite set of modes
environ. variables $\Phi(i,t,s,x) \in \mathbb{R}^d$ at time $t > s$ if $x \in \mathbb{R}^d$ at time s

Flow property $\boxed{\Phi(i,s_3,s_2,\Phi(s_2,s_1,x)) = \Phi(i,s_3,s_1,x), \quad \text{for all } 0 \le s_1 \le s_2 \le s_3,\ x}$

transition rate $\lambda(i,j,x)$ from state $(i,x) \to j$

prob. law $m(i,j,x)$ for state (j,x)

solution to (set $\Phi(t,s,x)$ in CSMP, see C. Cocozza-Thivent)

$$\sum_{i\in E}\int_{\mathbb{R}^d} f(i,x)\mathrm{d}\mu(t)(i,x) = \sum_{i\in E}\int_{\mathbb{R}^d} f(i,\Phi(i,t,t-\delta t,x))\mathrm{d}\mu_{t-\delta t}(i,x)$$
$$+\sum_{i,j\in E}\int_{t-\delta t}^{t}\int_{\mathbb{R}^d}\lambda(i,j,x)\left(\int_{\mathbb{R}^d} f(j,\Phi(j,t,s,y))\mathrm{d}m(i,j,x)(y) - f(i,\Phi(i,t,s,x))\right)\mathrm{d}\mu(s)(i,x)\,\mathrm{d}s$$

mesh $\mathcal{M}$ time step δt

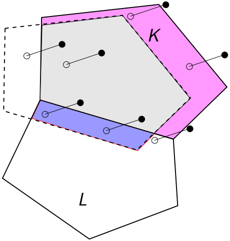
$$|K|\ u^0_{i,K} = \int_K \mu_0(i, dx),\ \forall K \in \mathcal{M},\ \forall i \in E$$

1. Transport step

$$v^{(i)}_{L,K} = |\{y \in L \text{ s.t. } \Phi\left(i, (n+1)\delta t, n\delta t, y\right) \in K\}|$$

$$\longrightarrow$$

$$|K|\,\widetilde{u}^n_{i,K} = \sum_{L\in\mathcal{M}} v^{(i)}_{L,K}\, u^n_{i,L} \text{ with } \sum_{K\in\mathcal{M}} v^{(i)}_{L,K} = |L|$$



2. Jump step

$$|L|\,\lambda^{(ij)}_{L,K} \quad = \int_L \lambda(j,i,y) \int_K \mathrm{d}m(j,i,y)(x)dy \text{ and } |K|\,\lambda^{(i)}_K \quad = \sum_{j\in E}\int_K \lambda(i,j,x)\mathrm{d}x$$

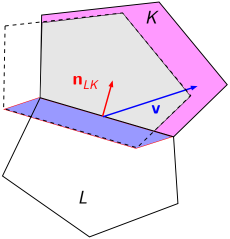
$$|K|\ u^{n+1}_{i,K} = \frac{1}{\delta t\,\lambda^{(i)}_K + 1}\,|K|\,\widetilde{u}^n_{i,K} + \delta t \sum_{j\in E}\sum_{L\in\mathcal{M}} \frac{\lambda^{(ij)}_{L,K}}{\delta t\,\lambda^{(j)}_L + 1}\,|L|\,\widetilde{u}^n_{j,L}$$

Flow given by regular O.D.E.

$$\frac{d\Phi}{dt} = \mathbf{v}(i,t,\Phi)$$
as long as $I(t) = i \in E$

with

$\mathbf{v}(i,\cdot,\cdot)$ Lipschitz continuous
$\longrightarrow$ Markov Flow property
$\Phi(i,s_3,s_2,\Phi(s_2,s_1,x)) = \Phi(i,s_3,s_1,x)$



test function in Chapman-Kolmogorov equation

$$f(i,x,s) = g(i,\Phi(i,t,s,x))$$

gives general formulation

Then

$$v^{(i)}_{L,K} = |\{y \in L \text{ s.t. } \Phi(i,(n+1)\delta t, n\delta t, y) \in K\}| \simeq \delta t \int_{\overline{K}\cap\overline{L}} (\mathbf{v}(i,s,x)\cdot \mathbf{n}_{LK})^+ \mathrm{d}s(x), \ \forall L \in \mathcal{N}_K$$

$$v^{(i)}_{K,K} = |\{y \in K \text{ s.t. } \Phi(i,(n+1)\delta t, n\delta t, y) \in K\}| \simeq |K| - \delta t \sum_{L\in\mathcal{N}_K} \int_{\overline{K}\cap\overline{L}} (\mathbf{v}(i,s,x)\cdot \mathbf{n}_{KL})^+ \mathrm{d}s(x)$$

$\longrightarrow$

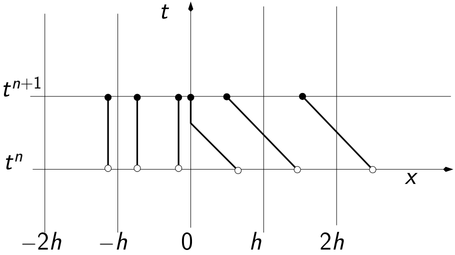
$$|K|(\tilde{u}^n_{i,K} - u^n_{i,K}) + \delta t \sum_{L\in\mathcal{N}_K} \int_{\overline{K}\cap\overline{L}} \left((\mathbf{v}(i,s,x)\cdot \mathbf{n}_{KL})^+ u^n_{i,K} - (\mathbf{v}(i,s,x)\cdot \mathbf{n}_{LK})^+ u^n_{i,L}\right) \mathrm{d}s(x) = 0$$

$\delta t = h$

$$\begin{aligned}
\Phi(s,t,x) &= x-(s-t) && \text{for all } s \geq t \text{ and } x \geq s-t,\\
\Phi(s,t,x) &= 0 && \text{for all } s \geq t \text{ and } 0 \leq x \leq s-t,\\
\Phi(s,t,x) &= x && \text{for all } s \geq t \text{ and } x \leq 0
\end{aligned}$$

initial data : uniform distribution on $[0,1]$

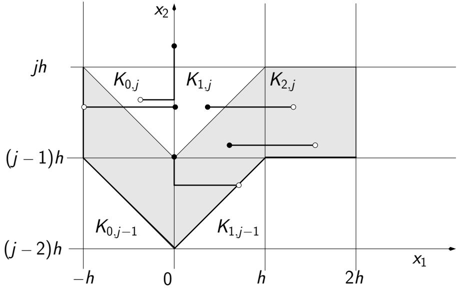
$t \in (0,1)$	$t \in [1,+\infty)$
$\mu_t = t\delta_0(x) + \mathbf{1}_{x\in(0,1-t)}\mathrm{d}x$	$\mu_t = \delta_0$



$$\begin{aligned}
u^{n+1}_{[0,h)} &= u^n_{[0,h)} + u^n_{[h,2h)} && \\
u^{n+1}_{[ih,(i+1)h)} &= u^n_{[(i+1)h,(i+2)h)} && i \geq 1\\
u^{n+1}_{[ih,(i+1)h)} &= u^n_{[ih,(i+1)h)} && i < 0
\end{aligned}$$

$$\Phi_\lambda(s,t,x_1,x_2) = \Big(\mathrm{sgn}(x_1)\max(0,|x_1|-(s-t)),\quad x_2+\lambda\max(0,s-t-|x_1|)\Big),$$
$$0\le t\le s,\ \ x_1,x_2\in\mathbb{R},$$

$\delta t = h\ \lambda = 1$



$$u_{ij}^{n+1} = \begin{cases} u_{i+1,j}^n & \text{if } i\ge 2,\\ u_{i-1,j}^n & \text{if } i\le 0,\end{cases}$$

and

$$u_{1,j}^{n+1} = \frac{1}{2}\left(u_{1,j-1}^n + u_{1,j}^n\right) + u_{2,j}^n + \frac{1}{2}\left(u_{0,j-1}^n + u_{0,j}^n\right)$$

when only non zero values are for $i=1$

$$u_{1,j}^{n+1} = \frac{1}{2}\left(u_{1,j-1}^n + u_{1,j}^n\right) = u_{1,j}^n - \frac{1}{2}\frac{\delta t}{h}\left(u_{1,j}^n - u_{1,j-1}^n\right),$$

convergence to the transport by the flow $\Phi_{1/2}$ $(\lambda = 1/2)$

convergence needs $\delta t\to 0$ and $h/\delta t\to 0$

we focus on the transport part

1/ forall $g \in \mathrm{Lip}_c(\mathbb{R}^N)$ Lipschitz continuous function with compact support

$$\left| \int_{\mathbb{R}^N} g(x) u^{n+1}_{\mathcal{M}}(x)\mathrm{d}x - \int_{\mathbb{R}^N} g(\Phi(t_{n+1}, t_n, x)) u^n_{\mathcal{M}}(x)\mathrm{d}x \right| \leq M_0 \,\mathrm{Lip}(g) h_{\mathcal{M}}$$

2/ $f \in \mathrm{Lip}_c(\mathbb{R}^N)$ Lipschitz continuous function with compact support

$$\left| \int_{\mathbb{R}^N} f(\Phi(t_{n_t+1}, t_{n+1}, x)) u^{n+1}_{\mathcal{M}}(x)\mathrm{d}x - \int_{\mathbb{R}^N} f(\Phi(t_{n_t+1}, t_n, x)) u^n_{\mathcal{M}}(x)\mathrm{d}x \right| \leq M_0 \,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f) h_{\mathcal{M}}$$

3/

$$\left| \int_{\mathbb{R}^N} f(x) u^{n_t+1}_{\mathcal{M}}(x)\mathrm{d}x - \int_{\mathbb{R}^N} f(\Phi(t_{n_t+1}, 0, x)) u^0_{\mathcal{M}}(x)\mathrm{d}x \right| \leq (n_t+1) M_0 \,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f) h_{\mathcal{M}}$$

4/ for $t \in]t_{n_t}, t_{n_t+1}]$,

$$\left| \int_{\mathbb{R}^N} f(\Phi(t_{n_t+1}, 0, x)) u^0_{\mathcal{M}}(x)\mathrm{d}x - \int_{\mathbb{R}^N} f(\Phi(t, 0, x))\mathrm{d}\mu_0(x) \right| \leq M_0 \,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f) (h_{\mathcal{M}} + \delta t)$$

5/

$$\left| \int_{\mathbb{R}^N} f(x) u_{\mathcal{M},\delta t}(t, x)\mathrm{d}x - \int_{\mathbb{R}^N} f(\Phi(t, 0, x))\mathrm{d}\mu_0(x) \right| \leq M_0 \,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f) \left(2T \frac{h_{\mathcal{M}}}{\delta t} + h_{\mathcal{M}} + \delta t \right)$$

control of Wasserstein distance $W_1(u_{\mathcal{M},\delta t}(t, \cdot)\mathrm{d}x - \mu_t)$

additional work for tightness in the case of jumps

$\varepsilon > 0$

$$\widehat{v}_{L,K}^n = v_{L,K}^n + \varepsilon\ \delta t|\sigma_{KL}|$$

$$\widehat{v}_{K,K}^n = v_{K,K}^n - \varepsilon\ \delta t \sum_{L\in\mathcal{M}\setminus\{K\}} |\sigma_{KL}|$$

and

$$|K|\ \widetilde{u}_K^{n+1} = \sum_{L\in\mathcal{M}} \widehat{v}_{L,K}^n\ u_L^n$$

instead of

$$|K|\ \widetilde{u}_K^{n+1} = \sum_{L\in\mathcal{M}} v_{L,K}^n\ u_L^n$$

Convergence study with $\delta t \to 0$ and $h/\delta t \leq C$ (CFL cond. + inv. CFL cond.) under some hypotheses on Φ

$$\left|\int_{\mathbb{R}^N} f(x)u_{\mathcal{M},\delta t}(t,x)\mathrm{d}x - \int_{\mathbb{R}^N} f(\Phi(t,0,x))\mathrm{d}\mu_0(x)\right|$$
$$\leq M_0\,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f)\left((A_1 + A_2)\frac{h_{\mathcal{M}}}{\delta t} + h_{\mathcal{M}} + \delta t\right)$$

with

$$A_1 = h_{\mathcal{M}} \sum_{n=0}^{n_t} \sum_{L\in\mathcal{M}} \sum_{K\in\mathcal{M}} \widehat{v}_{L,K}^n |u_L^n - u_K^n|$$

and

$$A_2 = h_{\mathcal{M}} \sum_{n=0}^{n_t} \sum_{K\in\mathcal{M}} u_K^n \int_{\{x,X(t_{n+1},t_n,x)\in K\}} |J(t_{n+1},t_n,x) - 1|\mathrm{d}x$$

needs of some control on space variations of u for passing to the limit

Main idea :

$$\partial_t u + \mathrm{div}\,(u\mathbf{v}) - \varepsilon\Delta u = 0,\ \ u(\cdot,0) = \mu_0$$

Multiplication by u gives $$\partial_t \int_{\mathbb{R}^d} \left(\frac{1}{2}u^2 + u^2\mathrm{div}\,(\mathbf{v}) + \varepsilon|\nabla u|^2\right) = 0$$

Problem : $\int_{\mathbb{R}^d} u^2 < \infty$ false with measures

method equivalent to multiplication by

$$\phi_m(u) = 1 - \frac{1}{(1+|u|)^{1+m}} \quad \text{with } m>0 \text{ control of concentration}$$

Then

$$\sum_{n=0}^{n_T-1} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} \widehat{v}_{L,K}^n\ \mathrm{d}(u_K^{n+1}, u_L^n, \tfrac{m+1}{2})^2 \le C$$

where

$$\begin{aligned} \mathrm{d}(x,y,\theta) &= \tfrac{|y-x|}{\max(|x|,|y|)^\theta} \\ \mathrm{d}(0,0,\theta) &= 0 \end{aligned}$$

$$A_1 \le C(h_{\mathcal{M}})^{r_1} \quad \text{and} \quad A_2 \le C(h_{\mathcal{M}})^{r_2}$$

Note : $1/\varepsilon$ in RHS of ineq.
Main tool : discrete Sobolev-Gagliardo-Nirenberg inequalities

pass to the limit $\delta t \to 0$ and $h/\delta t \le C$ in scheme considering regular test function

Explicit scheme : more precise - but asymptotic state out of reach

time step less expensive but much smaller

Implicit scheme : allows determination of asymptotic states (using O.D.E.)

resolution of large linear systems using BiCGSTAB and SOR preconditionner

Simplified example of population dynamics

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}(x,t) + \mathrm{div}(\mu(x,t)\mathbf{v}(x)) = \int_{\mathbb{R}^N} \lambda(y)\mathrm{d}m(y,x)\mu(y,t) - \lambda(x)\mu(x,t)$$

with the initial condition

$\mu(x,0) = \mu_{\mathrm{ini}}(x)$, for $x \in \mathbb{R}^N$.

$$v_{K,L} = \frac{1}{|\overline{K}\cap\overline{L}|}\int_{\overline{K}\cap\overline{L}} \mathbf{v}(x)\cdot\mathbf{n}_{KL}\mathrm{d}s(x)$$

and

$$w_{K,L} = \max(|v_{K,L}|,\varepsilon)$$

$$\lambda_{K,L} = \frac{1}{|K|}\int_K \lambda(x)\left(\int_L \mathrm{d}m(x)(\mathrm{d}y)\right)\mathrm{d}x$$

$$\lambda_K = \sum_{L\in\mathcal{M}} \lambda_{K,L} = \frac{1}{|K|}\int_K \lambda(x)\,\mathrm{d}x$$

$$u_K^{(0)} = \frac{1}{|K|}\int_K \mathrm{d}\mu_0(x)$$

and

$$\mathrm{m}(K)(u_K^{(n+1)} - u_K^{(n)}) + \delta t \sum_{L\in\mathcal{N}_K} |\overline{K}\cap\overline{L}| \left(v_{K,L}\frac{u_K^{(n+1)} + u_L^{(n+1)}}{2} + \frac{w_{K,L}}{2}(u_K^{(n+1)} - u_L^{(n+1)})\right) = -\delta t|K|\,\lambda_K u_K^{(n+1)} + \delta t \sum_{L\in\mathcal{M}} |L|\,\lambda_{L,K} u_L^{(n+1)}$$

conservation of probability mass $\longrightarrow$ weak convergence to Radon measure

multiplication by $\boxed{\phi_m(u) = 1 - \dfrac{1}{(1+|u|)^m} \text{with } m > 1}$

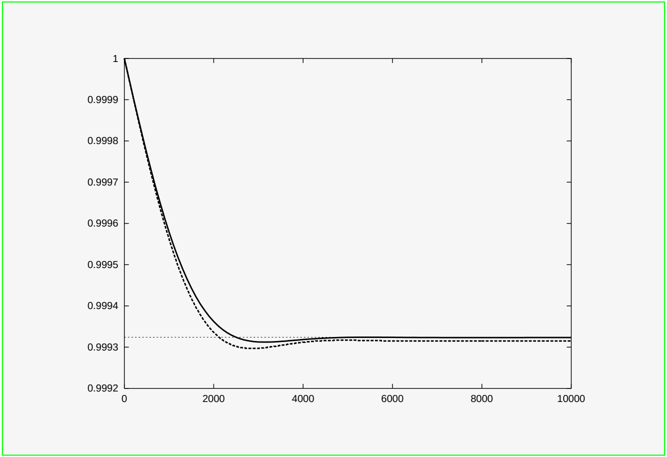
proof that
$$\boxed{\sum_{n=0}^{n_T-1} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} |\overline{K}\cap\overline{L}| \frac{(u_K^{(n+1)} - u_L^{(n+1)})^2}{(1+\max(u_K^{(n+1)}, u_L^{(n+1)}))^{m+1}} \leq C}$$

$$\boxed{\sum_{n=0}^{n_T-1} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} |\overline{K}\cap\overline{L}|\, |u_K^{(n+1)} - u_L^{(n+1)}| \leq C\, h_{\mathcal{M}}^{-1/q}}$$

$1/\varepsilon$ in RHS of ineq.
Main tool : discrete Sobolev inequalities

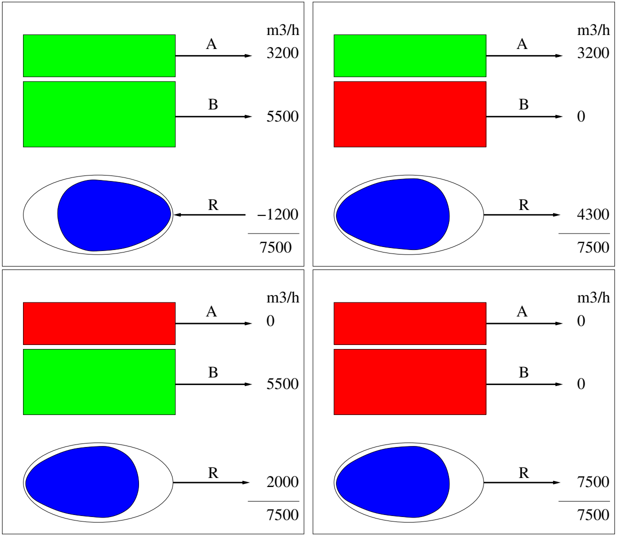
pass to the limit $\delta t \to 0$ and $h/\delta t \leq C$ in scheme considering regular test function

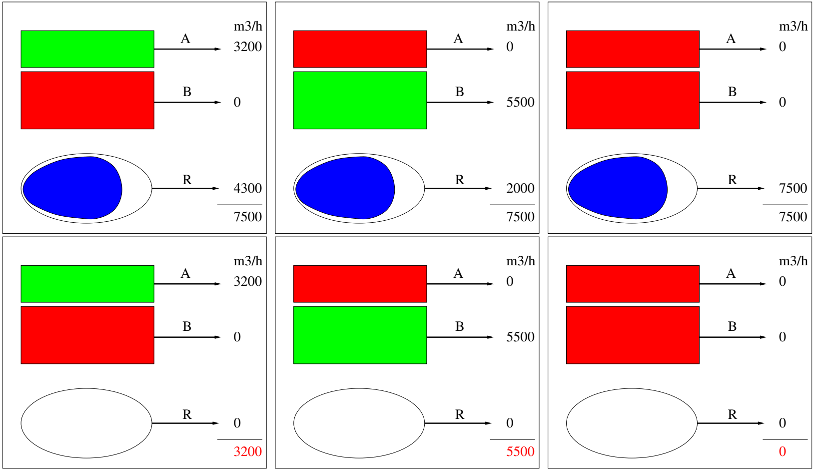
$E = \{1, 0\}$, Weibull law for working duration with average value 2000 hours, lognormal law for repair duration with average value 1.5 hours,



implicit scheme (solid line)
explicit scheme (dotted line)

Numerical comparison on a second example : the gas factory



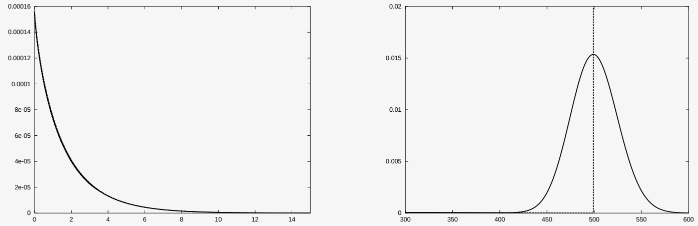

m3/h
A
3200
B
0
R
4300
7500
m3/h
A
0
B
5500
R
2000
7500
m3/h
A
0
B
0
R
7500
7500
m3/h
A
3200
B
0
R
0
3200
m3/h
A
0
B
5500
R
0
5500
m3/h
A
0
B
0
R
0
0

Monte Carlo simulation :

initialization at $t = 0$ in state $(1,1)$ $x_A = x_B = 0$, $x_R = R$
in each state, 2 random durations (w.r.t. to state)
in state $(1,1)$, compute duration until $x_R = R$
go to next jump, compute env. var.

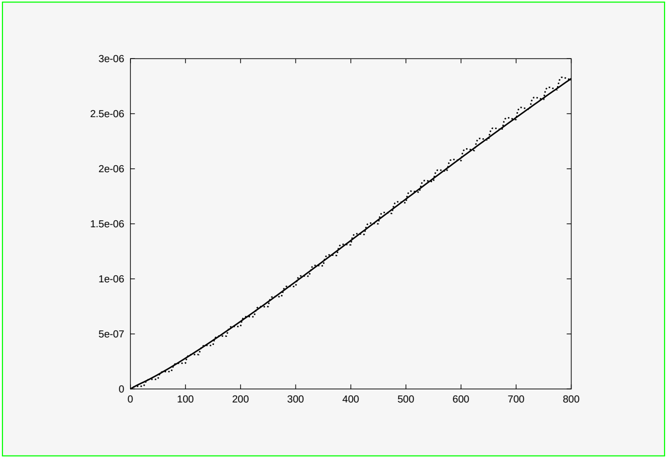
10^7 simulations until $t = 10^5$ hours

histogram of final state and env. var.
on one processor, computing time : 5 hours



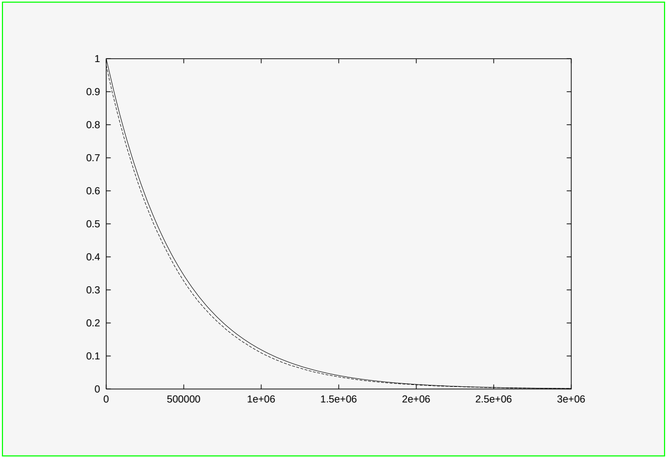
Marginal distributions at $t = 500$
repair (left) working (right).

$E = \{(1,1),(1,0),(0,1),(0,0)\}$



Probability of failure state $(0,0)$ w.r.t. to time
exp. scheme (dotted line) imp. scheme (solid line)

comparison only on short times...



solid line : imp. scheme
dotted line : Monte Carlo

FV finite volumes
MC Monte Carlo
PN Petri Networks

	FV	MC	PN
Case 1	0.9989952	0.9991	0.9991
Case 2	0.9896885	0.9907	0.9909
Case 3	0.9882614	0.9894	0.9892
Case 4	0.9731820	0.9772	0.9744

availability of contractual rate

	FV	MC	PN
Case 1	0.9995241	0.9996	0.9996
Case 2	0.9950731	0.9955	0.9956
Case 3	0.9939157	0.9945	0.9942
Case 4	0.9860841	0.9881	0.9867

Availability of the two production units

	FV	MC	PN
Case 1	0.0011344	0.001039	0.001024
Case 2	0.11396	0.1009	0.10013
Case 3	1.2059	1.0714	1.1150
Case 4	2.7053	2.2614	2.5837

Frequency of total production loss (in years^{-1})

Case 1	R immediately filled in state $(1, 1)$
Case 2	as Case 1, failure rates$\times$ 10
Case 3	as Case 2, failure rates$\times$ 10 if other unit down
Case 4	R to be filled, failure rates as in Case 3 if not full

Finite volume scheme may be cheap and accurate in some cases, but not all

Convergence analysis connected with numerical analysis of

scalar hyperbolic nonlinear equations

scalar parabolic equations with irregular data

discrete Sobolev - Gagliardo - Nirenberg inequalities

environmental variables :
working durations $x_A \in \mathbb{R}_+$, $x_B \in \mathbb{R}_+$, level of gas in the reservoir $x_R \in [0, R]$ with $R = 220000\ m^3$

If $\boxed{x_R < R}$

$\frac{dx_A}{dt} = 1,\ \frac{dx_B}{dt} = 1,\ \frac{dx_R}{dt} = 1200\ m^3.h^{-1}$
failure rate of A : $\lambda'_A = 1/200\ h^{-1}$ and failure rate of B : $\lambda'_B = 1/40\ h^{-1}$

otherwise, if $\boxed{x_R = R}$

$\frac{dx_A}{dt} = 1,\ \frac{dx_B}{dt} = 1,\ \frac{dx_R}{dt} = 0$
failure rate of A : $\lambda_A = 1/2000\ h^{-1}$ failure rate of B : $\lambda_B = 1/400\ h^{-1}$

failure of A : pass to state $(0, 1)$
failure of B : pass to state $(1, 0)$

environmental variables :
repair duration $x_A \in \mathbb{R}_+$, working duration $x_B \in \mathbb{R}_+$, level of gas in the reservoir $x_R \in [0, R]$

$$\frac{dx_A}{dt} = 1,\ \frac{dx_B}{dt} = 1$$

si $x_R > 0$, $\frac{dx_R}{dt} = -2000\ m^3.h^{-1}$

repair rate of A : $\mu_A(x_A)\ h^{-1}$

failure rate of B : $\lambda'_B = 1/40\ h^{-1}$

$$\mu_A(x) = d_A(x) / \int_x^{+\infty} d_A(s)\mathrm{d}s,\ d_A(x) = \mathcal{LN}\,(0.23, 2.25, x)$$

$$\mathcal{LN}(m, \sigma, s) = \frac{1}{s\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(s) - m}{\sigma}\right)^2\right)$$

repair of A : pass to state $(1, 1)$
failure of B : pass to state $(0, 0)$

environmental variables :
working duration $x_A \in \mathbb{R}_+$, repair duration $x_B \in \mathbb{R}_+$, level of gas in the reservoir $x_R \in [0, R]$

$\frac{dx_A}{dt} = 1$, $\frac{dx_B}{dt} = 1$
if $x_R > 0$, $\frac{dx_R}{dt} = -4300\ m^3.h^{-1}$
failure rate of A : $\lambda'_A = 1/200\ h^{-1}$
repair rate of B : $\mu_B(x_B)\ h^{-1}$

$$\mu_B(x) = d_B(x) / \int_x^{+\infty} d_B(s)\mathrm{d}s,\ d_B(x) = \mathcal{LN}\,(0.50, 1.83, x)$$

failure of A : pass to state $(0, 0)$
repair of B : pass to state $(1, 1)$

environmental variables :

repair durations $x_A \in \mathbb{R}_+$ and $x_B \in \mathbb{R}_+$, level of gas in the reservoir $x_R \in [0, R]$

$\frac{dx_A}{dt} = 1$, $\frac{dx_B}{dt} = 1$

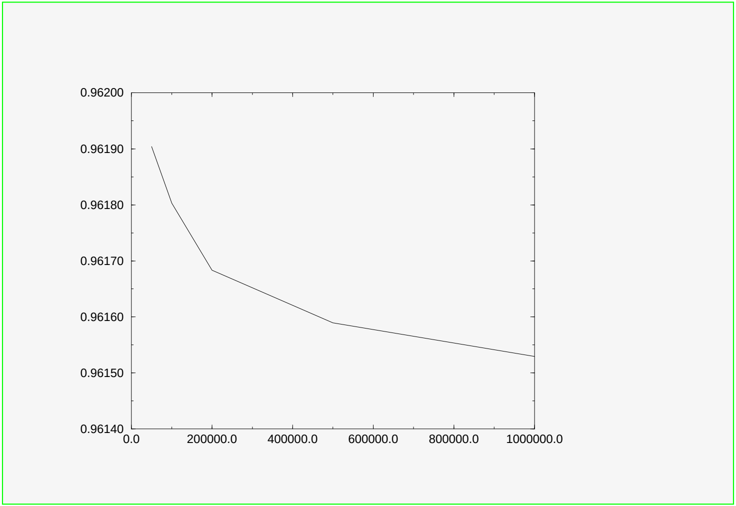
if $x_R > 0$, $\frac{dx_R}{dt} = -7500\ m^3.h^{-1}$

repair rate of A : $\mu_A(x_A)\ h^{-1}$

repair rate of B : $\mu_B(x_B)\ h^{-1}$

repair of A : pass to state $(1, 0)$

repair of B : pass to state $(0, 1)$



availibility of contractual rate
different meshes 500x100 1000x100 1000x200 1000x500 1000x1000