Un modèle PDMP multi-classes pour le contrôle de congestion implémenté par les connexions dans un grand réseau

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1. **Interacting multi-class transmissions in large stochastic networks (AAP 2009)**
2. **Self-adaptive congestion control for multi-class intermittent connections in a communication network (QUES 2011)**

et pour une étude poussée sur les lois invariantes pour 1.,

Maaike Verloop, CWI Amsterdam

3. **Stability properties of networks with interacting TCP flows (Proceedings of NET-COOP 2009)**

1 Modeling the Internet

The Internet is a vast, complex, and ill-known structure
in perpetual evolution.

Undisciplined users

access it for varied purposes
and have vastly different characteristics
and QoS (quality of service) requirements.

Connections (initiated by **users**) have to

self-adapt

to the **packet losses** due to the **congestion** they **create**

by **regulating** their **output**

using **algorithms** such as the

congestion control part of **TCP**

(**T**ransmission **C**ontrol **P**rotocol).

It is important to understand better this

feedback loop.

Highly nonlinear behavior is expected.

The situation is a fairly well understood for a

single node

with asymptotic formulæ for throughput, etc., even if

many problems remain open

depending on the level of precision and detail sought

(delayed information . . .)

Much less

is understood about the true issue,

Stochastic Networks,

***i.e.*, the coexistence, and hence the interaction,**

- of **varied data flows**

  (file transfers, telephony, video streaming, web browsing, MMORPG, P2P, . . .)

- with **varied characteristics**

  (routes, QoS requirements, rates, . . .)

each **using** in its **own fashion**

limited common resources

- constituted of **varied nodes**

  (links, routers, processors, buffers, . . .)

- with **varied purposes**.

It is natural to regroup data flows into a

reasonable number of classes,

according to their characteristics, and we focus on such

interacting multi-class network models.

One approach:

Introduced and studied by several distinguished authors, *e.g.*,

Kelly, Maulloo, and Tan (JORS 1998)

Massoulié and Roberts (INFOCOM 1999)

Kelly and Williams (AAP 2004)

Massoulié (AAP 2007)

Kang, Kelly, Lee, and Williams (AAP 2009)

Kelly, Massoulié, and Walton (QS 2009).

Optimisation problem: if there are x_k **connections** of class $k \in \{1, \ldots, K\}$, then class k obtains **throughput** λ_k^* such that (λ_k^*) achieves

$$\max_{\lambda \in \mathscr{C}} \sum_{k=1}^{K} x_k U_k(\lambda_k / x_k)$$

where U_k is some **utility function**, and $\mathscr{C}$ depends on the capacity of the resources.

The utility function is used to summarize in an idealized fashion the effect of TCP on the network.

This is a very high level macroscopic approach,

perhaps akin to Thermodynamics.

Our approach:

to devise a microscopic model
and then derive its macroscopic limit.

The idealization of TCP takes place at the micro level.

Akin to kinetic models in Statistical Physics.

2 Markovian modeling of user interaction

We propose

a Markovian model for a stochastic network

- **constituted of $J \geq 1$ nodes,**
- **hosting $K \geq 1$ classes of users (or connections),**
- **with $N_k \geq 1$ users in class k for $1 \leq k \leq K$.**

$$N = (N_1, \ldots, N_K) \qquad \textbf{class size vector},$$

$$|N| = N_1 + \cdots + N_K \qquad \textbf{total number of users}.$$

Each user (or connection) alternates between being

- **active, or on, when it has data to transmit**
- **and inactive, or off, when it has none**

and thus generates

intermittent (intermittent) transmissions.

Markov model and Itô-Skorohod equations

This model corresponds to a Markov process

$$W^N(t) = \left(W^N_{n,k}(t), 1 \le n \le N_k, 1 \le k \le K\right), \qquad t \ge 0,$$

where

$$W^N_{n,k}(t) \in \mathbb{R}_+ \cup \{-1\}$$

is the state (output, window) of the n-th transmission of class k at time t, with -1 corresponding to an inactive user.

We choose to represent it (in law) by the following

Itô-Skorohod SDE:

$$dW^N_{n,k}(t) = \mathbb{1}_{\left\{W^N_{n,k}(t-)=-1\right\}} \int (1+w)\mathbb{1}_{\{0<z<\lambda_k(U^N(t-))\}} \mathscr{A}_{n,k}(dw,dz,dt)$$

$$+ \mathbb{1}_{\left\{W^N_{n,k}(t-)\geq 0\right\}} \Big[a_k(W^N_{n,k}(t-), U^N(t-))\,dt$$

$$-(1-r_k)W^N_{n,k}(t-) \int \mathbb{1}_{\left\{0<z<b_k(W^N_{n,k}(t-),U^N(t-))\right\}} \mathscr{N}_{n,k}(dz,dt)$$

$$-(1+W^N_{n,k}(t-)) \int \mathbb{1}_{\left\{0<z<\mu_k(W^N_{n,k}(t-),U^N(t-))\right\}} \mathscr{D}_{n,k}(dz,dt) \Big]$$

with $U^N(t) = (U^N_j(t), 1 \leq j \leq J)$, $\quad U^N_j(t) = \sum_{k=1}^{K} A_{jk} \sum_{n=1}^{N_k} W^N_{n,k}(t)^+$,

where the $\mathscr{A}_{n,k}$ are Poisson point processes with intensity $\alpha_k(dw)dz\,dt$,

and the $\mathscr{N}_{n,k}$ and $\mathscr{D}_{n,k}$ are Poisson point processes with intensity $dz\,dt$.

The following can be proved using standard arguments.

If the functions a_k are Lipschitz

and b_k, λ_k, and μ_k are locally bounded,

then there is pathwise existence and uniqueness

of solution for the SDE,

and the corresponding Markov process is well defined.

The mean-field asymptotic regime

This SDE constitutes

a coupled system in very high dimension

and an asymptotic study is performed to render it

tractable, which reduces

the dimension to the number of classes K.

For $1 \leq k \leq K$, we assume that

$$N_k \to \infty, \qquad \frac{N_k}{|N|} := \frac{N_k}{N_1 + \cdots + N_K} \to p_k\,,$$

and, so as to scale the resource capacities adequately, that a factor $\frac{1}{|N|}$ is introduced inside the coefficients by

$$\text{replacing } U^N \text{ by } \overline{U}^N = \frac{1}{|N|} U^N\,.$$

This yields the following mean-field rescaled SDE:

$$dW^N_{n,k}(t) = \mathbb{1}_{\{W^N_{n,k}(t-)=-1\}} \int (1+w)\mathbb{1}_{\{0<z<\lambda_k(\overline{U}^N(t-))\}} \mathscr{A}_{n,k}(dw,dz,dt)$$
$$+ \mathbb{1}_{\{W^N_{n,k}(t-)\geq 0\}} \Big[a_k(W^N_{n,k}(t-), \overline{U}^N(t-))\, dt$$
$$-(1-r_k)W^N_{n,k}(t-) \int \mathbb{1}_{\{0<z<b_k(W^N_{n,k}(t-),\overline{U}^N(t-))\}} \mathscr{N}_{n,k}(dz,dt)$$
$$-(1+W^N_{n,k}(t-)) \int \mathbb{1}_{\{0<z<\mu_k(W^N_{n,k}(t-),\overline{U}^N(t-))\}} \mathscr{D}_{n,k}(dz,dt) \Big]$$

with $\overline{U}^N(t) = \frac{1}{|N|} U^N(t) = \left(\overline{U}^N_j(t), 1 \leq j \leq J\right)$,

$$\overline{U}^N_j(t) = \frac{1}{|N|} U^N_j(t) = \sum_{k=1}^{K} A_{jk} \frac{N_k}{|N|} \overline{W}^N_k(t), \quad \overline{W}^N_k(t) = \frac{1}{N_k} \sum_{n=1}^{N_k} W^N_{n,k}(t)^+.$$

This system is in multi-class mean-field interaction through

$$\left(\overline{W}_k^N(t), 1 \le k \le K\right) \text{ via } \overline{U}^N(t) = \frac{1}{|N|} U^N(t),$$

the vector of the empirical means of the class outputs

via

the rescaled throughput vector.

For $1 \leq k \leq K$, a natural quantity is the class k empirical measure

$$\Lambda_k^N = \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{\left(W_{n,k}^N(t), t \geq 0\right)}$$

where $\delta_{(x(t), t \geq 0)}$ denotes the Dirac mass at the sample path $(x(t), t \geq 0)$.

Notably

$$\overline{W}_k^N(t) = \langle w^+, \Lambda_k^N(t)(dw) \rangle .$$

3 Limit nonlinear Markov process and class interaction

In the mean-field asymptotic regime,

under adequate assumptions for the initial conditions,

a propagation of chaos phenomenon is expected:

- the **processes** $W_{n,k}^N$ should become **independent**, and each **converge in law** to a **process** W_k,
- the **empirical measures** Λ_k^N (**random laws on path space**) should **converge in law** (and in probability) to the **law** of the **same process** W_k,

where the limit process

$$W(t) = (W_k(t), 1 \leq k \leq K), \qquad t \geq 0,$$

solves the (adequately started) following equation:

McKean-Vlasov Itô-Skorohod SDE (Nonlinear Markov process)

$$
\begin{aligned}
dW_k(t) = {} & \mathbb{1}_{\{W_k(t-)=-1\}} \int (1+w)\mathbb{1}_{\{0<z<\lambda_k(u_W(t))\}}\, \mathscr{A}_k(dw,dz,dt) \\
& + \mathbb{1}_{\{W_k(t-)\geq 0\}} \Big[a_k(W_k(t-), u_W(t))\, dt \\
& \quad - (1-r_k) W_k(t-) \int \mathbb{1}_{\{0<z<b_k(W_k(t-),u_W(t))\}}\, \mathscr{N}_k(dz,dt) \\
& \quad - (1+W_k(t-)) \int \mathbb{1}_{\{0<z<\mu_k(W_k(t-),u_W(t))\}}\, \mathscr{D}_k(dz,dt) \Big]
\end{aligned}
$$

with

$$
u_W(t) = (u_{W,j}(t), 1 \leq j \leq J), \quad u_{W,j}(t) = \sum_{k=1}^{K} A_{jk} p_k \mathbb{E}(W_k(t)^+).
$$

The **interaction** between coordinates depends on

the **mean throughput** vector

$$(u_W(t), t \geq 0)$$

which is a **linear** functional of

the **mean class output** vector

$$\mathbb{E}(W(t)^+) = \mathbb{E}(W_k(t)^+, 1 \leq k \leq K) = \langle w^+, \mathscr{L}(W(t)) \rangle .$$

Notably, the infinitesimal generator of the Markov process $(W(t), t \geq 0)$ depends, at time t, on the

law $\mathscr{L}(W(t))$ of $W(t)$ itself

and not only on the value of the sample path.

Actually, only on the class marginals of $\mathscr{L}(W(t))$.

The Kolmogorov equations are

nonlinear integro-differential equations

and a

nonlinear martingale problem

can be associated to this process.

This is why it is called a

Nonlinear Markov process.

Using this **SDE representation**, we have adapted

contraction and **coupling** techniques developed

for **exchangeable** systems by **Sznitman (1980's, 1989)**

to this **multi-class** setting (**technical**), and

obtained existence and **uniqueness**

for this **fixed-point** problem,

as well as **propagation of chaos**.

This was done in G. and Robert (AAP 2009) for

persistent transmissions

and has been extended to the general case of

on-off users with intermittent transmissions

in G. and Robert (QUES 2011)

There were many difficulties, *e.g.*,

- lack of symmetry of multi-class systems w.r.t. exchangeable ones:

$$N_1! \cdots N_K! \ll |N|! = (N_1 + \cdots + N_K)!$$

  and rigorous results are rare for multi-class systems,

- quadratic behavior of $b_k(w,u) = w\beta_k(u)$,

- on-off users introduce discontinuities. Smart choices may simplify computations (-1 as cemetery state, ...).

The assumptions, notably on initial conditions, reflect this.

These must not be too stringent since

long-time and stationary behavior

are essential.

Gaussian moment assumptions were used.

4 Fixed points and invariant laws for the limit

Recall that solution $(W(t), t \geq 0)$ has in general

an infinitesimal generator at time t

depending not only on the state,

but also, through $u_W(t)$, on $\mathbb{E}(W(t)^+)$

and thus on the law $\mathscr{L}(W(t))$ itself.

This nonlinearity is an important complication in studying the behavior of $(W(t), t \geq 0)$, in particular for

existence and uniqueness of invariant laws.

Note that, starting from an invariant law,

$(\mathbb{E}(W(t)^+), t \geq 0)$ and hence $(u_W(t), t \geq 0)$ are constant,

and $(W(t), t \geq 0)$ corresponds to a

homogeneous Markov process

in equilibrium.

Interesting results on the invariant laws were obtained in G. and Robert (AAP 2009, QUES 2011) by reducing the corresponding

infinite-dimensional fixed-point problem

to a finite-dimensional one.

The results for on-off users being still preliminary,
we concentrate now on

users or connections

which emit persistently

and thus can be identified with their transmissions.

Assume that, for $1 \leq k \leq K$,

$$a_k(w,u) = a_k(u), \ b_k(w,u) = w\beta_k(u), \quad w \in \mathbb{R}_+, \ u \in \mathbb{R}_+^J,$$

where

$a_k : \mathbb{R}_+^J \longrightarrow \mathbb{R}_+$ is **Lipschitz bounded**

$\beta_k : \mathbb{R}_+^J \longrightarrow \mathbb{R}_+$ is **Lipschitz**.

Then, the invariant laws for the nonlinear SDE are in

one-to-one relation with

the solutions of the

finite-dimensional fixed-point problem

$$u = (u_j, 1 \le j \le J) \in \mathbb{R}_+^J, \quad u_j = \sum_{k=1}^{K} A_{jk} p_k \psi(r_k) \sqrt{\frac{a_k(u)}{\beta_k(u)}},$$

where

$$\psi(r) = \sqrt{\frac{2}{\pi}} \prod_{n=1}^{\infty} \frac{1 - r^{2n}}{1 - r^{2n-1}}.$$

Such a solution u^* corresponds to a

product-form invariant law with density

$$\prod_{k=1}^{K} H_{r_k,\rho_k}(w_k), \qquad w = (w_k, 1 \le k \le K) \in \mathbb{R}_+^K,$$

where $\rho_k = \dfrac{a_k(u^*)}{\beta_k(u^*)}$ **and, for** $x \in \mathbb{R}_+$,

$$H_{r,\rho}(x) = \frac{\sqrt{2\rho/\pi}}{\prod_{n=0}^{\infty}(1 - r^{2n+1})} \sum_{n=0}^{\infty} \frac{r^{-2n}}{\prod_{k=1}^{n}(1 - r^{-2k})} e^{-\rho r^{-2n} x^2/2}.$$

Note that the decay is appropriate for

Gaussian moment conditions.

The **limit equilibrium throughput** for **class k users** is given by

$$\lambda_k = \mathbb{E}(\overline{W}_k) = \int_{\mathbb{R}_+} x H_{r_k,\rho_k}(x)\,dx$$

$$= \psi(r_k)\sqrt{\rho_k} = \psi(r_k)\sqrt{\frac{a_k(u^*)}{\beta_k(u^*)}}$$

with u^* solving a **fixed-point problem**

which can be written as

$$H(u^*) = 0.$$

Back to an optimisation problem ?

Class $k \in \{1,\ldots,K\}$ receives throughput λ_k^*, such that (λ_k^*) achieves

$$\max_{\lambda \in \mathscr{C}} \sum_{k=1}^{K} p_k U_k(\lambda_k / p_k)$$

which can be put (under some conditions) in the form

$$\nabla G(\lambda^*/p) = 0.$$

How many solutions for the fixed-point problem ?

G., Robert, and Verloop (NETCOOP 2009) prove

existence and uniqueness of the invariant law

by contraction and monotonicity methods,

under some assumptions, for topologies such as

- **One node, several classes**
- **Linear networks with cross-traffic**
- **Trees**
- **Rings and toruses.**

Assume moreover that

$$A_{jk} = \begin{cases} 1 & \text{if node } j \text{ is used by a class } k \text{ user,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_k(u) = \beta_k \left(\sum_{j=1}^{J} A_{jk} u_j \right).$$

(With slight abuse of notations.)

The **fixed-point equation** becomes

$$u_j = \sum_{k=1}^{K} A_{jk} p_k \psi(r_k) \frac{\sqrt{a_k(u)}}{\sqrt{\beta_k\left(\sum_{j=1}^{J} A_{jk} u_j\right)}}, \qquad 1 \le j \le J.$$

We **assume** that $u \to \beta_k(u)$ is **strictly increasing** and **Lipschitz** and that $a_k(u) \equiv a_k$ is **constant**.

One node, several classes

There is a unique solution $u = u^*$ **for the fixed-point equation**

$$u = \sum_{k=1}^{K} \psi(r_k) p_k \sqrt{\frac{a_k}{\beta_k(u)}},$$

and a unique **invariant law** with density H_{r,ρ_k} and expectation

$$\psi(r_k)\sqrt{\frac{a_k}{\beta_k(u^*)}}$$

yielding the **mean equilibrium output**.

If moreover the classes vary only in their RTT's, *i.e.*,

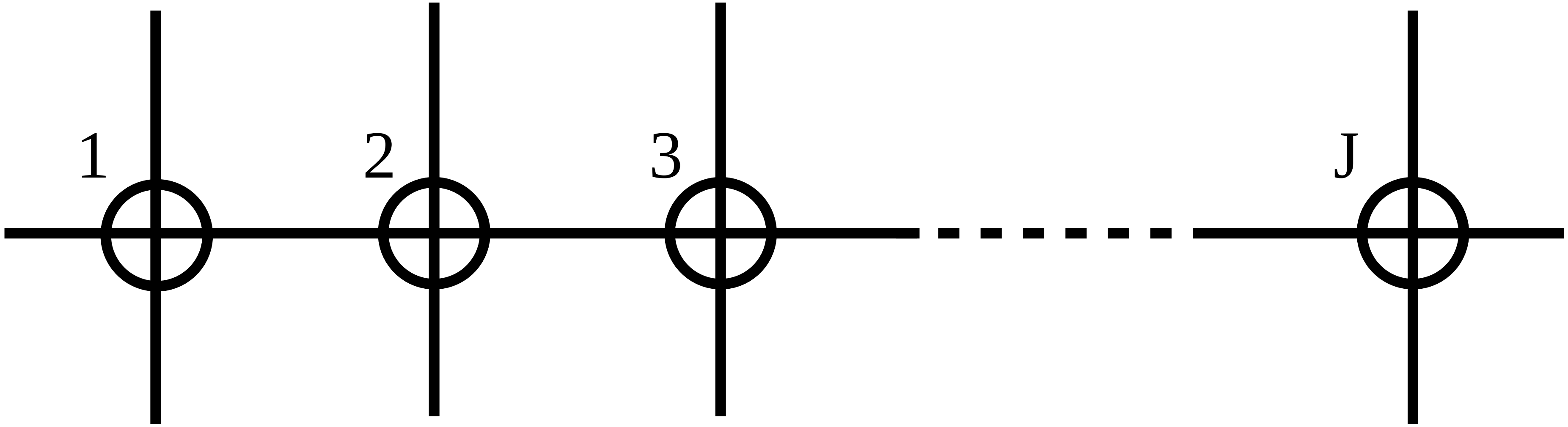
$$\beta_k \equiv \beta\,, \quad r_k \equiv r\,, \qquad 1 \leq k \leq K\,,$$

then the class **outputs** differ **only** by the factors

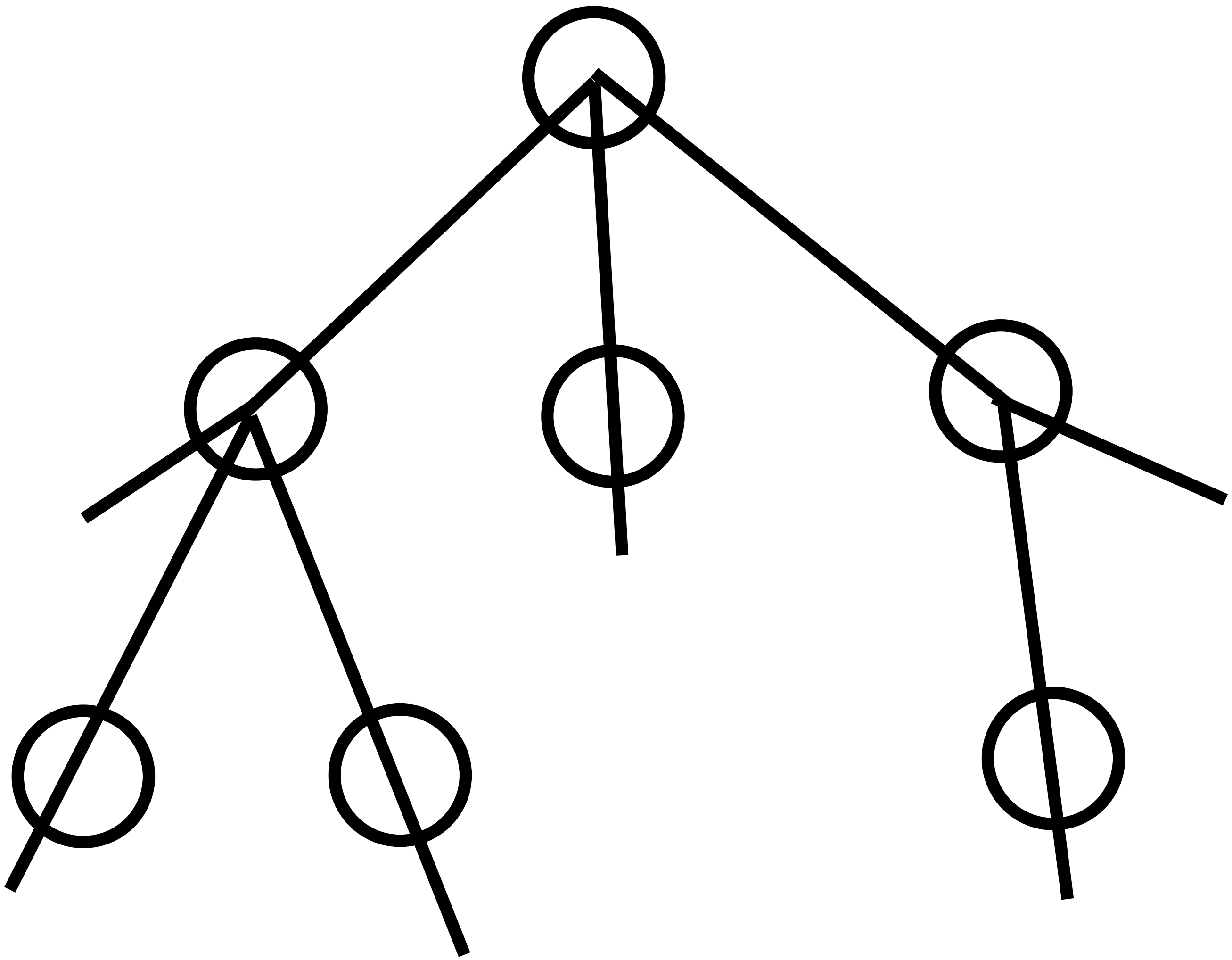
$$\sqrt{a_k} = 1/\sqrt{\mathrm{RTT}_k}\,.$$

Linear network with cross-traffic

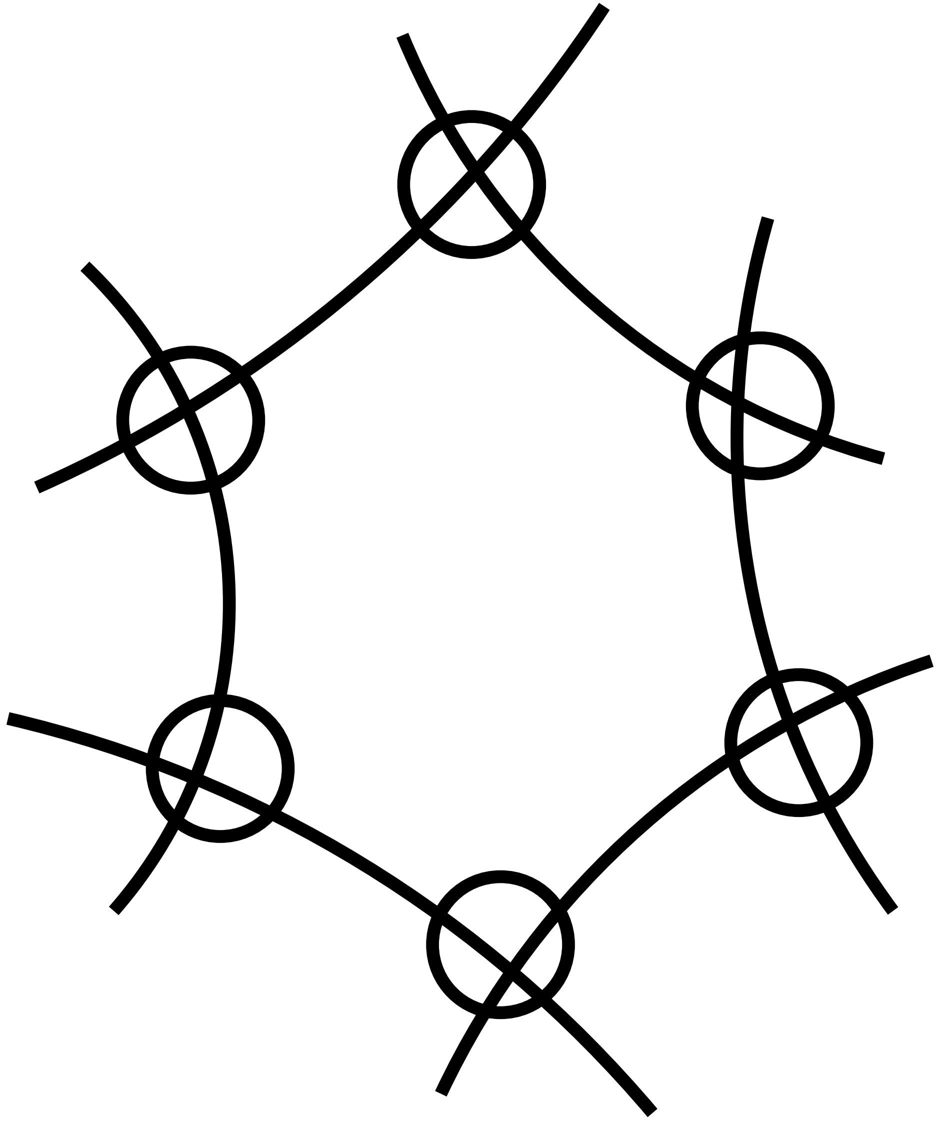
There are J nodes and $K = J + 1$ classes. For $1 \leq j \leq J$, class j transmissions use only node j. The transmissions of class $J+1$ use all J nodes.



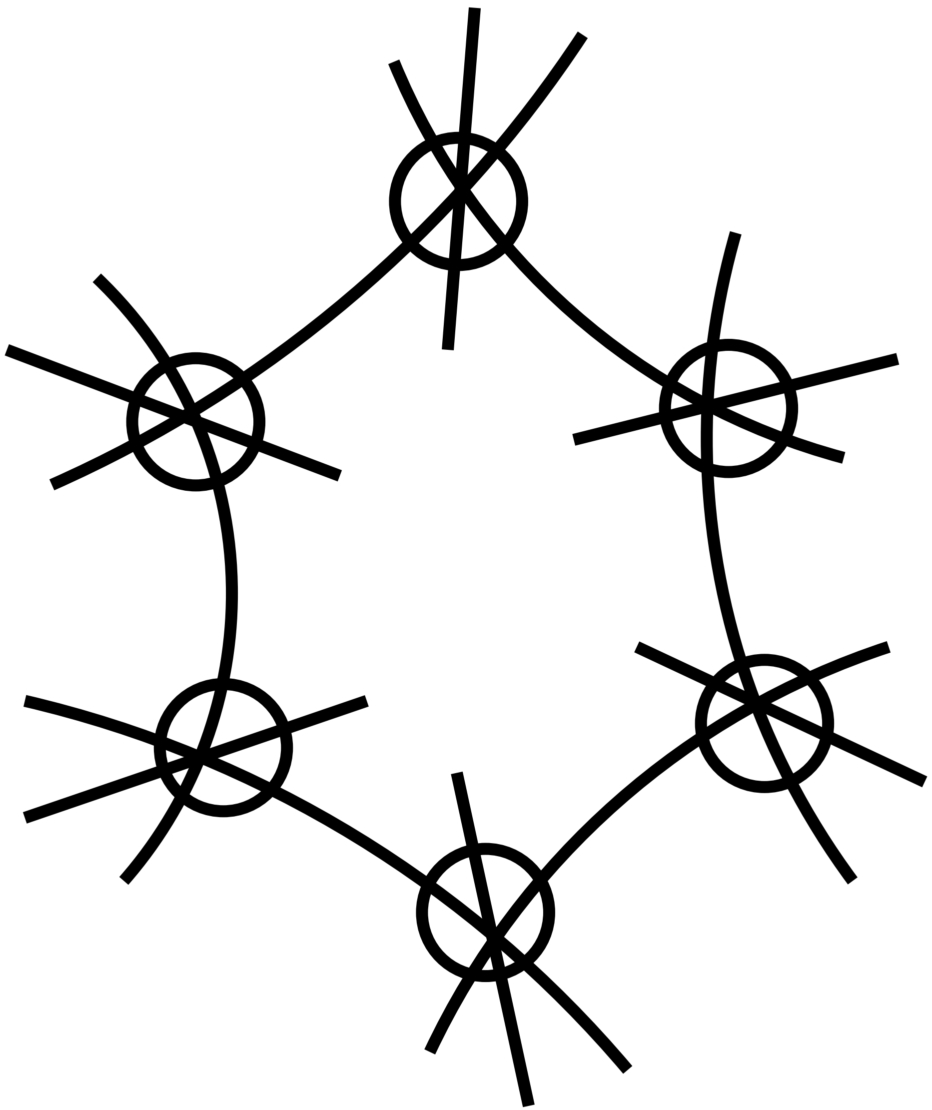
Trees



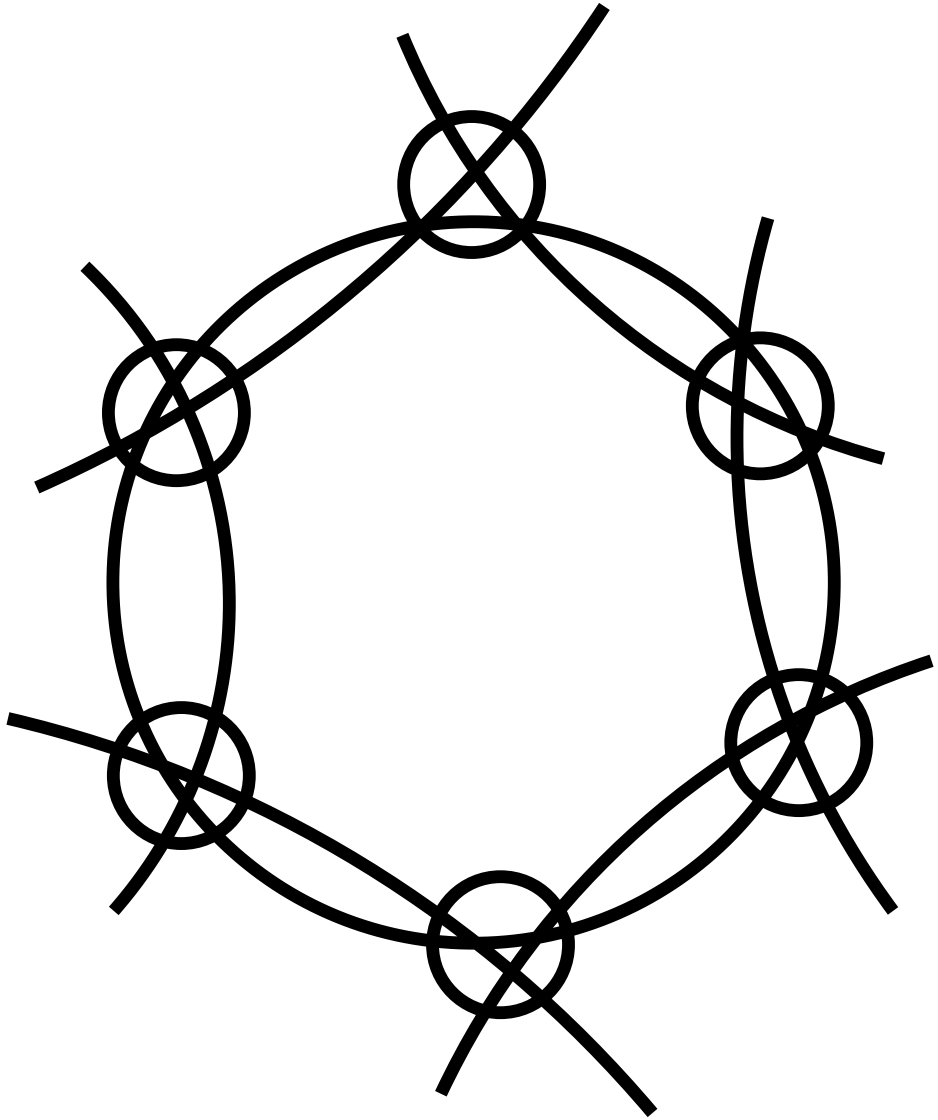
Ring topology, 1



Ring topology, 2



Ring topology, 3



Is uniqueness always true if

$$u \longrightarrow \beta_k(u)$$

is increasing ?

There exist examples when

not:

Raghunathan and Kumar (2007) in a wireless context.

Are there

meta-stability phenomena

in these networks ?

Known for

- **Loss Networks: Gibbens *et al.* (1990), Marbukh (1993),**
- **Wireless Networks: Antunes *et al.* (2008).**

Are there oscillations ?

Such possibilities have been suggested by the literature on the Internet and by simulations:

cyclic behavior in congested networks

***e.g.*, mean-field study for a single node and a single class and delay equations in Baccelli, McDonald, and Reynier (2002)**

5 A Conclusion

Representation of the interaction of flows

- Using an instantaneous **fluid picture** yields:

  An **optimisation** problem

  Data for Model : **Utility function**.

- Starting from **microscopic dynamics** yields:

  A **fixed point** equation

  Data for Model : **Equation coefficients**.

Perspectives

- **Modeling transmissions arriving from the outside world and disappearing after completion. A different scaling.**

- **More on the invariant laws for on-off users. Transient behavior of on periods needs to be assessed.**

- **Convergence of invariant laws:**

$$\lim_{N\to\infty}\lim_{t\to\infty} = \lim_{t\to\infty}\lim_{N\to\infty} ?$$

  Always difficult, especially with multiple equilibria.

- **Relations with the optimization problem approach.**

Merci

pour votre attention !