Constrained and Unconstrained Optimal Control of Piecewise Deterministic Markov Processes

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Introduction

Davis (80’s)

General class of non-diffusion dynamic stochastic hybrid models: deterministic trajectory punctuated by random jumps.

Applications

Engineering systems, biology, operations research, management science, economics, dependability and safety, . . .
Parameters of the model

- the state space: \( X \) open subset of \( \mathbb{R}^d \) (boundary \( \partial X \)).

- the flow: \( \phi(x, t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) satisfying
  \( \phi(x, t + s) = \phi(\phi(x, s), t) \) for all \( x \in \mathbb{R}^d \) and \( (t, s) \in \mathbb{R}^2 \).

- active boundary:
  \( \Delta = \{ x \in \partial X : x = \phi(y, t) \text{ for some } y \in X \text{ and } t \in \mathbb{R}^*_+ \} \). For \( x \in \bar{X} = X \cup \Delta \),
  \[ t^*(x) = \inf \{ t \in \mathbb{R}^+_0 : \phi(x, t) \in \Delta \}. \]

- \( A \) is the action space, assumed to be a Borel space.
  \( A^i \in \mathcal{B}(A) \) (respectively \( A^g \in \mathcal{B}(A) \)) is the set of impulsive (respectively gradual) actions satisfying \( A = A^i \cup A^g \) with \( A^i \cap A^g = \emptyset \).
Parameters of the model

- The set of feasible actions in state $x \in \overline{X}$ is $A(x) \subset A$. Let us introduce the following sets $K = K^i \cup K^g$ with

$$K^g = \{(x, a) \in X \times A^g : a \in A(x)\} \in \mathcal{B}(X \times A^g),$$

$$K^i = \{(x, a) \in \Delta \times A^i : a \in A(x)\} \in \mathcal{B}(\Delta \times A^i).$$

- The controlled jumps intensity $\lambda$ which is a $\mathbb{R}_+$-valued measurable function defined on $K^g$.

- The stochastic kernel $Q$ on $X$ given $K$ satisfying $Q(X \setminus \{x\}|x, a) = 1$ for any $(x, a) \in K^g$. It describes the state of the process after any jump.
Uncontrolled process

*Definition* of a PDMP

Parameters: flow $\phi$, intensity of the jumps $\lambda$, transition kernel $Q$
Uncontrolled process

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Definition of a PDMP
Parameters: flow $\phi$, intensity of the jumps $\lambda$, transition kernel $Q$
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*Definition* of a PDMP

Parameters: flow $\phi$, intensity of the jumps $\lambda$, **transition kernel $Q$**
Construction of the process

The canonical space $\Omega = \left( X \times (\mathbb{R}_+^* \times X)^\infty \right) \cup_{n=1}^{\infty} \Omega_n$ with $\Omega_n = X \times (\mathbb{R}_+^* \times X)^n \times (\{\infty\} \times \{x_\infty\})^\infty$.

Introduce the mappings $X_n : \Omega \rightarrow X_\infty = X \cup \{x_\infty\}$ by $X_n(\omega) = x_n$ and $\Theta_n : \Omega \rightarrow \mathbb{R}^*_+$ by $\Theta_n(\omega) = \theta_n$; $\Theta_0(\omega) = 0$ where

$$\omega = (x_0, \theta_1, x_1, \theta_2, x_2, \ldots) \in \Omega.$$

In addition $T_n(\omega) = \sum_{i=1}^{n} \Theta_i(\omega) = \sum_{i=1}^{n} \theta_i$ with $T_\infty(\omega) = \lim_{n \rightarrow \infty} T_n(\omega)$.

$H_n$ is the set of path up to $n$ and $H_n = (X_0, \Theta_1, X_1, \ldots, \Theta_n, X_n)$ is the $n$-term random history process.
Construction of the process

The random measure $\mu$ associated with $(\Theta_n, X_n)_{n \in \mathbb{N}}$ is a measure defined on $\mathbb{R}_+^* \times X$ by

$$\mu(dt, dx) = \sum_{n \geq 1} l\{T_n(\omega) < \infty\} \delta(T_n(\omega), X_n(\omega))(dt, dx).$$

The controlled process $\{\xi_t\}_{t \in \mathbb{R}_+}$:

$$\xi_t(\omega) = \begin{cases} \phi(X_n, t - T_n) & \text{if } T_n \leq t < T_{n+1} \text{ for } n \in \mathbb{N}; \\ x_\infty, & \text{if } T_\infty \leq t. \end{cases}$$

For $t \in \mathbb{R}_+$, define

$$\mathcal{F}_t = \sigma\{H_0\} \vee \sigma\{\mu([0, s] \times B) : s \leq t, B \in \mathcal{B}(X)\}.$$
Admissible strategies and conditional distribution

An admissible control strategy is a sequence $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}}$ such that, for any $n \in \mathbb{N}$,

- $\pi_n$ is a stochastic kernel on $A^g$ given $H_n \times \mathbb{R}_+$ satisfying
  $$\pi_n(A(\phi(x_n, t))|h_n, t) = 1 \text{ for } h_n = (x_0, \theta_1, x_1, \ldots \theta_n, x_n) \in H_n \text{ and } t \in ]0, t^*(x_n)[.$$

- $\gamma_n$ is a stochastic kernel on $A^i$ given $H_n$ satisfying
  $$\gamma_n(A(\phi(x_n, t^*(x_n)))|h_n) = 1 \text{ for } h_n = (x_0, \theta_1, x_1, \ldots \theta_n, x_n).$$

The set of admissible control strategies is denoted by $\mathcal{U}$. 
Admissible strategies and conditional distribution

When an admissible control strategy $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}}$ is considered then $\pi$ and $\gamma$ denote the random processes with values in $\mathcal{P}(A^g)$ and $\mathcal{P}(A^i)$ correspondingly as

$$\pi(da|t) = \sum_{n \in \mathbb{N}} I\{T_n < t \leq T_{n+1}\} \pi_n(da|H_n, t - T_n)$$

and

$$\gamma(da|t) = \sum_{n \in \mathbb{N}} I\{T_n < t \leq T_{n+1}\} \gamma_n(da|H_n),$$

for $t \in \mathbb{R}^*_+$. 
Admissible strategies and conditional distribution

For a strategy $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}} \in \mathcal{U}$, the intensity of jumps

$$\lambda^u_n(h_n, t) = \int_{\mathcal{A}^g} \lambda(\phi(x_n, t), a)\pi_n(da|h_n, t),$$

and the rate of jumps

$$\Lambda^u_n(h_n, t) = \int_{[0, t]} \lambda^u_n(h_n, s)ds,$$

the distribution of the state after a (stochastic) jump

$$Q^g_n(u)(dx|h_n, t) = \frac{1}{\lambda^u_n(h_n, t)} \int_{\mathcal{A}^g} Q(dx|\phi(x_n, t), a)\lambda(\phi(x_n, t), a)\pi_n(da|h_n, t),$$

the distribution of the state after a (boundary) jump

$$Q^i_n(u)(dx|h_n) = \int_{\mathcal{A}^i} Q(dx|\phi(x_n, t^*(x_n)), a)\gamma_n(da|h_n).$$
Admissible strategies and conditional distribution

Introduce the stochastic kernel $G_n$ on $\bar{\mathbb{R}}^*_+ \times X_{\infty}$ given $H_n$,

$$G_n(\Gamma|h_n) = \left[ I\{x_n=x_\infty\} + e^{-\Lambda_n^u(h_n, +\infty)} I\{x_n\in X\} I\{t^*(x_n)=\infty\} \right] \delta(+\infty, x_\infty)(\Gamma)$$

$$+ I\{x_n\in X\} \left[ \int_{\bar{\mathbb{R}}^*_+ \times X} l_\Gamma(t, x) \delta_{t^*(x_n)}(dt) Q_n^{i, u}(dx|h_n) e^{-\Lambda_n^u(h_n, t^*(x_n))} \right]$$

$$+ \int_{0, t^*(x_n)} \left[ \times X \right] l_\Gamma(t, x) Q_n^{g, u}(dx|h_n, t) \lambda_n^u(h_n, t) e^{-\Lambda_n^u(h_n, t)} dt \right],$$

where $\Gamma \in \mathcal{B}(\bar{\mathbb{R}}^*_+ \times X_{\infty})$ and $h_n = (x_0, \theta_1, x_1, \ldots, \theta_n, x_n) \in H_n$.

$G_n$ the joint distribution of the next sojourn time and state?
Admissible strategies and conditional distribution

Consider an admissible strategy $u \in \mathcal{U}$ and an initial state $x_0 \in \mathbf{X}$. There exists a probability $\mathbb{P}^u_{x_0}$ on $(\Omega, \mathcal{F})$ such that the restriction of $\mathbb{P}^u_{x_0}$ to $(\Omega, \mathcal{F}_0)$ is given by

$$\mathbb{P}^u_{x_0}(\{x_0\} \times (\mathbb{R}^*_+ \times \mathbf{X}_\infty)^\infty) = 1$$

and the positive random measure $\nu$ defined on $\mathbb{R}^*_+ \times \mathbf{X}$ by

$$\nu(dt, dx) = \sum_{n \in \mathbb{N}} \frac{G_n(dt - T_n, dx|H_n)}{G_n([t - T_n, +\infty) \times \mathbf{X}_\infty|H_n)} I\{T_n < t \leq T_{n+1}\}$$

is the predictable projection of $\mu$ with respect to $\mathbb{P}^u_{x_0}$.

→ The conditional distribution of $(\Theta_{n+1}, X_{n+1})$ given $\mathcal{F}_{T_n}$ under $\mathbb{P}^u_{x_0}$ is determined by $G_n(\cdot|H_n)$. 
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Unconstrained and constrained problems

Cost functions

- \((C^g_j)_{j \in \mathbb{N}_p}\) associated with a continuous action is a real-valued mapping defined on \(K^g\).
- \((C^i_j)_{j \in \mathbb{N}_p}\) associated with an impulsive action on the boundary \(\Delta\) is a real-valued mapping defined on \(K^i\).

The associated infinite-horizon discounted criteria corresponding to an admissible control strategy \(u \in \mathcal{U}\) are defined, for \(j \in \mathbb{N}_p\), by

\[
\mathcal{V}_j(u, x_0) = \mathbb{E}^u_{x_0} \left[ \int_{0, +\infty} e^{-\alpha s} \int_{A(\xi_s)} C^g_j(\xi_s, a) \pi(da|s) ds \right]
\]

\[
+ \mathbb{E}^u_{x_0} \left[ \int_{0, +\infty} e^{-\alpha s} l_{\xi_s \in \Delta} \int_{A(\xi_s^-)} C^i_j(\xi_s^-, a) \gamma(da|s) \mu(ds, X) \right]
\]

for any \(j \in \mathbb{N}_p\).
Unconstrained and constrained problems

- The optimization problem without constraint consists in minimizing the performance criterion

\[ \inf_{u \in U} \mathcal{V}_0(u, x_0). \]

- The optimization problem with \( p \) constraints consists in minimizing the performance criterion

\[ \inf_{u \in U} \mathcal{V}_0(u, x_0) \]

such that the constraint criteria

\[ \mathcal{V}_j(u, x_0) \leq B_j \]

are satisfied for any \( j \in \mathbb{N}_p^* \), where \((B_j)_{j \in \mathbb{N}_p^*}\) are real numbers representing the constraint bounds.
Different classes of strategies

- **non-randomized stationary**, if $\pi_n(\cdot|h_n, t) = \delta \varphi^s(\phi(x_n, t))(\cdot)$ and $\gamma_n(\cdot|h_n) = \delta \varphi^s(\phi(x_n, t))(\cdot)$, where $\varphi^s : \overline{X} \to A$ is a measurable mapping satisfying $\varphi^s(y) \in A(y)$ for any $y \in \overline{X}$.

- **stationary**, if for some $(\pi, \gamma) \in \mathcal{P}^g \times \mathcal{P}^i$ the control strategy $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}}$ is given by $\pi_n(da|h_n, t) = \pi(da|\phi(x_n, t))$ and $\gamma_n(db|h_n) = \gamma(db|\phi(x_n, t^*(x_n)))$.

- **feasible**, if $u \in \mathcal{U}$ and $\mathcal{V}_j(u, x_0) \leq B_j$, for $j \geq 1$. 
Hypotheses

**Assumption A.** There are constants $K \geq 0, \varepsilon_1 > 0$ and $\varepsilon_2 \in [0, 1]$ such that

(A1) For any $(x, a) \in K^g$, $\lambda(x, a) \leq K$

(A2) For any $(z, b) \in K^i$, $Q(A_{\varepsilon_1}|z, b) \geq 1 - \varepsilon_2$, where

$$A_{\varepsilon_1} = \{x \in X : t^*(x) > \varepsilon_1\}.$$  

**Assumption B.**

(B1) The set $A(y)$ is compact for every $y \in \overline{X}$.

(B2) The kernel $Q$ is weakly continuous.

(B3) The function $\lambda$ is continuous on $K^g$.

(B4) The flow $\phi$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^p$.

(B5) The function $t^*$ is continuous on $\overline{X}$.  

Assumption C.

(C1) The multifunction $\Psi^g$ from $X$ to $A$ defined by $\Psi(x) = A(x)$ is upper semicontinuous. The multifunction $\Psi$ from $\Delta$ to $A$ defined by $\Psi^i(z) = A(z)$ is upper semicontinuous.

(C2) The cost function $C^g_0$ (respectively, $C^i_0$) is bounded and lower semicontinuous on $K^g$ (respectively, $K^i$).
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Lemma

Suppose Assumption A is satisfied. Then there exists \( M < \infty \) such that, for any control strategy \( u \in \mathcal{U} \) and for any \( x_0 \in X \)

\[
\mathbb{E}^u_{x_0} \left[ \sum_{n \in \mathbb{N}^*} e^{-\alpha T_n} \right] \leq M \text{ and } \mathbb{P}^u_{x_0} (T_\infty < +\infty) = 0.
\]
Non-explosion

Elements of proof:

- For any control strategy $u$, $x_0 \in X$ we have for any $j \in \mathbb{N}$

\[
P^u_{x_0}(\Theta_{j+2} + \Theta_{j+1} > \varepsilon_1 | H_j) \geq e^{-2K\varepsilon_1}(1 - \varepsilon_2).
\]

- Now,

\[
\mathbb{E}^u_{x_0}\left[ e^{-\alpha(\Theta_{j+1}+\Theta_{j+2})} | H_j \right]
\]

\[
\leq P^u_{x_0}(\Theta_{j+1} + \Theta_{j+2} \leq \varepsilon_1 | H_j)
+ e^{-\alpha\varepsilon_1}P^u_{x_0}(\Theta_{j+1} + \Theta_{j+2} > \varepsilon_1 | H_j)
\]

\[
= 1 + [e^{-\alpha\varepsilon_1} - 1]P^u_{x_0}(\Theta_{j+1} + \Theta_{j+2} > \varepsilon_1 | H_j)
\]

\[
\leq 1 + [e^{-\alpha\varepsilon_1} - 1][1 - \varepsilon_2]e^{-2K\varepsilon_1} = \kappa < 1.
\]
Elements of proof:

- For any $j \in \mathbb{N}^*$,

\[
\mathbb{E}_x^u \left[ e^{-\alpha T_{2j+1}} \right] = \mathbb{E}_x^u \left[ e^{-\alpha T_{2j-1}} \mathbb{E}_x^u \left[ e^{-\alpha (\Theta_{2j+1} + \Theta_{2j+2})} | H_{2j-1} \right] \right] \\
\leq \kappa \mathbb{E}_x^u \left[ e^{-\alpha T_{2j-1}} \right],
\]

and so

\[
\mathbb{E}_x^u \left[ e^{-\alpha T_{2j+1}} \right] \leq \kappa^j \mathbb{E}_x^u \left[ e^{-\alpha T_1} \right] \leq \kappa^j.
\]

Similarly,

\[
\mathbb{E}_x^u \left[ e^{-\alpha T_{2j+2}} \right] \leq \kappa^j \mathbb{E}_x^u \left[ e^{-\alpha T_2} \right] \leq \kappa^j.
\]

for any $j \in \mathbb{N}$.

- Therefore,

\[
\mathbb{E}_x^u \left[ \sum_{n \in \mathbb{N}^*} e^{-\alpha T_n} \right] \leq \frac{2}{1 - \kappa}.
\]
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The unconstrained problem and the DP approach

Notation and preliminary results:

- ▶ $A \overline{X}$ is the set of functions $g \in \mathbb{B}(\overline{X})$ such that for any $x \in \overline{X}$, the function $g(\phi(x, \cdot))$ is absolutely continuous on $[0, t^*(x)] \cap \mathbb{R}_+$. 

- ▶ Let $g \in A(\overline{X})$, there exists a real-valued measurable function $\lambda g$ defined on $\overline{X}$ satisfying for any $t \in [0, t^*(x)]$

$$g(\phi(x, t)) = g(x) + \int_{[0,t]} \lambda g(\phi(x, s))ds.$$

- ▶ Let $R \in \mathcal{P}(X | Y)$. Then $Rf(y) \overset{\cdot}{=} \int_X f(x)R(dx|y)$ for any $y \in Y$ and measurable function $f$. For any measure $\eta$ on $(Y, \mathcal{B}(Y))$, $\eta R(\cdot) \overset{\cdot}{=} \int_Y R(\cdot|y)\eta(dy)$.

- ▶ $q(dy|x, a) \overset{\cdot}{=} \lambda(x, a)[Q(dy|x, a) - \delta_x(dy)]$
Sufficient conditions for the existence of a solution for the HJB equation associated with the optimization problem.

**Theorem**

*Suppose assumptions A, B and C hold. Then there exist* \( W \in \mathbb{A}(X) \) *and* \( \mathcal{X} W \in \mathbb{B}(X) \) *satisfying*

\[
- \alpha W(x) + \mathcal{X} W(x) + \inf_{a \in A^g(x)} \left\{ C^g_0(x, a) + q W(x, a) \right\} = 0,
\]

*for any* \( x \in X \), *and*

\[
W(z) = \inf_{b \in A^i(z)} \left\{ C^i_0(z, b) + Q W(z, b) \right\},
\]

*for any* \( z \in \Delta \). *Moreover, for any* \( x \in X \)

\[
W(x) = \inf_{u \in \mathcal{U}} \mathcal{V}_0(u, x).
\]
Sufficient conditions for the existence of an optimal strategy.

**Theorem**

Suppose assumptions A, B and C hold. There exists a measurable mapping \( \hat{\varphi} : \overline{X} \rightarrow A \) such that \( \hat{\varphi}(y) \in A(y) \) for any \( y \in \overline{X} \) and satisfying

\[
C^g_0(x, \hat{\varphi}(x)) + qW(x, \hat{\varphi}(x)) = \inf_{a \in A(x)} \left\{ C^g_0(x, a) + qW(x, a) \right\}
\]

for any \( x \in \mathbf{X} \), and

\[
C^i_0(z, \hat{\varphi}(z)) + QW(z, \hat{\varphi}(z)) = \inf_{b \in A(z)} \left\{ C^i_0(z, b) + QW(z, b) \right\}.
\]

for any \( z \in \Delta \). Moreover, the stationary non-randomized strategy \( \hat{\varphi} \) is optimal.
Elements of proof:

- Define recursively \( \{W_i\}_{i \in \mathbb{N}} \) as

\[
W_{i+1}(y) = \mathcal{B} W_i(y),
\]

with \( W_0(y) = -K_A I_{A_1}(y) - (K_A + K_B) I_{A_1}(y) \) and

\[
\mathcal{B} V(y) = \int_{[0,t^*(y)[} e^{-(K+\alpha)t} \mathcal{R} V(\phi(y, t)) dt
\]

\[
+ e^{-(K+\alpha)t^*(y)} \mathcal{I} V(\phi(y, t^*(y)))
\]

where

\[
\mathcal{R} V(x) = \inf_{a \in A(x)} \left\{ C_0^g(x, a) + q V(x, a) + K V(x) \right\},
\]

and

\[
\mathcal{I} V(z) = \inf_{b \in A(z)} \left\{ C_0^i(z, b) + Q V(z, b) \right\}.
\]
\( W_i \) is lower semicontinuous and
\[
|W_i(y)| \leq K_A I_{A_{\varepsilon_1}}(y) + (K_A + K_B) I_{A_{\varepsilon_1}}(y).
\]

\( \mathcal{B} \) is monotone \((V_1 \leq V_2 \Rightarrow \mathcal{B} V_1 \leq \mathcal{B} V_2)\), \( \{ W_i \}_{i \in \mathbb{N}} \) is increasing and \( W_i \to W \) and \( W \) is bounded and lower semicontinuous.

\[
\lim_{i \to \infty} \mathbb{R} W_i(x) = \mathbb{R} W(x), \text{ for any } x \in X
\]
\[
\lim_{i \to \infty} \mathbb{I} W_i(z) = \mathbb{I} W(z), \text{ for any } z \in \Delta.
\]
The unconstrained problem and the DP approach

By using the bounded convergence Theorem,

\[ W(y) = \mathcal{B}W(y) \]

\[ = \int_{[0,t^*(y)]} e^{-(K+\alpha)t} \mathcal{K}W(\phi(y, t)) \, dt \]

\[ + e^{-(K+\alpha)t^*(y)} \mathcal{T}W(\phi(y, t^*(y))), \]

where \( y \in \bar{X} \).

Then \( W \in A(\bar{X}) \) and there exists \( \mathcal{K}W \in \mathcal{B}(X) \)

\[ -\alpha W(x) + \mathcal{K}W(x) + \inf_{a \in A^g(x)} \left\{ C^g_0(x, a) + qW(x, a) \right\} = 0, \]

for any \( x \in X \), and

\[ W(z) = \inf_{b \in A^i(z)} \left\{ C^i_0(z, b) + QW(z, b) \right\}, \]

for any \( z \in \Delta \).
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Occupation measure

For any admissible control strategy $u \in \mathcal{U}$, the occupation measure $\eta_u \in \mathcal{M}(\mathcal{K})$ associated with $u$ is defined as follows

$$
\eta_u(\Gamma) = \mathbb{E}^u_{x_0} \left[ \int_{\Gamma \cap \mathcal{K}^g} \int_{0,\infty} e^{-\alpha s} \delta_{\xi_s}(dx) \pi(da|s) ds \right] 
$$

$$
+ \mathbb{E}^u_{x_0} \left[ \int_{\Gamma \cap \mathcal{K}^i} \sum_{n \in \mathbb{N}^*} e^{-\alpha T_n} \delta_{\xi_{T_n}}(dz) \gamma(db|T_n-) \right].
$$

for any $\Gamma \in \mathcal{B}(\mathcal{K})$. 
Linear programming approach

The infinite-horizon discounted criteria can be rewritten as

\[
V_j(u, x_0) = \mathbb{E}_x^u \left[ \int_{0, \infty} e^{-\alpha s} \int_{A(\xi_s)} C^g_j(\xi_s, a) \pi(da|s) ds \right]
\]

\[
+ \mathbb{E}_x^u \left[ \int_{0, \infty} e^{-\alpha s} \mathbb{1}_{\{\xi_s \in \Delta\}} \int_{A(\xi_s)} C^i_j(\xi_s, a) \gamma(da|s) \mu(ds, \mathbf{X}) \right]
\]

\[
= \eta^g_u(C^g_j) + \eta^i_u(C^i_j)
\]

where the restriction of \( \eta_u \) to \( K^g \) (resp. \( K^i \)) is denoted by \( \eta^g_u \) (resp. \( \eta^i_u \)).
Admissible measure

A finite measure $\eta \in \mathcal{M}(K)$ is called admissible if, for any $(W, X W) \in \mathcal{A}(X) \times \mathcal{B}(X)$, the following equality holds

$$
\int_X \left( \alpha W(x) - X W(x) \right) \hat{\eta}^g(dx) + \int_{\Delta} W(z) \hat{\eta}^i(dz) = W(x_0) + \int_{K^g} qW(x, a) \eta^g(dx, da) + \int_{K^i} QW(z, b) \eta^i(dz, db).
$$

with $\hat{\eta}^g$ (resp. $\hat{\eta}^i$) denotes the marginal of $\eta^g$ (resp. $\eta^i$) w.r.t. to $X$. 
Occupation and admissible measures

The next important result shows the link between the set of admissible measures and the set of occupation measures.

Theorem
Suppose Assumption A is satisfied. Then the following assertions hold.

i) For any control strategy $u \in \mathcal{U}$, the occupation measure $\eta_u$ is admissible.

ii) Suppose that the measure $\eta$ is admissible. Then there exist stochastic kernels $\pi \in \mathcal{P}^g$ and $\gamma \in \mathcal{P}^i$ for which the stationary control strategy $u = (\pi, \gamma) \in \mathcal{U}_s$ satisfies $\eta = \eta_u$. 
The constrained linear program, labeled $\mathbb{LP}$, is defined as

$$\inf_{(\eta^g,\eta^i) \in \mathbb{M}} \eta^g(C_0^g) + \eta^i(C_0^i)$$

where $\mathbb{M}$ is the set of measures $(\eta^g,\eta^i)$ in $\mathcal{M}(K^i) \times \mathcal{M}(K^g)$ such that $\eta^g + \eta^i$ is admissible and satisfies

$$\eta^g(C_j^g) + \eta^i(C_j^i) \leq B_j.$$
Linear programming approach

Theorem
Suppose Assumption A holds and the cost functions $C^g_j$ and $C^i_j$ are bounded from below for any $j \in \mathbb{N}_p$. Then the values of the constrained control problem and the linear program $\mathbb{LP}$ are equivalent:

$$\inf_{\eta^g, \eta^i \in \mathcal{M}} \eta^g(C^g_0) + \eta^i(C^i_0) = \inf_{u \in \mathcal{U}^f} \mathcal{V}_0(u, x_0).$$
Linear programming approach

Theorem

Suppose Assumptions A, B and (C1) are satisfied. Assume the cost functions \( C_g^j \) (resp. \( C_i^j \)) are bounded from below and lower semicontinuous on \( K_g^j \) (resp. \( K_i^j \)) for any \( j \in \mathbb{N}_p \).

If the set of feasible strategies is non empty then the \( \text{LP} \) is solvable and there exists a stationary feasible strategy \( u^* \) satisfying

\[
\eta_{u^*}^g(C_0^g) + \eta_{u^*}^i(C_0^i) = \inf_{(\eta^g, \eta^i) \in M} \eta^g(C_0^g) + \eta^i(C_0^i) = \inf_{u \in U^f} \mathcal{V}_0(u, x_0) = \mathcal{V}_0(u^*, x_0).
\]