Retour exponentiel à l'équilibre pour l'équation de croissance-fragmentation

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joint work with Étienne Bernard

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The growth-fragmentation equation

Consider a population of individuals structured by their size $x > 0$ (*e.g.* cells or polymers) which:

- grow in a deterministic way with the speed $\tau(x)$
- split randomly with the rate $B(x)$.

The splitting produces new individuals of size zx $(0 < z < 1)$ according to the kernel $\wp(z)$ which is a positive measure on $[0, 1]$.

The distribution of size $f(t, x)$ satisfies the following integro-PDE:

$$\partial_t f(t,x) = -\partial_x\big(\tau(x)f(t,x)\big) - B(x)f(t,x) + \int_0^1 B\Big(\frac{x}{z}\Big)f\Big(t,\frac{x}{z}\Big)\frac{\wp(dz)}{z}$$

In this talk we will consider a constant growth rate $\tau(x) \equiv \tau > 0$.

The growth-fragmentation equation

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To guarantee the size conservation during the fragmentation, we consider kernels $\wp$ which satisfy $\int_0^1 z\, \wp(dz) = 1$.

Examples: $\wp = 2\,\delta_{\frac{1}{2}}$ (mitosis), $\wp \equiv 2$ (homogeneous fragmentation)

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Then the size balance only evolves by growth

$$\frac{d}{dt}\int_0^\infty x\,f(t,x)\,dx = \int_0^\infty \tau f(t,x)\,dx$$

and the total number of individuals only evolves by fragmentation

$$\frac{d}{dt}\int_0^\infty f(t,x)\,dx = \big(\wp([0,1]) - 1\big)\int_0^\infty B(x)f(t,x)\,dx.$$

Eigenelements (Perthame and Ryzhik 2003 ; Michel 2005 ; Doumic and G. 2010)

The growth-fragmentation operator

$$Lf(x) = -\tau f'(x) - B(x)f(x) + \int_0^1 B\left(\frac{x}{z}\right) f\left(\frac{x}{z}\right) \frac{\wp(dz)}{z}$$

has a principal eigenvalue $\lambda > 0$ associated with a positive eigenfunction G, unique up to normalization, *i.e.* a unique solution to

$$LG = \lambda G, \qquad G \geqslant 0, \qquad \int_0^\infty G(x)\,dx = 1.$$

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There exists also a unique solution ϕ to the dual problem

$$L^* \phi = \lambda \phi, \qquad \phi \geqslant 0, \qquad \int_0^\infty G(x)\phi(x)\, dx = 1$$

where

$$L^*\varphi(x) = \tau \varphi'(x) + B(x)\left[\int_0^1 \varphi(zx)\wp(dz) - \varphi(x)\right].$$

The rescaled equation

To investigate the long-time profile of the solutions, it is convinient to consider the *rescaled* growth-fragmentation equation

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The eigenfunction G is a **steady-state** for the rescaled growth-fragmentation equation.

General Relative Entropy (Michel, Mischler and Perthame 2005)

For $H : \mathbb{R} \to \mathbb{R}$ define the functional

$$\mathcal{H}[g] = \int_0^\infty \phi(x) G(x) H\left(\frac{g(x)}{G(x)}\right) dx$$

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and compute for $g(t,x)$ a solution to the rescaled equation

$$\frac{d}{dt}\mathcal{H}[g(t,\cdot)] = -\mathcal{D}[g(t,\cdot)]$$

where

$$\mathcal{D}[g] = \int_0^\infty \int_0^1 B(x)\phi(zx)G(x) \times \Big[H\Big(\frac{g(zx)}{G(zx)}\Big) - H\Big(\frac{g(x)}{G(x)}\Big) + H'\Big(\frac{g(x)}{G(x)}\Big)\Big(\frac{g(x)}{G(x)} - \frac{g(zx)}{G(zx)}\Big)\Big]\wp(dz)dx.$$

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If H is **convex**, this functional is an **entropy**

$$\frac{d}{dt}\mathcal{H}[g(t,\cdot)] \leqslant 0.$$

Conservation law and convergence

Particular choices of convex functions H :

- using $H(x) = x$ we get the conservation law

$$\forall t \geqslant 0, \quad \int_0^\infty g(t,x)\phi(x)\,dx = \int_0^\infty g(0,x)\phi(x)\,dx := \varrho$$

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- for $H(x) = |\varrho - x|^p$ the entropy $\mathcal{H}_p$ is the p-power of a weighted L^p distance to the equilibrium ϱG

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Theorem (Michel, Mischler and Perthame 2005)

$$\|g(t,\cdot) - \varrho G\|_{L^p(\phi G^{1-p}dx)} \underset{t\to+\infty}{\searrow} 0$$

Exponential decay by Poincaré inequality

If one is able to prove the functional inequality

$$\nu\,\mathcal{H} \leqslant \mathcal{D}$$

for some constant $\nu > 0$, then the Grönwall lemma gives immediatly

$$\mathcal{H}[g(t,\cdot)] \leqslant \mathcal{H}[g(0,\cdot)]\,e^{-\nu t}.$$

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Theorem (Cáceres, Cañizo & Mischler 2011 ; Balagué, Cañizo & G. 2013)

Assmue that $\wp(z) \geqslant \underline{\wp} > 0$. There exists $\nu > 0$ such that

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and so

$$\|g(t,\cdot) - \varrho G\|_{L^2(\phi G^{-1} dx)} \leqslant e^{-\nu t}\, \|g(0,\cdot) - \varrho G\|_{L^2(\phi G^{-1} dx)}$$

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Is such a Poincaré inequality valid for $\mathcal{H}_1$???

Exponential decay in $L^1_\phi = L^1$

Theorem (Perthame and Ryzhik 2003 ; Laurençot and Perthame 2009 ; Bardet, Christen, Guillin, Malrieu and Zitt 2013)

Assume that B is constant.
There exist explicit constants $C > 0$ and $\nu > 0$ such that

$$\|g(t,\cdot) - \varrho G\|_{L^1} \leqslant Ce^{-\nu t}\big(\|g(0,\cdot) - \varrho G\|_{L^1} + \|M\|_{L^1}\big)$$

where

$$M(x) = \int_0^x \big(g(0,y) - \varrho G(y)\big)\,dy.$$

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Exponential decay in smaller weighted L^1 spaces

For stronger weights than the dual eigenfunction ϕ, the exponential decay occurs for general coefficients.

<u>Theorem</u> (Mischler and Scher 2013 ; Grigorescu and Kang 2012)

For $\psi(x) = 1 + x^\alpha$ with α larger than an explicit value $\alpha^* \geqslant 1$, there exist $C > 0$ and $\nu > 0$ such that

$$\|g(t,\cdot) - \varrho G\|_{L^1_\psi} \leqslant Ce^{-\nu t}\|g(0,\cdot) - \varrho G\|_{L^1_\psi}.$$

Notice that $L^1_\psi \subsetneq L^1_\phi$ since $\phi(x) \leqslant C(1 + x)$.

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Question: can we have the same result in L^1_ϕ ?

Semigroup formulation of the problem

Denote by $(T_t)_{t\geqslant 0}$ the semigroup generated by $L-\lambda$

$$T_t\, g = e^{t(L-\lambda)} g$$

and by P the projector on $Span(G)$

$$Pg = \left(\int g\phi\right) G.$$

Recall that from the GRE we have the punctual convergence of T_t to P

$$\forall g \in L^1_\phi, \qquad \|T_t g - Pg\|_{L^1_\phi} \underset{t\to+\infty}{\searrow} 0.$$

Question: do we have uniform convergence

$$|||\, T_t - P \,|||_{\mathcal{L}(L^1_\phi)} \xrightarrow[t\to+\infty]{} 0 \quad ?$$

$\longrightarrow$ The answer depends on the fragmentation rate B.

Main results

Theorem (Bernard and G.)

Assume that the fragmentation rate B is **bounded** and that

$$\exists B_\infty > 0, \qquad B(x) = B_\infty \qquad \text{for } x \text{ large enough.}$$

Then

$$\forall t \geqslant 0, \qquad |||T_t - P|||_{\mathcal{L}(L^1_\phi)} \geqslant 1.$$

Theorem (Bernard and G.)

Assume that

$$\exists \gamma, B_\infty > 0, \qquad B(x) \sim B_\infty x^\gamma \qquad \text{when } x \to +\infty.$$

Then

$$\exists \nu, C > 0, \qquad \forall t \geqslant 0, \qquad |||T_t - P|||_{\mathcal{L}(L^1_\phi)} \leqslant Ce^{-\nu t}.$$

Merci pour votre attention !