On Jump/Diffusion and Jump-Diffusion Stochastic Differential Equations in Open Quantum System

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Overview

- **I) Mathematical Formalism of Quantum Mechanics**
  - Axioms of Quantum Mechanics
  - Quantum Indirect Measurement: Discrete Time Quantum Trajectories
- **II) Continuous Time model: Stochastic Schrödinger Equations, Stochastic Master Equations.**
  - Diffusion Model
  - Piecewise Deterministic Model
  - Jump-Diffusion Model
- **III) From Discrete to Continuous Time Model**

I) Mathematical Formalism of Quantum Mechanics

DISCRETE TIME QUANTUM TRAJECTORIES

Axioms

- 1st axiom A quantum system is described by a complex Hilbert space $\mathcal{H}$. We shall only consider finite dimensional space $\mathcal{H} = \mathbb{C}^N$
- 2nd axiom A state of quantum system is described by a **state** $\rho \in \mathcal{S}$ where

$$\mathcal{S} = \{\rho \in \mathbb{M}_N(\mathbb{C}) | \rho = \rho^*, \rho \geqslant 0, \mathrm{Tr}(\rho) = 1\}$$

- 3rd axiom The evolution of a quantum system is described by an Hamiltonian H. We have the celebrated **Schrödinger Equation**

$$d\rho(t) = -i[H, \rho(t)]dt,$$

where $[A, B] = AB - BA$

$$\rho \to \rho(t) = U_t \rho U_t^*$$

where $U_t = e^{-itH}$.

Axioms

- 4th axiom **Measurement**: A physical quantity is described by an **observable** A which is a self-adjoint operator $A = A^*$. In particular we get the spectral decomposition

$$A = \sum_{i=1}^{p} \lambda_i \, P_i, \quad \text{with} \quad P_i P_j = \delta_{ij} P_i$$

A measurement of the observable A gives a random result on the spectrum of A

$$\mathbb{P}[\text{to observe } \lambda_i] = \mathrm{Tr}[\rho P_i]$$

Remark: since $\sum_i P_i = I$ and $P_i P_i^* \geqslant 0$

$$\mathrm{Tr}[\rho P_i] = \mathrm{Tr}[\rho P_i P_i^*] \geqslant 0 \qquad (1)$$

$$\sum_i \mathrm{Tr}[\rho P_i] = \mathrm{Tr}[\rho] = 1 \qquad (2)$$

Wavepacket Reduction

- Conditionaly to the result of the observation of A, the state of the system is modified. Assume we have observed λ_i the state ρ becomes

$$\rho \to \frac{P_i \rho P_i}{\mathrm{Tr}[\rho P_i]}$$

- The result of a measurement gives rise of a random evolution of the state

$$\rho \to \rho_i = \frac{P_i \rho P_i}{\mathrm{Tr}[\rho P_i]},$$

with probability $p_i = \mathrm{Tr}[\rho P_i]$ (Schrödinger cat).

- What happens if you want to make a **second observation** of A. Since $P_i P_j = \delta_{ij} P_i$, if the first result was λ_i the second result will be λ_i with probability 1.

$$\mathbb{P}[\text{to observe } \lambda_j] = \mathrm{Tr}[\rho_i P_j] = \mathrm{Tr}\left[\frac{P_i \rho P_i}{\mathrm{Tr}[\rho P_i]} P_j\right] = \delta_{ij}$$

- Such measurement is called direct measurement.

Quantum Zenon effect

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Quantum Zenon effect

QUANTUM REPEATED INDIRECT MEASUREMENT

Coupling

- Consider a **coupling** of $\mathcal{H}$ with an auxiliary system $\mathcal{E}$. The coupling system is described by the Hilbert space $\mathcal{H} \otimes \mathcal{E}$. Consider an initial state ρ of $\mathcal{H}$ and a reference state β of $\mathcal{E}$. The initial state of $\mathcal{H} \otimes \mathcal{E}$ is $\rho \otimes \beta$
- Interaction: total Hamiltonian H_{tot}:

$$H_{\mathrm{tot}} = H_S \otimes I + I \otimes H_E + \lambda H_{\mathrm{int}}$$

  where H_{int} is an Hamiltonian of interaction.
- Interaction during a time τ:

$$\rho \otimes \beta \to U(\rho \otimes \beta)U^*,$$

  where $U = e^{-i\tau H_{\mathrm{tot}}}$
- In general $U(\rho \otimes \beta)U^*$ is not a product of operator => **quantum entanglement**

Indirect Measurement

- Let $A = \sum \lambda_i P_i$ be an observable of $\mathcal{E}$ that we want to observe. We promote it as observable of $\mathcal{H} \otimes \mathcal{E}$ with

$$I \otimes A = \sum_i \lambda_i (I \otimes P_i)$$

- Probability distribution:

$$\mathbb{P}[\text{to observe } \lambda_i] = \mathrm{Tr}[U(\rho \otimes \beta)U^* \, I \otimes P_i]$$

- Wave Packet reduction:

$$\rho \otimes \beta \to \mu_1(i) = \frac{I \otimes P_i \, U(\rho \otimes \beta)U^* \, I \otimes P_i}{\mathrm{Tr}[U(\rho \otimes \beta)U^* \, I \otimes P_i]}$$

- Partial trace over $\mathcal{E}$: $\mathrm{Tr}_{\mathcal{E}}$

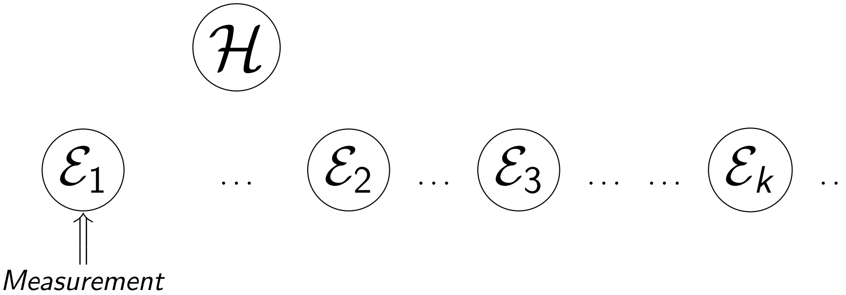
$$\rho_1(i) = \mathrm{Tr}_{\mathcal{E}}[\mu_1(i)] = \frac{\mathcal{L}_i(\rho)}{\mathrm{Tr}[\mathcal{L}_i(\rho)]}$$

- This gives the transition (in general $\mathcal{L}_j \circ \mathcal{L}_i \neq \delta_{ij}\mathcal{L}_i$). The system $\mathcal{E}$ disappears and a second copy can appear and we can consider a similar framework.

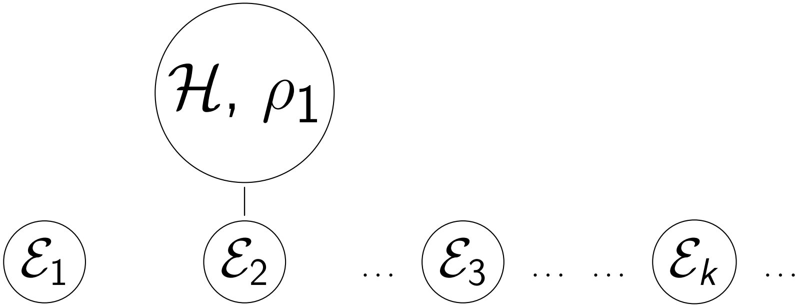
1st interaction

We put our system in contact with an infinite chain of auxiliary quantum systems

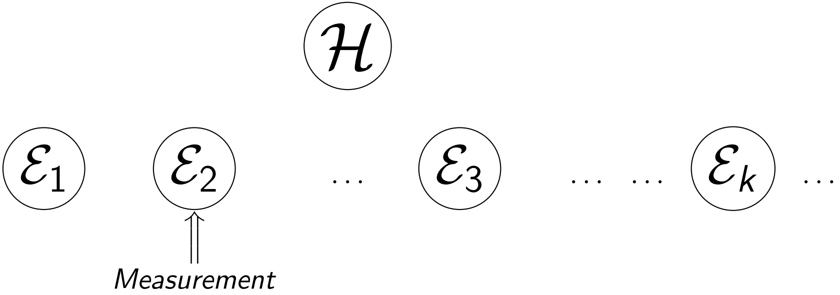
1st measurement



2nd interaction



2nd measurement



and so on $\Rightarrow$ discrete quantum trajectory (ρ_n)

Proposition

Let A be an observable of the form $A = \sum_{i=0}^{p} \lambda_i P_i$.

Then there exists a probability space $(\Omega, \mathcal{C}, P)$, where the the discrete quantum trajectory (ρ_k), describing the quantum repeated measurement of A, is a Markov chain.

More precisely if $\rho_k = \theta$ is a state on $\mathcal{H}_0$, then ρ_{k+1} takes the values

$$\frac{\mathcal{L}_i(\theta)}{Tr[\mathcal{L}_i(\theta)]}, \quad i \in \{0, \ldots, p\}$$

where $\mathcal{L}_i(\theta) = Tr_{\mathcal{H}}[(I \otimes P_i)\ U(\theta \otimes \beta)U^\star\ (I \otimes P_i)]$. Each state appears with probability

$$p_i(\theta) = Tr[\mathcal{L}_i(\theta)].$$

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$$p_i(\theta) = Tr[\mathcal{L}_i(\theta)].$$

- The previous result can be summarized by writing the following evolution equation:

$$\rho_{k+1} = \sum_{i=0}^{p} \frac{\mathcal{L}_i(\rho_k)}{Tr[\mathcal{L}_i(\rho_k)]} \mathbf{1}_i^{k+1},$$

with $\mathcal{L}_i(\theta) = \mathrm{Tr}_{\mathcal{H}}[(I \otimes P_i)\ U(\theta \otimes \beta)U^\star\ (I \otimes P_i)]$.

- **Remark**: The operator U depends on the time interaction τ

$$U = e^{-i\tau H_{tot}}.$$

Remark

What gives the limit τ goes to 0?

What is the limit evolution?

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II) Stochastic Master Equations
Continuous Quantum Trajectories

- Let us start with model where only one measurement apparatus is concerned
- Evolution of $\mathcal{H}_0$ without measurement : Master Equation in Lindblad form
$$d\rho_t = L(\rho_t)dt.$$
- Effect of measurement = perturbation of this ode under the form of stochastic differential equations

Stochastic Master Equations

Diffusive Equation

- **Diffusive Equation**

$$d\rho_t = L(\rho_t)dt + [C\rho_t + \rho_t C^\star - Tr[\rho_t(C + C^\star)]\rho_t]dW_t$$

1. The process (W_t) is a standard Brownian motion.
2. C is an arbitrary operator appearing in the Lindblad operator, L.

- Often this equation appears on the following form

$$\begin{aligned} d\rho_t &= L(\rho_t)dt \\ &\quad + [C\rho_t + \rho_t C^\star - Tr[\rho_t(C + C^\star)]\rho_t](dy_t - \mathrm{Tr}[\rho_t(C + C^\star)]dt), \end{aligned}$$

where

$$dy_t = dW_t + \mathrm{Tr}[\rho_t(C + C^\star)]dt$$

- The process (y_t) represents the measurement process recorded by the measurement apparatus (Homodyne/Heterodyne detection in Quantum Optics).

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Jump Equation

- **Jump Equation**

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right]\left(d\tilde{N}_t - Tr\left[\mathcal{J}(\rho_t)\right]dt\right)$$

  1. The process $(\tilde{N}_t)$ is a counting process of stochastic intensity

$$t \longrightarrow \int_0^t Tr[\mathcal{J}(\rho_s)]ds.$$

  2. $\mathcal{J}(\rho) = C\rho C^\star$.

- The process $(\tilde{N}_t)$ represents the number of photon detected up to time t by a photo detector.

- First fact: $d\mathbb{E}[\rho(t)] = L(\mathbb{E}[\rho(t)])dt$. That is $(\mathbb{E}[\rho(t)])$ reproduces the solution of the Lindblad master equation
- In the previous cases if

$$L(\rho) = -i[H,\rho] - \frac{1}{2}\{C^\star C,\rho\} + C\rho C^\star$$

and if at time 0 $\rho_0 = |\psi_0\rangle\langle\psi_0|$ then there exists ψ_t such that

$$\rho(t) = |\psi_t\rangle\langle\psi_t|, \forall t$$

The equation satisfied by ψ_t is called a Stochastic Schrödinger Equation

- In general we have $\rho(t) = \rho^*(t)$ and $\mathrm{Tr}[\rho(t)] = 1$, then if there is a solution and if the initial condition is a density matrix then the solution is self-adjoint and of trace 1. What is very difficult to show is that the solution is positive.

Existence and uniqueness: the diffusive case

Theorem

Let $(\Omega, \mathcal{F}, P)$ a probability space where (W_t) is a standard Brownian motion.

the equation

$$\begin{aligned} d\rho_t &= L(\rho_t)dt \\ &\quad + \Big[C\rho_t + \rho_t C^\star - Tr[\rho_t(C + C^\star)]\rho_t\Big] dW_t \end{aligned}$$

admits a unique solution (ρ_t) with values in the set of states of $\mathcal{H}_0$.

- Non Lipschitz coefficients

  One can use a truncature method

- We show that this equation preserves the property of being a state.
- Here we can use the approximation procedure to show the positivity of the solution

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Processus de comptage $(\tilde{N}_t)$?

- Jump equation

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right](d\tilde{N}_t - Tr[\mathcal{J}(\rho_t)]dt)$$

Recall: $(\tilde{N}_t)$ is a counting process of intensity $\int_0^t Tr[\mathcal{J}(\rho_s)]ds$.

Remark

This is clearly a PDMP.

Existence and uniqueness

Theorem

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space equipped with a random Poisson measure μ on $\mathbb{R}^+ \times \mathbb{R}$ whose the intensity measure is the Lebesgue measure $ds \otimes dx$. The SDE

$$\rho_t = \rho_0 + \int_0^t \left[L(\rho_{s-}) - \mathcal{J}(\rho_{s-}) + Tr[\mathcal{J}(\rho_{s-})]\rho_{s-} \right] ds$$
$$+ \int_0^t \int_{\mathbb{R}} \left[\frac{\mathcal{J}(\rho_{s-})}{Tr[\mathcal{J}(\rho_{s-})]} - \rho_{s-} \right] \mathbf{1}_{0<x<tr[\mathcal{J}(\rho_{s-})]} \, \mu(ds, dx)$$

admits a unique solution (ρ_t). The process $(\tilde{N}_t)$ defined by

$$\tilde{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<tr[\mathcal{J}(\rho_{s-})]} \, \mu(ds, dx)$$

is a counting process of intensity $\int_0^t Tr[\mathcal{J}(\rho_{s-})]ds$.

Generalisation

One can generalise

$$\begin{aligned}\rho_t &= \rho_0 + \int L(\rho_{s-})ds + \sum_{i=0}^{l} \int_0^t h_i(\rho_{s-})dW_i(s) \\ &\quad + \sum_{i=0}^{q} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-})\mathbf{1}_{0<x<v_i(\rho_{s-})} \left[\mu_i(dx, ds) - dxds\right],\end{aligned}$$

where $(W_t = (W_0(t), \ldots, W_p(t))$ are a p-dimensional Brownian motion and μ_i are $p+1$ random measure of intensity $ds \otimes dx$. All the processes are independent.

Remark The functions h_i et g_i are given by

$$\begin{aligned}h_i(\rho) &= C_i\rho + \rho C_i^\star - Tr[\rho(C_i + C_i^\star)]\rho \\ g_i(\rho) &= \frac{\mathcal{J}_i(\rho)}{Tr[\mathcal{J}_i(\rho)]} - \rho\end{aligned}$$

III) Convergence Result

FROM DISCRETE TO CONTINUOUS QUANTUM TRAJECTORIES

Back to the discrete setup

- Setup
  1. Case $\mathcal{H} = \mathcal{E} = \mathbb{C}^2$
  2. An observable of $\mathcal{H}$ is of the form $A = \lambda_0 P_0 + \lambda_1 P_1$.
  3. The discrete stochastic Schrödinger equation

$$\rho_{k+1} = \frac{\mathcal{L}_0(\rho_k)}{p_0(\rho_k)} \mathbf{1}_0^{k+1} + \frac{\mathcal{L}_1(\rho_k)}{p_1(\rho_k)} \mathbf{1}_1^{k+1}.$$

- Let us introduce $X_{k+1} = \dfrac{\mathbf{1}_1^{k+1} - p_1(\rho_k)}{\sqrt{p_0(\rho_k) p_1(\rho_k)}}$.
- In terms of X_{k+1}, we get

$$\rho_{k+1} = \mathcal{L}_0(\rho_k) + \mathcal{L}_1(\rho_k) + \left[-\sqrt{\frac{p_0}{p_1}} \mathcal{L}_0(\rho_k) + \sqrt{\frac{p_1}{p_0}} \mathcal{L}_1(\rho_k) \right] X_{k+1}.$$

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Asymptotic assumptions

- Recall that (ρ_k) is defined through the quantity

$$\mathcal{L}_i(\rho) = \mathrm{Tr}_{\mathcal{H}}[\,(I \otimes P_i)\;\; U(n)(\rho \otimes \beta)U^\star(n)\;\; (I \otimes P_i)\,]$$

$$U(n) = e^{i\frac{1}{n}H_{tot}}$$

Naturally the asymptotic assumptions are going to appear in $U(n)$.

- Now, we fix a basis $\{\Omega_0, \Omega_1\}$
The reference state of the chain will be $\beta = |\Omega_0\rangle\langle\Omega_0|$.

Expression de U(n)

- Let us write $U(n)$ as

$$U(n) = \begin{pmatrix} U_0^0(n) & U_0^1(n) \\ U_1^0(n) & U_1^1(n) \end{pmatrix}$$

  where the $U_{ij}(n)$ are operators $\mathcal{H}_0$.
- **S. Attal-Y. Pautrat**: From repeated to continuous quantum interactions, "*Annales Henri Poincaré*"
  In the previous article, the authors gives a precise description of the asymptotic conditions that we need to impose to $U_{ij}(n) \Longrightarrow$ in order to obtain a non-trivial limit for the quantum repeated interactions model (interms of quantum stochastic calculus).
- In our context, we naturally adopt their conditions and we need

$$\begin{aligned} U_0^0(n) &= I + \frac{1}{n}\left(-iH_0 - \frac{1}{2}C^\star C\right) + \circ\left(\frac{1}{n}\right) \\ U_1^0(n) &= \frac{1}{\sqrt{n}}C + \circ\left(\frac{1}{n}\right) \end{aligned}$$

Limit evolution in the case of a diagonal A

- If A is **diagonal** in $\{\Omega_0, \Omega_1\}$. For example $A = 1 \times |\Omega_0\rangle\langle\Omega_0| + 0 \times |\Omega_1\rangle\langle\Omega_1|$, then we have

$$\mathcal{L}_0(\rho) = U_0^0 \rho (U_0^0)^\star = \rho + \frac{1}{n}\left[(-iH_0 - \frac{1}{2}C^\star C)\rho + \rho(-iH_0 - \frac{1}{2}C^\star C)\right]$$

$$\mathcal{L}_1(\rho) = U_1^0 \rho (U_1^0)^\star = \frac{1}{n} C \rho C^\star$$

The **transition probabilities** satisfies

$$\begin{aligned} p_0(\rho_k) &= 1 - \frac{1}{n} Tr[\mathcal{J}(\rho_k)] + \circ\left(\frac{1}{n}\right) \\ p_1(\rho_k) &= \frac{1}{n} Tr[\mathcal{J}(\rho_k)] + \circ\left(\frac{1}{n}\right). \end{aligned}$$

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$$\begin{aligned}\rho_{k+1} &= \rho_k + \frac{1}{n}[L(\rho_k) + \circ(1)] \\ &\quad + \left(\frac{\mathcal{J}(\rho_k)}{Tr[\mathcal{J}(\rho_k)]} - \rho_k + \circ(1)\right)(\mathbf{1}_1^{k+1} - p_1(\rho_k)).\end{aligned}$$

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In case of a non diagonal A

- If $A = \lambda_0 P_0 + \lambda_1 P_1$ de $\mathcal{H}$ **is not diagonal** in $\{\Omega_0, \Omega_1\}$. For example $A = |\Omega_0\rangle\langle\Omega_1| + |\Omega_1\rangle\langle\Omega_0|$ we get the following asymptotic expression

$$\mathcal{L}_0(\rho) = \frac{1}{2}\left(U_0^0\rho(U_0^0)^\star + U_0^0\rho(U_1^0)^\star + U_1^0\rho(U_0^0)^\star + U_1^0\rho(U_1^0)^\star\right) \quad (3)$$

$$= \frac{1}{2}\left(\rho + \frac{1}{\sqrt{n}}(C\rho + \rho C^\star) + \frac{1}{n}L(\rho)\right) \quad (4)$$

$$\mathcal{L}_1(\rho) = \frac{1}{2}\left(U_0^0\rho(U_0^0)^\star - U_0^0\rho(U_1^0)^\star - U_1^0\rho(U_0^0)^\star + U_1^0\rho(U_1^0)^\star\right)$$

- Here the **probabilities** are

$$p_0(\rho_k) = \frac{1}{2} + \frac{1}{\sqrt{n}}\left[Tr[\rho_k(C + C^\star)] + \circ(1)\right]$$

$$p_1(\rho_k) = 1 - p_0(\rho_k).$$

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$$\rho_{k+1} = \rho_k + \frac{1}{n}[L(\rho_k) + \circ(1)]$$
$$+\frac{1}{\sqrt{n}}\Big[C\rho_k + \rho_k C^\star - Tr[\rho_k(C + C^\star)]\rho_k + \circ(1)\Big]X_{k+1}.$$

Here the **probabilities** are

$$\begin{aligned} p_0(\rho_k) &= \xi + \frac{1}{\sqrt{n}}\nu\Big[Tr[\rho_k(C + C^\star)] + \circ(1)\Big] \\ p_1(\rho_k) &= 1 - p_0(\rho_k). \end{aligned}$$

Convergence to the diffusive case

- From the previous description, we put

$$\rho_{[nt]} = \rho_0 + \sum_{k=0}^{[nt]-1} \frac{1}{n}[L(\rho_k) + \circ(1)] + \sum_{k=0}^{[nt]-1} \frac{1}{\sqrt{n}}[\mathcal{H}(\rho_k) + \circ(1)]X_{k+1}.$$

- Putting

$$\rho_n(t) = \rho_{[nt]}, \qquad V_n(t) = \frac{[nt]}{n}, \qquad W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]-1} X_{k+1}.$$

- We have that $(\rho_n(t))$ satisfies

$$\rho_n(t) = \rho_0 + \int_0^t L(\rho_n(s-))dV_n(s) + \int_0^t \mathcal{H}(\rho_n(s-))dW_n(s) + \varepsilon_n(t).$$

Convergence - Kurtz-Protter

Theorem

The process $(W_n(t), V_n(t), \varepsilon_n(t))$ *converge in distribution to* $(W_t, V_t, 0)$ *where* (W_t) *is a standard Brownian motion and* $V_t = t$ *for all* t.
Moreover, we have

$$\sup_n \mathbf{E}\Big[[W_n(t), W_n(t)]\Big] < \infty$$

Then the process $(\rho_n(t))$ *satisfying*

$$\rho_n(t) = \rho_0 + \int_0^t L(\rho_n(s-))dV_n(s) + \int_0^t \mathcal{H}(\rho_n(s-))dW_n(s) + \varepsilon_n(t)$$

converge in ditribution to (ρ_t) *the unique solution of*

$$\rho_t = \rho_0 + \int_0^t L(\rho_s)ds + \int_0^t \mathcal{H}(\rho_s)dW_s.$$

The jump case

- Again

$$\rho_{[nt]} = \rho_0 + \sum_{k=0}^{[nt]-1} \frac{1}{n}\Big[L(\rho_k) - \mathcal{J}(\rho_k) + Tr[\mathcal{J}(\rho_k)]\rho_k + \circ(1)\Big] + \sum_{k=0}^{[nt]-1}\left(\frac{\mathcal{J}(\rho_k)}{Tr[\mathcal{J}(\rho_k)]} - \rho_k + \circ(1)\right)\mathbf{1}_1^{k+1}.$$

- Again, we put

$$\rho_n(t) = \rho_{[nt]}\,, \qquad V_n(t) = \frac{[nt]}{n}\,, \qquad N_n(t) = \sum_{k=0}^{[nt]-1} \mathbf{1}_1^{k+1}.$$

- We get the discrete SDE

$$\rho_n(t) = \rho_0 + \int_0^t \Theta(\rho_n(s-))dV_n(s) + \int_0^t \Phi(\rho_n(s-))dN_n(s) + \varepsilon_n(t).$$

Convergence

- Kurtz-Protter?

- In the jump case, **we can not directly show that** $(N_n(t))$ converge in distribution to the process $(\tilde{N}_t)$.
- Method:
  1. Coupling.
  2. Comparison with a Euler scheme.

Theorem

The process $(\rho_n(t))$ defined from the quantum repeated measurement of a diagonal observable converge in distribution to (ρ_t) solution of the jump equation.

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General case

- How can we compare the two method?
- How can we show that

$$\rho_{k+1} = \sum_{i=0}^{p} \frac{\mathcal{L}_i(\rho_k)}{Tr[\mathcal{L}_i(\rho_k)]} \mathbf{1}_i^{k+1}$$

converges to

$$\begin{aligned} \rho_t &= \rho_0 + \int L(\rho_{s-})ds + \sum_{i=0}^{l} \int_0^t h_i(\rho_{s-})dW_i(s) \\ &\quad + \sum_{i=0}^{q} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-})\mathbf{1}_{0<x<v_i(\rho_{s-})} \left[\mu_i(dx, ds) - dxds\right]. \end{aligned}$$

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Conclusion

- Long time behaviour
- Estimation

MERCI