Risk processes in dimension 2.

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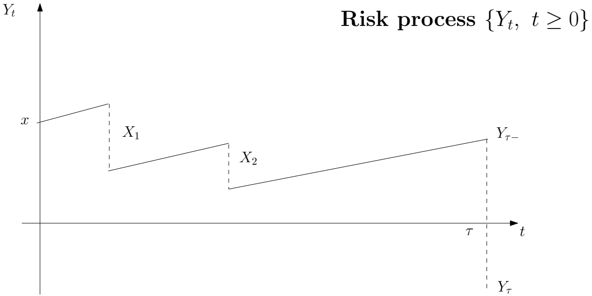
Content

Content

Sparre Andersen model (1957)



$$Y_t = x + pt - S_t, \quad S_t = \sum_{i=1}^{N_t} X_i,$$

x initial reserve, p premium rate, X_i i.i.d. claim amounts, $\{N_t,\ t \geq 0\}$ renewal process.

$$\boxed{\tau := \inf\{t \geq 0|\ Y_t < 0\}.}$$

Safety loading $\rho := \frac{p}{E(S_1)} - 1$ is positive.

More general model and Gerber-Shiu function

$$\begin{cases} dY_t &= b(Y_t)dt + \sigma(Y_t)dB_t - dS_t \\ Y_0 &= x \end{cases}$$

where $\{B_t,\ t \geq 0\}$ is a brownian motion. Ex.

- $b(Y_t) = \delta Y_t + p$, $\sigma(Y_t) = 0$ model with interest rate $\delta \geq 0$,
- $b(Y_t) = \delta Y_t + p$, $\sigma(Y_t) = \sigma > 0$ generalized Ornstein-Uhlenbeck.

Modulation by an *exterior environment* $\{X(t),\ t \geq 0\}$ is also possible : $b(X(t), Y_t)$, $\sigma(X(t), Y_t)$.

- Gerber-Shiu function $\boxed{E_x(e^{-\beta\tau}\mathbf{1}_{\{\tau<+\infty\}}w(Y_{\tau-}, Y_\tau)),}$
- Ruin probabilities :

$$\boxed{\psi(x, T) := P_x(\tau \leq T),\ \psi(x) := P_x(\tau < +\infty).}$$

Obtaining the ruin probability $\psi(x)$

In the case where $\{S_t,\ t \geq 0\}$ is a compound Poisson process :

- Establishing a renewal equation for $\psi(x)$ (by conditioning on the first claim),
- Establishing an integro-differential equation of the form

$$0 = b(x)\frac{\partial \bar{\psi}}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 \bar{\psi}}{\partial x^2} - \lambda \bar{\psi}(x) + \lambda \int_0^x \bar{\psi}(x-y)F(dy) \quad (1)$$

with boundary conditions $\bar{\psi}(0)$ and $\bar{\psi}(+\infty)$, where $\bar{\psi}(x) := 1 - \psi(x)$, and $F(.)$ distribution of claims.

But (1) not always solvable !

An alternative approach : duality with queues

In the case where $\sigma(.) = 0$, let $\{Q(t),\ t \geq 0\}$ be defined by

$$\begin{cases} dQ(t) &= -b(Q(t))dt + dS_t + dL_t \\ L_t &= \int_0^t \mathbf{1}_{\{Q(s)=0\}} dL_s, \end{cases}$$

with $Q(0) = 0$. Then

$$\boxed{\psi(x,t) = P(Q(t) > x)}$$

(Asmussen and Schock Petersen (89)).

$\longrightarrow$ Transforms a problem related to the infimum of a process on $[0,t]$ into a problem relating the distribution of a process at time t.
$\longrightarrow$ Link between queueing and ruin theory.

Content

Model

Capitals Y_t^1 and Y_t^2 of an insurer and reinsurer.
$\{Y_t = (Y_t^1, Y_t^2),\ t \geq 0\}$ satisfies

$$\begin{cases} dY_t &= b(Y_t)dt - dS(t) \\ Y_0 &= x = (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

Ex : $b(Y_t) = \delta Y_t + p$, $\delta \geq 0$ interest rate, $p = (p_1, p_2) \in \mathbb{R}_+^2$ premium rates,

$\{S(t) = (S_1(t), S_2(t)),\ t \geq 0\}$ claims process, of which components are *correlated* jump processes.

$$\boxed{\tau := \inf\{t \geq 0 |\ Y_t \in \mathcal{S}\}, \quad \text{where} \quad \mathcal{S} \subset \mathbb{R}^2,}$$

Relatively few works on this topic, see Avram et al (2008), Biard (2013), Hu and Jiang (2013),...

Examples of sets $\mathcal{S}$

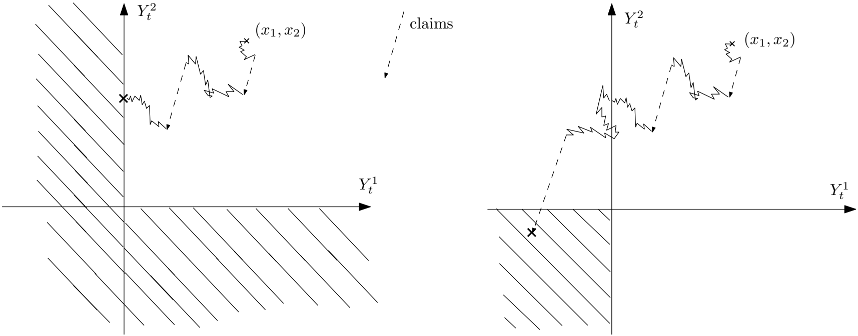


FIGURE : Sets $\mathcal{S} = \mathbb{R}^2 \setminus [0, +\infty)^2$ and $\mathcal{S} = (-\infty, 0)^2$.

Content

A model with interest rate : R.(2009)

$$\begin{cases} dY_t^1 = \delta Y_t^1 dt + p_1 dt - \alpha ds(t) \\ dY_t^2 = \delta Y_t^2 dt + p_2 dt - (1-\alpha) ds(t) \\ Y_0 = (Y_0^1, Y_0^2) = (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

where $\{s(t),\ t \geq 0\}$ is a compound Poisson process, and $\alpha \in (0,1)$. We let

- $\tau_i := \inf\{t \geq 0 |\ Y_t^i < 0\}$, $i = 1, 2$,
- $s_i^* := -p_i/\delta$, $i = 1, 2$, *definitive ruin* threshold.

NB : cdf's $P_{x_i}(\tau_i \leq t)$, $i = 1, 2$, are available e.g. in Wu et al (2005).

Goal : Determine closed expression of

$$\boxed{\psi(x_1, x_2, t) = P_{(x_1,x_2)}(\tau \leq t)}.$$

Here $\tau := \inf\{t \geq 0|\ Y_t \in \mathbb{R}^2 \setminus [0, +\infty)^2\} = \min(\tau_1, \tau_2)$.

How to obtain $\psi(x_1, x_2, t)$?

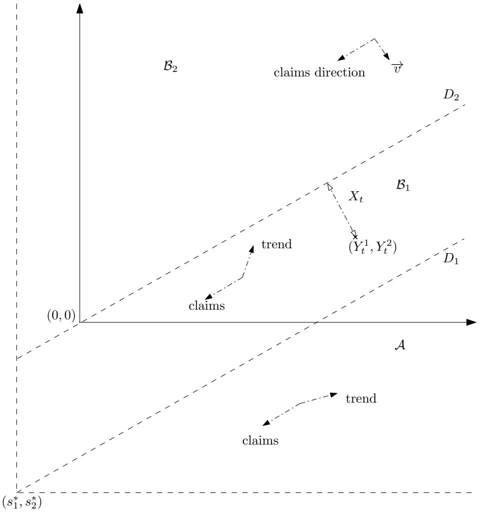


FIGURE : Partitioning $[s_1^*, +\infty) \times [s_2^*, +\infty)$.

How to obtain $\psi(x_1, x_2, t)$?

Let $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$ defined as on graph :

- $[s_1^*, +\infty) \times [s_2^*, +\infty) = \mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2$,
- one can prove that $\mathcal{A}$ and $\mathcal{B}_2$ are absorbing,
- $\mathcal{B}_1$ is transient.

$\Longrightarrow$ If $x = (x_1, x_2) \in \mathcal{A}$ then
$\psi(x_1, x_2, t) = P_{(x_1,x_2)}(\tau \leq t) = P_{x_2}(\tau_2 \leq t)$.

$\Longrightarrow$ If $x = (x_1, x_2) \in \mathcal{B}_2$ then $\psi(x_1, x_2, t) = P_{x_1}(\tau_1 \leq t)$.

How to obtain $\psi(x_1, x_2, t)$?

But if $x = (x_1, x_2) \in \mathcal{B}_1$ then competition between Y_t^2 et $X_t :=< Y_t, \overrightarrow{v} >$ distance between (Y_t^1, Y_t^2) and D_2.

In fact, one can show that X_t is deterministic, decreasing, and $X_t = 0$ for a (non random) $t = T(x_1, x_2)$. Hence

$$\psi(x_1, x_2, t)$$

$$= \begin{cases} P_{x_2}(\tau_2 \leq t), & t < T(x), \\ P_{x_2}(\tau_2 \leq T(x)) + P_{x_1}(\tau_1 \leq t) - P_{x_1}(\tau_1 \leq T(x)), & t \geq T(x). \end{cases}$$

Content

Introduction
- One dimensional model
- Ruin problems in dim. 2

Three solvable cases in dim. $K \geq 2$
- A 2-dim. model with interest rate an proportional reinsurance
- A K-dim. model with interest rate and common shocks
- A 2-dim. model with proportional reinsurance

Asymptotics for a non Markovian dim. 2 model

Model

$$\begin{cases} dY_t &= (\delta Y_t + p)dt - dS(t), \\ Y_0 &= (Y_0^1, ..., Y_0^K) = (x_1, ..., x_K) \in \mathbb{R}^K. \end{cases}$$

Here $Y_t = (Y_t^1, ..., Y_t^K) \in \mathbb{R}^K$, $p \in \mathbb{R}_+^K$, $\{S(t),\ t \geq 0\}$ CPP with *common shocks* :

$$S(t) = \sum_{j=1}^{N_t} Z_j$$

with N_t Poisson process and $(Z_j)_{j\in\mathbb{N}}$ i.i.d. but with $Z_j \in \mathbb{R}_+^K$ having correlated components.

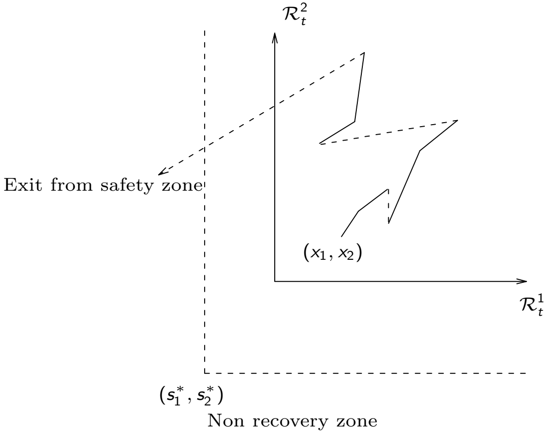
- $s_i^* := -p_i/\delta$, $i = 1, ..., K$, *definitive ruin* threshold.
- $\tau_i^*(x_i) := \inf\{t \geq 0 |\ Y_t^i < s_i^*\}$, $i = 1, 2$,

We are here interested in

$$\boxed{\tau^*(x_1, ..., x_K) = \inf\{t \geq 0, Y_t \notin \mathcal{D}\}, \quad \mathcal{D} = \prod_{i=1}^K [s_i^*, +\infty) \subset \mathbb{R}^K}$$

i.e. $\tau^*(x_1, ..., x_K) = \min(\tau_i^*(x_i), i = 1, ..., K)$.

Sample path of the multidimensional risk process



We set

$$\psi^*(x_1, ..., x_K, t) = P(\tau^*(x_1, ..., x_K) < t)$$

the c.d.f. of $\tau^*(x_1, ..., x_K)$.

LT of Dual process

Goal : Expression of the (multiple) Laplace Transform $\int_{x=(x_1,\ldots,x_K)\in\mathcal{D}} \psi^*(x_1,\ldots,x_K,t)e^{v'x}dx,\ v \in \mathbb{R}_+^K$.

Tool : Use of Dual process $\{Q(t) \in \mathbb{R}^K,\ t \geq 0\}$:

$$\begin{cases} dQ(t) &= -(\delta Q(t) + p)dt + dS_t \\ Q(0) &= s^* = (s_1^*,\ldots,s_K^*), \end{cases}$$

Proposition

One has

$$\int_{x=(x_1,\ldots,x_K)\in\mathcal{D}} \psi^*(x_1,\ldots,x_K,t)e^{v'x}dx = \sum_{A\subset\{1,\ldots,K\}} C_A\ E\left(e^{v_A' Q(t)}\right)$$

where $v_A \in \mathbb{R}_+^K$ defined by $[v_A]_i = v_i \mathbf{1}_{\{i\in A\}}$, and for some explicit constants C_A, $A \subset \{1,\ldots,K\}$.

Content

No interest rate and prop. reinsurance on one claim : A.Badescu, E.C.K.Cheung and L.R. (2011)

$Y_t = (Y_t^1, Y_t^2)$ verifies :

$$\begin{cases} dY_t^1 = p_1\, dt - a\, dL_t - dS_t, \\ dY_t^2 = p_2\, dt - (1-a)\, dL_t, \\ (Y_0^1, Y_0^2) = (x_1, x_2), \end{cases} \tag{2}$$

- Two types of independent claims $\{L_t,\ t \geq 0\}$ and $\{S_t,\ t \geq 0\}$,
- Proportion $a \in (0,1)$ of claims L_t taken in charge by Y^1 (insurer), $1-a$ taken in charge by Y^2 (reinsurer).

Recall that : $\tau := \inf\{t \geq 0 |\ Y_t \in \mathbb{R}^2 \setminus [0, +\infty)^2\}$.

Large and small claims

Suppose that the following assumption holds

$$\frac{p_2}{p_1} > \frac{1-a}{a}, \tag{3}$$

"Reinsuring large claims", i.e. $E(L_1) \gg E(S_1)$.

$\Longrightarrow$ Existence of absobing set $\mathcal{A}^- \subset [0, +\infty)^2$.

Large and small claims

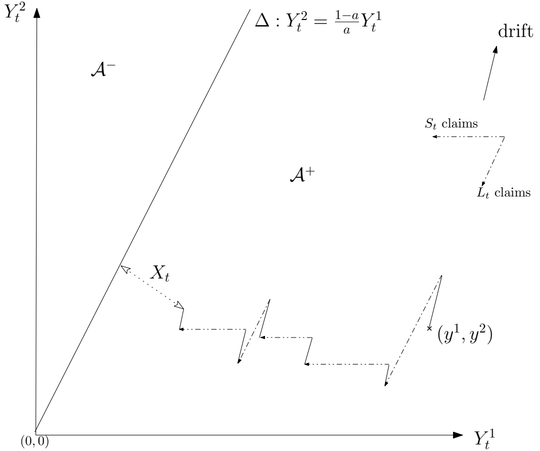


FIGURE : Partitioning $[0, +\infty)^2$

Partitioning

X_t (algebraic) distance from $Y_t = (Y_t^1, Y_t^2)$ to Δ :

$$X_t = X_0 - ct - (1-a)S_t$$

for a certain $c > 0$. This time, $\{X_t,\ t\ \geq 0\}$ is a random process.

- $X_t < 0$ (resp. > 0) iff $Y_t \in \mathcal{A}^-$ (resp. $\mathcal{A}^+$),
- $(X_t)_{t\geq 0}$ is a.s. decreasing,
- $(X_t)_{t\geq 0}$ and $(Y_t^2)_{t\geq 0}$ are independent.

Let $\tau_X := \inf\{t \geq 0 |\ X_t < 0\}$ hitting time of 0 of X_t.

How to obtain Laplace Transform of τ ?

Two cases :

- $Y_t \in \mathcal{A}^+$ ($\iff$ $X_t > 0$) : potential ruin of Y_t^2,
- $Y_t \in \mathcal{A}^-$ ($\iff$ $X_t < 0$) : potential ruin of Y_t^1

Laplace (or Fourier) transform of τ starting from $Y_0 \in \mathcal{A}^+$ by considering the two following different scenarios :

$$E_{(x_1,x_2)}\left[e^{-\beta\tau}\mathbf{1}_{\{\tau<\infty\}}\right]$$

$$= E_{(x_1,x_2)}\left[e^{-\beta\tau_2}\mathbf{1}_{\{\tau_X>\tau_2\}}\right] + E_{(x_1,x_2)}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_X\leq\tau_2,\tau_1<\infty\}}\right]$$

1. Y_t^2 hits 0 before $X_t \Longrightarrow \tau = \tau_2$ jointly to event $[\tau_X > \tau_2]$,
2. X_t hits 0 before $Y_t^2 \Longrightarrow$ "$\tau =$ ruin time of Y^1 starting from $Y^1_{\tau_X}$ given $\tau_X < \tau_2$, + excursion time within $\mathcal{A}^+$".

Content

Asymptotics : R.(2012)

Model :

$$\begin{cases} Y_t^1 & = & x_1 + \int_0^t p_1(J(s))ds - aS_t - bB_t^H, \\ Y_t^2 & = & x_2 + \int_0^t p_2(J(s))ds - (1-a)S_t - (1-b)B_t^H. \end{cases}$$

- $\{J(t),\ t \geq 0\}$ irreducible finite Markov chain, $\{S_t,\ t \geq 0\}$ additive Markov process, with light tailed claims,
- $\{B_t^H,\ t \geq 0\}$ fractional brownian motion with Hurst parameter $H \in (1/2, 1)$ (long range dependence).
- $p_i(.)$, $i = 1, 2$ premium rates,
- proportional reinsurance with rates $a \in (0,1)$ and $b \in (0,1)$.

We assume that safety loadings ρ_i, $i = 1, 2$, are positive.

Asymptotics

Goal : Determine asymptotics

$$\boxed{-\frac{1}{x_1^{2-2H}} \ln \psi_{\text{or}}(x_1, x_2) \longrightarrow C^*_{\text{or}}(\beta, H),}$$

$$\boxed{-\frac{1}{x_1^{2-2H}} \ln \psi_{\text{sim}}(x_1, x_2) \longrightarrow C^*_{\text{sim}}(\beta, H).}$$

as $||(x_1, x_2)|| \longrightarrow +\infty$ with $x_2/x_1 = \beta$.

$$\begin{aligned} \psi_{\text{or}}(x_1, x_2) &= P_{(x_1,x_2)}(\tau_{\text{or}} < +\infty) \\ \psi_{\text{sim}}(x_1, x_2) &= P_{(x_1,x_2)}(\tau_{\text{sim}} < +\infty) \end{aligned}$$

Here :

$$\begin{aligned} \tau_{\text{or}} &:= \inf\{t \geq 0 |\ Y_t \in \mathbb{R}^2 \setminus [0, +\infty)^2\} \\ \tau_{\text{sim}} &:= \inf\{t \geq 0 |\ Y_t \in (-\infty, 0)^2\}. \end{aligned}$$

Asymptotics.

Tools :

- Reduction to a one dimensional first passage problem,
- Use result by Duffield and O'Connell (1995).

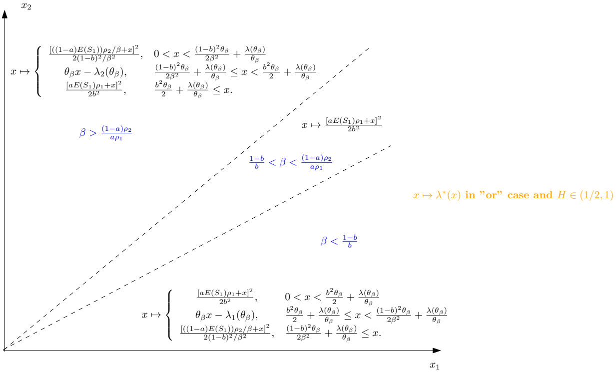
Prior remarks :

- Avram et al (2008) provide sharp asymptotics in the Lévy case.
- Few results concerning first passage times distribution of FBM (even in the one dimensional case !), see Molchan (2000), Decreusefond and Nualart (2008).

Result

Theorem (R.(2012))

Positive constant $C^*_{or}(\beta, H)$ ($H \in (1/2, 1)$) is equal to $\inf_{z>0} z^{-(2-2H)} \lambda^*(z)$, where $\lambda^*(.)$ is given in the following figure.



Similar result for $C^*_{sim}(\beta, H)$ ($H \in (5/6, 1)$)

Merci !